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Weak and Strong Convergence of Hybrid Subgradient Method for Pseudomonotone Equilibrium Problems and Nonspreading-Type Mappings in Hilbert Spaces

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Abstract. In this paper, we introduce a hybrid subgradient method for finding an element common to both the solution set of a class of pseudomonotone equilibrium problems, and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space. We establish some weak and strong convergence theorems of the sequences generated by our iterative method under some suitable conditions. These convergence theorems are investigated without the Lipschitz condition for bifunctions. Our results complement many known recent results in the literature.

1. Introduction

Let H be a real Hilbert space in which the inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$ and we denote the set of fixed points of T by $F(T)$. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C,$$

and it is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x \in C, \text{ and } y \in F(T).$$

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A mapping $T : C \to C$ is said to be *a strict pseudocontraction* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C,$$

where I is the identity mapping on H. If $k = 0$, then T is nonexpansive on C.

In 2008, Kohsaka and Takahashi [15] defined a mapping T in a in Hilbert spaces H to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \text{ for all } x, y \in C.$$

Following the terminology of Browder-Petryshyn [10], Osilike and Isiogugu [17] called a mapping T of C into itself *κ-strictly pseudononspreading* if there exists $\kappa \in [0, 1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2+2\langle x-Tx, y-Ty\rangle+\kappa\|x-Tx-(y-Ty)\|^2, \text{ for all } x, y \in C.$$

Clearly, every nonspreading mapping is κ-strictly pseudononspreading but the converse is not true; see [17]. We note that the class of strict pseudocontraction mappings and the class of κ-strictly pseudononspreading mappings are independent.

In 2010, Kurokawa and Takahashi [16] obtained a weak mean ergodic theorem of Baillon's type [7] for nonspreading mappings in Hilbert spaces. Furthermore, using the idea of mean convergence in Hilbert spaces, they also proved a strong convergence theorem of Halpern's type [12] for this class of mappings. After that, in 2011, Osilike and Isiogugu [17] introduced the concept of κ-strictly pseudononspreading mappings and they proved a weak mean convergence theorem of Baillon's type similar to [16]. They further proved a strong convergence theorem using the idea of mean convergence. This theorem extended and improved the main theorems of [16] and gave an affirmative answer to an open problem posed by Kurokawa and Takahashi [16] for the case when the mapping T is averaged. In 2013 Kangtunyakarn [14] proposed a new technique, using the projection method, for κ-strictly pseudononspreading mappings. He obtained a strong convergence theorem for finding the common element of the set of solutions of a variational inequality, and the set of fixed points of κ-strictly pseudononspreading mappings in a real Hilbert space.

On the other hand, let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F, C)$. It is well known that there are several problems, such as complementarity problems, minimax problems, the Nash equilibrium problem in noncooperative games, fixed point problems, optimization problems, that can be written in the form of an EP. In other words, the EP is a unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.; see [6, 8, 11] and the references therein.

In recent years the problem of finding an element common to the set of solutions of a equilibrium problems, and the set of fixed points of nonlinear mappings, has become a fascinating subject, and various methods have been developed by many authors for solving this problem (see [1, 4, 5, 20]). Most of all the existing algorithms for this problem are based on applying the proximal point method to the equilibrium problem $EP(F,C)$, and using a Mann's iteration to the fixed point problems of nonexpansive mappings. The convergence analysis has been considered when the bifunction F is monotone. This is because the proximal point method is not valid when the underlying operator F is pseudomonotone.

Recently, Anh [2] introduced a new hybrid extragradient iteration method for finding a element common to the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone bifunctions. In this algorithm the equilibrium bifunction is not required to satisfy any monotonicity property, but it must satisfy a Lipschitz-type continuous bifunction i.e. there are two Lipschitz constants $c_1 > 0$ and $c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \ \forall x,y,z \in C. \tag{1.2}$$

They obtained strongly convergent theorems for the sequences generated by these processes in a real Hilbert space.

Anh and Muu [3] reiterated that the Lipschitz-type condition (1.2) is not in general satisfied, and if it is, that finding the constants c_1 and c_2 is not easy. They further observed that solving strongly convex programs is also difficult except in special cases when C has a simple structure. They introduced and studied a new algorithm, which is called a hybrid subgradient algorithm for finding a common point in the set of fixed points of nonexpansive mappings and the solution set of a class of pseudomonotone equilibrium problems in a real Hilbert space. The proposed algorithm is a combination of the well-known Mann's iterative scheme for fixed point and the projection method for equilibrium problems. Furthermore, the proposed algorithm uses only one projection and does not require any Lipschitz condition for the bifunctions. To be more precise, they proposed the following iterative method:

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \gamma_n w_n), \ \gamma_n = \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tu_n, \ \text{for each } n = 1,2,3,..., \end{cases} \tag{1.3}$$

where $\partial_\epsilon F(x,\cdot)(x)$ stands for ϵ-subdifferential of the convex function $F(x,\cdot)$ at x and $\{\epsilon_n\}$, $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\}$, and $\{\alpha_n\}$ were chosen appropriately. Under certain conditions, they prove that $\{x_n\}$ converges strongly to a common point in the set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mapping. Using the idea of Anh and Muu [3], Thailert et al. [21] proposed a new algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of common fixed points of a family of strict pseudocontraction mappings in a real Hilbert space. Then Thailert et al. [22] introduced new general iterative methods for finding a common element in the solution set of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings which is a solution of a certain optimization problem related to a strongly positive linear operator. Under suitable control conditions, They proved the strong convergence theorems of such iterative schemes in a real Hilbert space.

In this paper, motivated by Anh and Muu [3], Kangtunyakarn [14], and other research going on in this direction, we proposed a hybrid subgradient method for the pseudomonotone equilibrium problem and the finite family of κ-strictly pseudononspreading mapping in a real Hilbert space. The weak and strong convergence of the proposed methods is investigated under certain assumptions. Our results improve and extend many recent results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. It is well-known that for all $x, y, z \in H$ and α, β, $\gamma \in [0, 1]$, with $\alpha+\beta+\gamma = 1$ there holds

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{2.1}$$

and

$$\| \alpha x + \beta y + \gamma z \|^2 = \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \|z\|^2 - \alpha\beta \| x - y \|^2 - \beta\gamma \|y - z\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\| x - P_C x \| \leq \| x - y \|$ for all $y \in C$. Such a P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that, $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \text{for all } x \in H \text{ and } z \in C. \tag{2.3}$$

It is easy to see that (2.3) equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2, \quad \text{for all } x \in H \text{ and } z \in C. \tag{2.4}$$

Lemma 2.1.([19]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u,$$

where P_C is the metric projection of H onto C.

Recall that a bifunction $F : C \times C \to \mathbb{R}$ is said to be

(i) *η-strongly monotone* if there exists a number $\eta > 0$ such that

$$F(x, y) + F(y, x) \le -\eta\|x - y\|^2, \text{ for all } x, y \in C,$$

(ii) *monotone* on C if

$$F(x, y) + F(y, x) \le 0, \text{ for all } x, y \in C,$$

(iii) *pseudomonotone* on C with respect to $x \in C$ if

$$F(x, y) \ge 0 \text{ implies } F(y, x) \le 0, \text{ for all } y \in C.$$

It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), for every $x \in C$. Moreover, F is said to be *pseudomonotone* on C with respect to $A \subseteq C$, if it is pseudomonotone on C with respect to every $x \in A$. When $A \equiv C$, F is called pseudomonotone on C.

The following example, taken from [18], shows that a bifunction may not be pseudomonotone on C, but yet is pseudomonotone on C with respect to the solution set of the equilibrium problem defined by F and C:

$$F(x, y) := 2y|x|(y - x) + xy|y - x|, \text{ for all } x, y \in \mathbb{R},\ C := [-1, 1].$$

Clearly, $EP(F) = \{0\}$. Since $F(y, 0) = 0$ for every $y \in C$, this bifunction is pseudomonotone on C with respect to the solution $x^* = 0$, However, F is not pseudomonotone on C. In fact, both $F(-0.5, 0.5) = 0.25 > 0$ and $F(0.5, -0.5) = 0.25 > 0$.

For solving the equilibrium problem (1.1), let us assume that Δ is an open convex set containing C and the bifunction $F : \Delta \times \Delta \to \mathbb{R}$ satisfies the following assumptions:

(A1) $F(x, x) = 0$ for all $x \in C$ and $F(x, \cdot)$ is convex and lower semicontinuous on C;

(A2) for each $y \in C$, $F(\cdot, y)$ is weakly upper semicontinuous on the open set Δ;

(A3) F is pseudomonotone on C with respect to $EP(F, C)$ and satisfies the strict paramonotonicity property, i.e., $F(y, x) = 0$ for $x \in EP(F, C)$ and $y \in C$ implies $y \in EP(F, C)$;

(A4) if $\{x_n\} \subseteq C$ is bounded and $\epsilon_n \to 0$ as $n \to \infty$, then the sequence $\{w_n\}$ with $w_n \in \partial_n F(x_n, \cdot)x_n$ is bounded, where $\partial_\epsilon F(x, \cdot)x$ stands for the ϵ-subdifferential of the convex function $F(x, \cdot)$ at x.

The following idea of the ϵ-subdierential of convex functions can be found in the work of Bronsted and Rockafellar [9] but the theory of ϵ-subdierential calculus was given by Hiriart-Urruty [13].

Definition 2.2. Consider a proper convex function $\phi : C \to \overline{\mathbb{R}}$. For a given $\epsilon > 0$, the ϵ-subdierential of ϕ at $x_0 \in Dom\phi$ is given by

$$\partial_\epsilon \phi(x_0) = \{x \in C : \phi(y) - \phi(x_0) \ge \langle x, y - x_0\rangle - \epsilon,\ \forall y \in C\}.$$

Remark 2.3. It is known that if the function ϕ is proper lower semicontinuous convex, then for every $x \in Dom\phi$, the ϵ-subdierential $\partial_\epsilon \phi(x)$ is a nonempty closed convex set (see [13]).

Next, throughout this paper, weak and strong convergence of a sequence $\{x_n\}$ in H to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. In order to prove our main results, we need the following lemmas.

Lemma 2.4.([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a κ-strictly pseudonospreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Remark 2.5. If $T : C \to C$ is a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$, then from Lemma 2.8 in [14] and Lemma 2.1, we have $F(T) = VI(C,(I-T)) = F(P_C(I - \lambda(I - T)))$, for all $\lambda > 0$.

Lemma 2.6. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For every $i = 1, 2, ..., N$, let $T_i : C \to C$ be a finite family of κ_i-strictly pseudononspreading mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{a_1, a_2, ..., a_n\} \subset (0, 1)$ with $\Sigma_{i=1}^N a_i = 1$, let $\bar{\kappa} = max\{\kappa_1, \kappa_2, ..., \kappa_N\}$ and let $\lambda \in (0, 1 - \bar{\kappa})$. Then*

(i) $\bigcap_{i=1}^N F(T_i) = F(\Sigma_{i=1}^N a_i P_C(I - \lambda(I - T_i)))$.

(ii) $\|\Sigma_{i=1}^N a_i P_C(I-\lambda(I-T_i))x-y\|^2 \leq \|x-y\|^2$, *for all* $x \in C$ *and* $y \in \bigcap_{i=1}^N F(T_i)$, *i.e.* $\Sigma_{i=1}^N a_i P_C(I - \lambda(I - T_i))$ *is quasi-nonexpansive.*

Proof. (i) It easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(\Sigma_{i=1}^N a_i P_C(I - \lambda(I - T_i)))$. Let $x \in F(\Sigma_{i=1}^N a_i P_C(I-\lambda(I-T_i)))$ and let $x^* \in \bigcap_{i=1}^N F(T_i) \subseteq F(\Sigma_{i=1}^N a_i P_C(I-\lambda(I-T_i)))$. Note that for every $i = 1, 2, 3, ..., N$ we have

$$\begin{aligned} \|P_C(I - \lambda(I - T_i))x - x^*\|^2 &\leq \|x - x^* - \lambda(I - T_i)\|^2 \\ &= \|x - x^*\|^2 - 2\lambda\langle x - x^*, (I - T_i)x\rangle \\ &\quad + \lambda^2\|(I - T_i)x\|^2. \end{aligned} \tag{2.5}$$

Put $A_i = I - T_i$, for all $i = 1, 2, ..., N$, we have $T_i = I - A_i$ and

$$\begin{aligned} \|T_i x - T_i x^*\|^2 &= \|(I - A_i)x - (I - A_i)x^*\|^2 \\ &= \|(x - x^*) - A_i x\|^2 \\ &= \|x - x^*\|^2 - 2\langle x - x^*, A_i x\rangle + \|A_i x\|^2 \\ &\leq \|x - x^*\|^2 + \kappa_i\|(I - T_i)x - (I - T_i)x^*\|^2 + 2\langle x - T_i x, x^* - T_i x^*\rangle \\ &= \|x - x^*\|^2 + \kappa_i\|(I - T_i)x\|^2, \end{aligned} \tag{2.6}$$

which implies that

$$(1 - \kappa_i)\|(I - T_i)x\|^2 \leq 2\langle x - x^*, A_i x\rangle, \quad \text{for all } i = 1, 2, 3, ..., N$$

From (2.5) and (2.6), we have

$$\begin{aligned}\|P_C(I-\lambda(I-T_i))x-x^*\|^2 &\le \|x-x^*\|^2-2\lambda\langle x-x^*,(I-T_i)x\rangle\\ &\quad +\lambda^2\|(I-T_i)x\|^2\\ &\le \|x-x^*\|^2-\lambda(1-\kappa_i)\|(I-T_i)x\|^2\\ &\quad +\lambda^2\|(I-T_i)x\|^2\\ &= \|x-x^*\|^2-\lambda[(1-\kappa_i)-\lambda]\|(I-T_i)x\|^2\\ &\le \|x-x^*\|^2, \qquad (2.7)\end{aligned}$$

for all $i=1,2,3,\dots,N$.
From the definition of x and (2.7), we have

$$\begin{aligned}\|x-x^*\|^2 &= \|\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i))x-x^*\|^2\\ &= a_1\|P_C(I-\lambda(I-T_1))x-x^*\|^2+a_2\|P_C(I-\lambda(I-T_2))x-x^*\|^2+\cdots\\ &\quad +a_N\|P_C(I-\lambda(I-T_N))x-x^*\|^2-a_1a_2\|P_C(I-\lambda(I-T_1))x\\ &\quad -P_C(I-\lambda(I-T_2))x\|^2-a_2a_3\|P_C(I-\lambda(I-T_2))x-\\ &\quad P_C(I-\lambda(I-T_3))x\|^2-\cdots-a_{N-1}a_N\|P_C(I-\lambda(I-T_{N-1}))x-\\ &\quad P_C(I-\lambda(I-T_N))x\|^2\\ &\le \|x-x^*\|^2-a_1a_2\|P_C(I-\lambda(I-T_1))x-P_C(I-\lambda(I-T_2))x\|^2\\ &\quad -a_2a_3\|P_C(I-\lambda(I-T_2))x-P_C(I-\lambda(I-T_3))x\|^2-\cdots\\ &\quad -a_{N-1}a_N\|P_C(I-\lambda(I-T_{N-1}))x-P_C(I-\lambda(I-T_N))x\|^2.\end{aligned}$$

This implies that

$$P_C(I-\lambda(I-T_1))x=P_C(I-\lambda(I-T_2))x=\cdots=P_C(I-\lambda(I-T_N))x$$

Since $x\in F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))$, we get that $x=P_C(I-\lambda(I-T_i))x$, for all $i=1,2,3,...,N$ From Remark 2.5, we have $x\in F(T_i)$, for all $i=1,2,3,...,N$. That is $x\in\bigcap_{i=1}^N F(T_i)$. Hence $F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))\subseteq\bigcap_{i=1}^N F(T_i)$.
(ii) Let $x\in C$ and $y\in\bigcap_{i=1}^N F(T_i)=F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))$
As the same argument as in (i), we can show that

$$\|P_C(I-\lambda(I-T_i))x-y\|^2\le\|x-y\|^2, \qquad (2.8)$$

for all $i=1,2,3,...,N$. Thus

$$\begin{aligned}\|\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i))x-y\|^2 &\le a_1\|P_C(I-\lambda(I-T_1))x-y\|^2\\ &\quad +a_2\|P_C(I-\lambda(I-T_2))x-y\|^2+\cdots\\ &\quad +a_N\|P_C(I-\lambda(I-T_N))x-y\|^2\\ &\le \Sigma_{i=1}^N a_i\|x-y\|^2=\|x-y\|^2. \qquad (2.9)\end{aligned}$$

□

Lemma 2.7.([23]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \ \ n \geq 1,$$

where $\sum_{n=0}^{\infty} b_n < \infty$. Then the sequence $\{a_n\}$ is convergent.

3. Weak Convergence Theorem

In this section, we prove weak convergence theorem for finding a common element in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{\kappa_1, \kappa_2, ..., \kappa_N\} \subset [0, 1)$ and $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i \leq \min\{1 - \kappa_1, 1 - \kappa_2, ..., 1 - \kappa_N\}$ *and* $\Sigma_{n=1}^{\infty} \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$;

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$;

(iii) $\sum_{n=0}^{\infty} \delta_n = \infty$, $\sum_{n=0}^{\infty} \delta_n^2 < \infty$, *and* $\sum_{n=0}^{\infty} \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

Proof. First, we will show that $\{x_n\}$ is bounded. Let $p \in \Omega$. Then we have

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\langle x_n - u_n, p - u_n\rangle \\ &\leq \|x_n - p\|^2 + 2\langle x_n - u_n, p - u_n\rangle. \end{aligned} \tag{3.2}$$

Since $u_n = P_C(x_n - \rho_n w_n)$ and $p \in C$, we get that

$$\langle x_n - u_n, p - u_n\rangle \leq \rho_n \langle w_n, p - u_n\rangle. \tag{3.3}$$

Substuting (3.3) into (3.2), we have

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 + 2\rho_n\langle w_n, p - u_n\rangle \\
&= \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\rho_n\langle w_n, x_n - u_n\rangle \\
&\le \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\rho_n\|w_n\|\|x_n - u_n\| \\
&\le \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\delta_n\|x_n - u_n\|. \qquad (3.4)
\end{aligned}$$

By using $u_n = P_C(x_n - \rho_n w_n)$ and $x_n \in C$ again, we get

$$\begin{aligned}
\|x_n - u_n\|^2 &= \langle x_n - u_n, x_n - u_n\rangle \\
&\le \rho_n\langle w_n, x_n - u_n\rangle \\
&\le \rho_n\|w_n\|\|x_n - u_n\| \\
&\le \delta_n\|x_n - u_n\|, \qquad (3.5)
\end{aligned}$$

which implies that

$$\|x_n - u_n\| \le \delta_n. \qquad (3.6)$$

By condition (iii), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \qquad (3.7)$$

Combining (3.4) and (3.6), we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\delta_n^2. \qquad (3.8)$$

Since $w_n \in \partial_{\epsilon_n}F(x_n, \cdot)x_n$, $p \in C$ and $F(x, x) = 0$ for each $x \in C$, we obtain that

$$\begin{aligned}
\langle w_n, p - x_n\rangle &\le F(x_n, p) - F(x_n, x_n) + \epsilon_n \\
&= F(x_n, p) + \epsilon_n. \qquad (3.9)
\end{aligned}$$

Thus, it follows from (3.8) and (3.9) that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + 2\rho_n F(x_n, p) + 2\rho_n\epsilon_n + 2\delta_n^2. \qquad (3.10)$$

Form Lemma 2.6 (ii), we have

$$\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - p\|^2 \le \|x_n - p\|^2. \qquad (3.11)$$

From (3.1), (3.10) and (3.11), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + \beta_n\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + \beta_n\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - p\|^2 \\
&\quad + \gamma_n\|u_n - p\|^2 - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\Big(\|x_n - p\|^2 + 2\rho_n F(x_n, p) \\
&\quad + 2\rho_n\epsilon_n + 2\delta_n^2\Big) - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\
&= \|x_n - p\|^2 + 2\gamma_n\rho_n F(x_n, p) + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\
&\quad - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2. \qquad (3.12)
\end{aligned}$$

Since $p \in EP(F,C)$ and F is pseudomonotone on F with respect to p, we get that $F(x_n,p) \leq 0$ for all $n \in \mathbb{N}$. Then from (3.12) it follows that

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \|x_n - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\quad -\alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I-\lambda_n^i(I-T_i))x_n\|^2 \\ &\leq \|x_n - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2. \end{aligned} \tag{3.13}$$

Let $\eta_n = 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2$ for all $n \geq 0$. From condition (ii) and (iii), we get that

$$\Sigma_{n=0}^\infty \eta_n = \Sigma_{n=0}^\infty(2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2) \leq 2b\Sigma_{n=0}^\infty \rho_n\epsilon_n + 2b\Sigma_{n=0}^\infty \delta_n^2 < +\infty$$

Now applying Lemma 2.7 to (3.13), we obtain that the $\lim_{n\to\infty}\|x_n - p\|$ exists, i.e. $\lim_{n\to\infty}\|x_n - p\| = \bar{a}$ for some $\bar{a} \in C$. Thus $\{x_n\}$ is bounded. Also, it easy to verify that $\{u_n\}$ and $\{\Sigma_{i=1}^N a_i P_C(I-\lambda_n^i(I-T_i))x_n\}$ are also bounded.

Next, we will show that $\limsup_{n\to\infty} F(x_n,p) = 0$ for any $p \in \Omega$. Since F is pseudomonotone on C and $F(p,x_n) \geq 0$, we have $-F(x_n,p) \geq 0$. From (3.12) and condition (ii), we have

$$\begin{aligned} 2\gamma_n\rho_n[-F(x_n,p)] &\leq \|x_n - p\|^2 - \|x_{n+1}-p\|^2 \\ &\quad +2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1}-p\|^2 + 2b\rho_n\epsilon_n + 2b\delta_n^2. \end{aligned} \tag{3.14}$$

Summing up (3.14) for every n, we obtain

$$\begin{aligned} 0 &\leq 2\sum_{n=0}^\infty \gamma_n\rho_n[-F(x_n,p)] \\ &\leq \|x_0 - p\|^2 + 2b\sum_{n=0}^\infty \rho_n\epsilon_n + 2b\sum_{n=0}^\infty \delta_n^2 < +\infty. \end{aligned} \tag{3.15}$$

By the assumption (A_4), we can find a real number w such that $\|w_n\| \leq w$ for every n. Setting $\mathit{\Gamma} := \max\{\sigma, w\}$, where σ is a real number such that $0 < \sigma_n < \sigma$ for every n, it follows from (ii) that

$$0 \leq \frac{2a}{\mathit{\Gamma}}\sum_{n=0}^\infty \delta_n[-F(x_n,p)] \tag{3.16}$$

$$\leq 2\sum_{n=0}^\infty \gamma_n\rho_n[-F(x_n,p)] < +\infty, \tag{3.17}$$

which implies that

$$0 \leq \sum_{n=0}^\infty \delta_n[-F(x_n,p)] < +\infty. \tag{3.18}$$

Combining with $-F(x_n,p) \geq 0$ and $\sum_{n=0}^{\infty} \delta_n = \infty$, we can deduced that $\limsup_{n\to\infty} F(x_n,p) = 0$ as desired.

Next, we will show that $\omega_\omega(x_n) \subset \Omega$, where $\omega_\omega(x_n) = \{x \in H : x_{n_i} \rightharpoonup x$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}\}$. In deed since $\{x_n\}$ is bounded and H is reflexive, $\omega_\omega(x_n)$ is nonempty. Let $\bar{x} \in \omega_\omega(x_n)$. Then there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$. converging weakly to $\bar{x}$, that is $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$. By the convexity, C is weakly closed and hence $\bar{x} \in C$. Since $F(\cdot,p)$ is weakly upper semicontinuous for $p \in \Omega$, we obtain

$$\begin{aligned} F(\bar{x},p) &\geq \limsup_{i\to\infty} F(x_n,p) \\ &= \lim_{i\to\infty} F(x_{n_i},p) \\ &= \limsup_{n\to\infty} F(x_n,p) \\ &= 0. \end{aligned} \tag{3.19}$$

Since F is pseudomontone with respect to p and $F(p,\bar{x}) \geq 0$, we obtain $F(\bar{x},p) \leq 0$. Thus $F(\bar{x},p) = 0$. Furthermore, by assumption (A_3), we get that $\bar{x} \in EP(F,C)$. On the other hand, from (3.13) and conditions (ii)–(iii), we have

$$\begin{aligned} &\alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I-\lambda_n^i(I-T_i))x_n\|^2 \\ &\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2b\rho_n\epsilon_n + 2b\delta_n^2 \end{aligned} \tag{3.20}$$

taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I-T_i))x_n\| = 0. \tag{3.21}$$

Now, we will show that $\bar{x} \in \bigcap_{i=1}^N F(T_i)$. Assume that $\bar{x} \notin \bigcap_{i=1}^N F(T_i)$. By Lemma 2.6, we have $\bar{x} \notin F(\Sigma_{i=1}^N a_i P_C(I - \lambda_n(I-T_i)))$. From the Opial's condition, (3.21) and condition (i), we can write

$$\begin{aligned} \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| &< \liminf_{i\to\infty} \|x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I-T_i))\bar{x}\| \\ &\leq \liminf_{i\to\infty} \Big(\|x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I-T_i))x_{n_i}\| \\ &\quad + \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I-T_i))x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I-T_i))\bar{x}\| \Big) \\ &\leq \liminf_{i\to\infty} \Big(\|x_{n_i} - \bar{x}\| + \Sigma_{i=1}^N a_i \lambda_n^i \|(I-T_i)x_{n_i} - (I-T_i)\bar{x}\| \Big) \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\|. \end{aligned}$$

This is a contradiction. Then $\bar{x} \in \bigcap_{i=1}^N F(T_i)$. Thus $\bar{x} \in EP(F,C) \cap F(T) = \Omega$ and so $\omega_\omega(x_n) \subset \Omega$.

Finally, we prove that $\{x_n\}$ converge weakly to an element of Ω. It's sufficient to show that $\omega_\omega(x_n)$ is a single point set. Taking $z_1, z_2 \in \omega_\omega(x_n)$ arbitrarily, and let $\{x_{n_k}\}$ and $\{x_{n_m}\}$ be subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z_1$ and $x_{n_m} \rightharpoonup z_2$ respectively. Since $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$ and z_1, $z_2 \in \Omega$, we get that $\lim\limits_{n\to\infty} \|x_n - z_1\|$ and $\lim\limits_{n\to\infty} \|x_n - z_2\|$ exist. Now, assume that $z_1 \neq z_2$, then by the Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{k\to\infty} \|x_{n_k} - z_1\| \\
&< \lim_{k\to\infty} \|x_{n_k} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{m\to\infty} \|x_{n_m} - z_2\| \\
&< \lim_{m\to\infty} \|x_{n_m} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|, \qquad (3.22)
\end{aligned}$$

which is a contradiction. Thus $z_1 = z_2$. This show that $\omega_\omega(x_n)$ is single point set. i.e. $x_n \rightharpoonup \bar{x}$. This completes the proof. □

If we set $\kappa_i = 0$ for all $i = 1, 2, ..., N$ then we get the following Corollary.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$(3.23) \quad \begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i < 1$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$*;*

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$*;*

(ii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

4. Strong Convergence Theorem

In this section, to obtain strong convergence result, we add the control condition $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$, and then we get the strong convergence theorem for finding a common element in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{\kappa_1, \kappa_2, ..., \kappa_N\} \subset [0, 1)$ and $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n),\ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n,\ \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i \leq \min\{1 - \kappa_1, 1 - \kappa_2, ..., 1 - \kappa_N\}$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$*;*

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$ *and* $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$*;*

(iii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$.

Proof. By a similar argument to the proof of Theorem 3.1 and (2.4), we have

$$\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - P_\Omega(x_n)\|^2 \leq \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2$$

and

$$\|u_n - P_\Omega(x_n)\|^2 \leq \|u_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2. \tag{4.2}$$

It follows from (4.2) and condition (ii) that

$$
\begin{aligned}
&\|x_{n+1} - P_\Omega(x_{n+1})\|^2 \\
&\quad \leq \|\alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n + \gamma_n u_n - P_\Omega(x_n)\|^2 \\
&\quad \leq \alpha_n \|x_n - P_\Omega(x_n)\|^2 + \beta_n \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - P_\Omega(x_n))\|^2 \\
&\qquad +\gamma_n \|u_n - P_\Omega(x_n)\|^2 \\
&\quad \leq \alpha_n \|x_n - P_\Omega(x_n)\|^2 + \beta_n \Big(\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - x_n\|^2 \\
&\qquad -\|x_n - P_\Omega(x_n)\|^2 \Big) + \gamma_n \Big(\|u_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2 \Big) \\
&\quad = (\alpha_n - (\beta_n + \gamma_n)) \|x_n - P_\Omega(x_n)\|^2 + \beta_n \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - x_n\|^2 \\
&\qquad +\gamma_n \|u_n - x_n\|^2. \\
&\quad \leq (2\alpha_n - 1) \|x_n - P_\Omega(x_n)\|^2 + b \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n - x_n\|^2 \\
&\qquad +b \|u_n - x_n\|^2.
\end{aligned}
$$

Combining (3.7), (3.21), conditions (ii)–(iii), and the boundedness of the sequence $\{x_n - P_\Omega(x_n)\}$, we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - P_\Omega(x_{n+1})\| = 0 \tag{4.3}
$$

Since Ω is convex, for all $m > n$, we have $\frac{1}{2}(P_\Omega(x_m) + P_\Omega(x_n)) \in \Omega$, and therefore

$$
\begin{aligned}
\|P_\Omega(x_m) - P_\Omega(x_n)\|^2 &= 2\|x_m - P_\Omega(x_m)\|^2 + 2\|x_m - P_\Omega(x_n)\|^2 \\
&\quad -4\|x_m - \frac{1}{2}(P_\Omega(x_m) + P_\Omega(x_n))\|^2 \\
&\leq 2\|x_m - P_\Omega(x_m)\|^2 + 2\|x_m - P_\Omega(x_n)\|^2 \\
&\quad -4\|x_m - P_\Omega(x_m)\|^2 \\
&= 2\|x_m - P_\Omega(x_n)\|^2 - 2\|x_m - P_\Omega(x_m)\|^2. \qquad (4.4)
\end{aligned}
$$

Using (3.13) with $p = P_\Omega(x_n)$, we have

$$
\begin{aligned}
\|x_m - P_\Omega(x_n)\|^2 &\leq \|x_{m-1} - P_\Omega(x_n)\|^2 + \eta_{m-1} \\
&\leq \|x_{m-2} - P_\Omega(x_n)\|^2 + \eta_{m-1} + \eta_{m-2} \\
&\leq \ldots \\
&\leq \|x_n - P_\Omega(x_n)\|^2 + \sum_{j=n}^{m-1} \eta_j, \qquad (4.5)
\end{aligned}
$$

where $\eta_j = 2\gamma_j \rho_j \epsilon_j + 2\gamma_j \delta_j^2$. It follows from (4.4) and (4.5) that

$$
\|P_\Omega(x_m) - P_\Omega(x_n)\|^2 \leq 2\|x_n - P_\Omega(x_n)\|^2 + 2\sum_{j=n}^{m-1} \eta_j - 2\|x_m - P_\Omega(x_m)\|^2. \tag{4.6}
$$

Together with (4.3) and $\sum_{j=0}^{\infty} \eta_j < +\infty$, this implies that $\{P_\Omega(x_n)\}$ is a Cauchy sequence, Hence $\{P_\Omega(x_n)\}$ strongly converges to some point $x^* \in \Omega$. Moreover, we obtain

$$x^* = \lim_{i\to\infty} P_\Omega(x_{n_i}) = P_\Omega(\bar{x}) = \bar{x}, \tag{4.7}$$

which implies that $P_\Omega(x_i) \to x^* = \bar{x} \in \Omega$. Then from (4.3) and (4.7), we can conclude that $x_n \to \bar{x}$. This completes the proof. □

If we set $\kappa_i = 0$ for all $i = 1, 2, ..., N$ then we get the following Corollary.

Corollary 4.2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i (I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i < 1$ *and* $\Sigma_{n=1}^{\infty} \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$*;*

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$ *and* $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$*;*

(iii) $\sum_{n=0}^{\infty} \delta_n = \infty$, $\sum_{n=0}^{\infty} \delta_n^2 < \infty$, *and* $\sum_{n=0}^{\infty} \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

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On Some Identities and Generating Functions for (s,t)-Pell and (s,t)-Pell-Lucas Numbers

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Abstract

In this paper, we obtain the Binet's formula for (s,t)-Pell and (s,t)-Pell-Lucas numbers and then we get some identities for these numbers by using the Binet's formula. Moreover, we obtain the generating functions for (s,t)-Pell and (s,t)-Pell-Lucas sequences and another expression for the general term of the sequences by using the ordinary generating functions.

Keywords: Pell number; Pell-Lucas number; (s,t)-Pell number; (s,t)-Pell-Lucas number; Binet's formula; Generating function.

1. Introduction

It is well-known that the Fibonacci and Lucas numbers are the most famous of the recursive sequences that have been studied in the literature over several years. They are widely used in a variety of research areas such as Engineering, Architecture, Nature and Art. (see: [6-10]). For $n \geq 2$, the classical Fibonacci $\{F_n\}$ and Lucas $\{L_n\}$ sequences are defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, with the initial conditions $F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1$ respectively. On the other hand, other sequences that also important are Pell and Pell- Lucas sequences. The Pell and Pell-Lucas sequences are

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defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ and $Q_0 = 0, Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$, respectively. For more detailed information about Pell and Pell-Lucas sequences can be found in [4, 6]. Recently, Pell and Pell-Lucas numbers were generalized and studied by many authors in the different ways to derive many identities. For a lot of identities of Pell, Pell-Lucas numbers and their generalization can be found in [1, 2, 5-7] and the references therein.

In this paper we investigate the generalization of Pell and Pell-Lucas numbers, which is called (s,t)-Pell and (s,t)-Pell-Lucas numbers and then we obtain the Binet's formula and some identities for these numbers. Also, we give the generating functions for the (s,t)-Pell and (s,t)-Pell-Lucas sequences and another expression for the general term of the sequences, by using the ordinary generating functions.

2. (s,t)-Pell and (s,t)-Pell-Lucas Numbers and some identities

In this section, a new generalization of Pell and Pell-Lucas numbers are introduced and it's Binet's formula are obtained. After that, by using the Binet's formula, we obtain some identities for these numbers. We begin this section with the following definition.

Definition 2.1 [2] Let s,t be any real number with $s^2 + t > 0, s > 0$ and $t \neq 0$. Then the (s,t)-Pell sequences $\{P_n(s,t)\}_{n=0}^{\infty}$ and the (s,t)-Pell-Lucas sequences $\{Q_n(s,t)\}_{n=0}^{\infty}$ are defined respectively by

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t), \text{ for } n \geq 2, \quad (2.1)$$

$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t), \text{ for } n \geq 2, \quad (2.2)$$

with initial conditions $P_0(s,t) = 0, P_1(s,t) = 1$ and $Q_0(s,t) = 2,\ Q_1(s,t) = 2s.$

The first few terms of $\{P_n(s,t)\}_{n=0}^{\infty}$ are $0, 1, 2s, 4s^2 + t,\ 8s^3 + 4st$ and so on. Also, the first few terms of $\{Q_n(s,t)\}_{n=0}^{\infty}$ are $2, 2s, 4s^2 + 2t, 8s^3 + 6st$ and so on. The terms of (s,t)-Pell and (s,t)-Pell-Lucas sequences are called (s,t)-Pell-numbers and (s,t)-Pell-Lucas-numbers respectively.

Throughout this paper, for convenience we will use the symbol P_n and Q_n instead of $P_n(s,t)$ and $Q_n(s,t)$ respectively. Also, we denoted the set of whole numbers by $\mathbb{N}_0$ (i.e. $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$).

Particular case of the Definition 2.1 are :

- If $s = \frac{1}{2}, t = 1$ then the classical Fibonacci and Lucas sequence are obtained.
- If $s = t = 1$ then the classical Pell and Pell-Lucas sequence are obtained.
- If $s = \frac{1}{2}, t = 2$ then the classical Jacobsthal and Jacobsthal-Lucas sequence are obtained.

Next, we give the explicit formula for the term of order n of (s,t)-Pell and (s,t)-Pell-Lucas numbers. From the Definition 2.1, we have that the characteristic equation of (2.1) and (2.2) are in the form

$$x^2 = 2sx + t \quad (2.3)$$

and the root of equation (2.3) are $\alpha = s + \sqrt{s^2 + t}$ and $\beta = s - \sqrt{s^2 + t}$. We note that $\alpha + \beta = 2s$, $\alpha - \beta = 2\sqrt{s^2 + t}$ and $\alpha\beta = -t$ and we get the following theorem.

Theorem 2.2 (Binet's formula)

The n^{th} (s,t) -Pell number and the n^{th} (s,t) -Pell-Lucas number are given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.4}$$

and

$$Q_n = \alpha^n + \beta^n \tag{2.5}$$

respectively, where α, β are the roots of the characteristic equation (2.3) and $\alpha > \beta$.

Proof. Since the characteristic equation (2.3) has two distinct roots, the closed form of $\{P_n\}$ is given by

$$P_n = c_1\alpha^2 + c_2\beta^n,$$

for some coefficients c_1 and c_2. Giving to n the values $n = 0$ and $n = 1$ then solving this system of linear equations

we obtain $c_1 = \dfrac{1}{\alpha - \beta} = -c_2$, and therefore

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Similarly, the closed form of Q_n is given by

$$Q_n = c_1\alpha^n + c_2\beta^n,$$

for some coefficients c_1 and c_2. By the same fashion as above, we obtain $c_1 = c_2 = 1$, and hence

$$Q_n = \alpha^n + \beta^n. \qquad \square$$

Theorem 2.3 (Catalan's identity)

Let $n, r \in \mathbb{N}_0$ with $n > r$. Then

$$P_{n+r}P_{n-r} - P_n^2 = -(-t)^{n-r} P_n^2 \tag{2.6}$$

and

$$Q_{n+r}Q_{n-r} - Q_n^2 = (-t)^{n-r}\left(Q_r^2 - 4(-t)^r\right) \tag{2.7}$$

Proof. Using the Binet's formula (2.4), we have

$$\begin{aligned} P_{n+r}P_{n-r} - P_n^2 &= \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha - \beta} \cdot \frac{\alpha^{n-r} - \beta^{n-r}}{\alpha - \beta} \\ &\quad - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 \\ &= \frac{-(\alpha\beta)^{n-r}(\alpha^r - \beta^r)^2}{(\alpha - \beta)^2} \\ &= -(-t)^{n-r} P_r^2. \end{aligned}$$

Using the Binet's formula (2.5), we have

$$\begin{aligned} Q_{n+r}Q_{n-r} - Q_n^2 &= (\alpha^{n+r} + \beta^{n+r})(\alpha^{n-r} + \beta^{n-r}) \\ &\quad - (\alpha^n + \beta^n)^2 \\ &= (\alpha\beta)^{n-r}\left((\alpha^r + \beta^r)^2 - 4(\alpha\beta)^r\right) \\ &= (-t)^{n-r}\left(Q_r^2 - 4(-t)^r\right). \qquad \square \end{aligned}$$

Note that for $r = 1$, equation (2.6) and (2.7) give Cassini's identity for the (s,t) -Pell and (s,t) -Pell-Lucas numbers respectively.

Theorem 2.4 (Cassini's identity)

Let $n \in \mathbb{N}_0$. Then

$$P_{n+1}P_{n-1} - P_n^2 = -(-t)^{n-1} \tag{2.8}$$

and

$$Q_{n+1}Q_{n-1} - Q_n^2 = 4(s^2 + t)(-t)^{n-1} \tag{2.9}$$

Proof. By letting $r = 1$ in Theorem 2.3, we get the results. $\square$

Theorem 2.5 (d'Ocagne's identity)

Let $m,n \in \mathbb{N}_0$ *with* $m > n$.. *Then*

$$P_m P_{n+1} - P_{m+1}P_n = (-t)^n P_{m-n} \qquad (2.10)$$

and

$$Q_m Q_{n+1} - Q_{m+1}Q_n = 2(-t)^n \sqrt{s^2+t}\left(Q_{m-n} - 2(s+\sqrt{s^2+t})^{m-n}\right). \qquad (2.11)$$

Proof. Using the Binet's formula (2.4), we have

$$\begin{aligned} P_m P_{n+1} - P_{m+1}P_n &= \frac{\alpha^m - \beta^m}{\alpha-\beta}\cdot\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\cdot\frac{\alpha^n-\beta^n}{\alpha-\beta} \\ &= \frac{(\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n}-\beta^{m-n})}{(\alpha-\beta)^2} \\ &= (\alpha\beta)^n\cdot\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta} \\ &= (-t)^n P_{m-n}. \end{aligned}$$

Using the Binet's formula (2.5), we have

$$\begin{aligned} Q_m Q_{n+1} - Q_{m+1}Q_n &= (\alpha^m+\beta^m)(\alpha^{n+1}+\beta^{n+1}) - (\alpha^{m+1}+\beta^{m+1})(\alpha^n+\beta^n) \\ &= (\alpha\beta)^n(\alpha-\beta)(\alpha^{m-n}+\beta^{m-n}-2\alpha^{m-n}). \end{aligned}$$

Since $\alpha = s+\sqrt{s^2+t}$ and $\beta = s-\sqrt{s^2+t}$, we get

$$Q_m Q_{n+1} - Q_{m+1}Q_n = 2(-t)^n\sqrt{s^2+t}(Q_{m-n} - 2(s+\sqrt{s^2+t})^{m-n}) .\square$$

Theorem 2.6 *Let* $\{P_n\}$ *and* $\{Q_n\}$ *be* (s,t) *-Pell and* (s,t) *-Pell-Lucas sequences. Then*

$$\lim_{n\to+\infty}\frac{P_n}{P_{n-1}} = \alpha \qquad (2.12)$$

and

$$\lim_{n\to+\infty}\frac{Q_n}{Q_{n-1}} = \alpha. \qquad (2.13)$$

Proof. By using the Binet's formula (2.4), we have

$$\lim_{n\to+\infty}\frac{P_n}{P_{n-1}} = \lim_{n\to+\infty}\frac{\alpha^n-\beta^n}{\alpha^{n-1}-\beta^{n-1}} = \lim_{n\to+\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}-\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}.$$

Since $\left|\frac{\beta}{\alpha}\right| < 1, \lim_{n\to+\infty}\left(\frac{\beta}{\alpha}\right)^n = 0$,and therefore $\lim_{n\to+\infty}\frac{P_n}{P_{n-1}} = \alpha.$

On the other hand, using the Binet's formula (2.5) and using the same way as above, we obtain

$$\begin{aligned} \lim_{n\to+\infty}\frac{Q_n}{Q_{n-1}} &= \lim_{n\to+\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}} \\ &= \lim_{n\to+\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n} \\ &= \alpha. \end{aligned}$$

$\square$

3. Generating Functions for (s,t)-Pell and (s,t)-Pell-Lucas Numbers

In this section, the generating functions for the (s,t) -Pell and (s,t) -Pell-Lucas sequences are given. First, we shall give the generating functions for the (s,t) -Pell sequences. We shall write the (s,t) -Pell sequence as a power series where each term of the sequence correspond to coefficients of the series and from that fact, we find the generating function. Let us consider the (s,t) -Pell sequences for any positive integer s,t . By definition of ordinary generating function of some sequence, considering this sequence, the ordinary generating function associated is defined by

$$\begin{aligned} G(P_n;x) &= \sum_{n=0}^{\infty} P_n x^n \\ &= P_0 + P_1x + P_2x^2 + \ldots + P_nx^n + \ldots \qquad (3.1) \end{aligned}$$

From (2.1) and $P_0 = 0, P_1 = 1,$ we have

$$\sum_{n=0}^{\infty} P_n x^n = x + \sum_{n=2}^{\infty}\left(2sP_{n-1} + tP_{n-2}\right)x^n$$

$$= x + 2s\sum_{n=2}^{\infty} P_{n-1}x^n + t\sum_{n=2}^{\infty} P_{n-2}x^n$$

$$= x + 2sx\sum_{n=2}^{\infty} P_{n-1}x^{n-1} + tx^2\sum_{n=2}^{\infty} P_{n-2}x^{n-2}. \quad (3.2)$$

Now, consider that $k = n-1$ and $j = n-2$, the equation (3.2) can be written by

$$\sum_{n=0}^{\infty} P_n x^n = x + 2sx\sum_{k=1}^{\infty} P_k x^k + tx^2\sum_{j=0}^{\infty} P_j x^j$$

$$= x + 2sx\sum_{k=0}^{\infty} P_k x^k + tx^2\sum_{j=0}^{\infty} P_j x^j .$$

Thus,

$$\sum_{n=0}^{\infty} P_n x^n = x + 2sx\sum_{n=0}^{\infty} P_n x^n + tx^2\sum_{n=0}^{\infty} P_n x^n.$$

Therefore, the ordinary generating function of the (s,t) -Pell sequence can be written as

$$G\left(P_n; x\right) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2sx-tx^2}. \quad (3.3)$$

Applying the ratio test for absolute convergence and using (2.12), we have

$$\lim_{n\to+\infty}\left|\frac{P_n x^n}{P_{n-1}x^{n-1}}\right| = |x|\lim_{n\to+\infty}\frac{P_n}{P_{n-1}} = \alpha|x|, \quad (3.4)$$

and so the series converges absolutely if $|x| < \frac{1}{\alpha}$ and diverges if $|x| > \frac{1}{\alpha}$. Thus it's radius of convergence R is equal to $\frac{1}{\alpha}$. Now, by the similar argument as above, we get that the ordinary generating function of (s,t) -Pell-Lucas sequence can be written as

$$G\left(Q_n; x\right) = \sum_{n=0}^{\infty} Q_n x^n = \frac{2-2sx}{1-2sx-tx^2}, \quad (3.5)$$

and it's radius of convergence R is equal to $\frac{1}{\alpha}$.

Finally, we give another expression for the general term of the (s,t) -Pell sequence using the ordinary generating function.

Theorem 3.1

Let $p(x) = G(P_n; x) = \sum_{n=0}^{\infty} P_n x^n$, *for* $x \in (-\frac{1}{\alpha}, \frac{1}{\alpha})$.

Then
$$P_n = \frac{p^{(n)}(0)}{n!}, \quad (3.6)$$

where $p^{(n)}(x)$ *denotes the* n^{th} *order derivative of the function* $p(x)$.

Proof. Since $p(x) = \sum_{n=0}^{\infty} P_n x^n$, we have

$$p'(x) = \sum_{n=1}^{\infty} nP_n x^{n-1},$$

$$p''(x) = \sum_{n=2}^{\infty} n(n-1)P_n x^{n-2},$$

$$p'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)P_n x^{n-3},$$

$$\vdots$$

$$p^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-(k-1))P_n x^{n-k}$$

$$= k\left(k-1\right)\left(k-2\right)\cdots 1\cdot P_k + \sum_{n=k+1}^{\infty} n(n-1)(n-2)\cdots\left(n-(k-1)\right)P_n x^{n-k}$$

$$= k!P_k + \sum_{n=k+1}^{\infty} n(n-1)(n-2)\cdots\left(n-(k-1)\right)P_n x^{n-k},$$

then $p^{(k)}(0) = k!P_k$. Thus for all $n \geq 1$, we have

$$P_n = \frac{p^{(n)}(0)}{n!}. \qquad \square$$

By using the same approximation as in Theorem 3.1, we obtain the following theorem.

Theorem 3.2 *Let*

$$q(x) = G(Q_n; x) = \sum_{n=0}^{\infty} Q_n x^n, \textit{ for } x \in (-\frac{1}{\alpha}, \frac{1}{\alpha}).$$

Then

$$Q_n = \frac{q^{(n)}(n)}{n!}, \qquad (3.7)$$

where $q^{(n)}(x)$ *denotes the* n^{th} *order derivative of the function* $q(x)$.

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Some identities for (s, t)-Pell and (s, t)-Pell-Lucas numbers and its application to Diophantine equations

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Abstract

In this paper, some new identities for *(s,t)*-Pell and *(s,t)*-Pell-Lucas numbers are obtained by using matrix methods. Moreover, the solutions of some Diophantine equations are presented by applying these identities.

Keywords: Pell-numbers; Pell-Lucas number; *(s,t)*-Pell number; *(s,t)*-Pell-Lucas numbers; Matrix method

1. Introduction

Let s,t be any real number with $s^2+t>0$, $s>0$ and $t\neq 0$. Then the (s,t)-Pell sequences $\{P_n(s,t)\}_{n\in\mathbb{N}}$ [1] is defined by

$$P_n(s,t)=2sP_{n-1}(s,t)+tP_{n-2}(s,t),\quad \text{for all } n\geq 2, \tag{1}$$

with initial conditions $P_0(s,t)=0$ and $P_1(s,t)=1$. The first few terms of $\{P_n(s,t)\}_{n\in\mathbb{N}}$ are $0,\ 1,\ 2s,\ 4s^2+t,\ 8s^3+4st$ and so on. The terms of this sequence are called (s,t)-Pell numbers and we denoted the n^{th} (s,t)-Pell numbers by $P_n(s,t)$. The (s,t)-Pell numbers for negative subscripts can be defined as $P_{-n}(s,t)=\dfrac{-P_n(s,t)}{(-t)^n}$, for all $n\geq 1$. Then it follows that $P_n(s,t)=2sP_{n-1}(s,t)+\mathrm{t}P_{n-2}(s,t)$, for all $n\in\mathbb{Z}$. Also, (s,t)-Pell-Lucas sequences $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ [1] is defined by $Q_0(s,t)=2$, $Q_1(s,t)=2s$ and

$$Q_n(s,t)=2sQ_{n-1}(s,t)+tQ_{n-2}(s,t),\quad \text{for all } n\geq 2, \tag{2}$$

The first few terms of $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ are $2,\ 2s,\ 4s^2+2t,\ 8s^3+6st$ and so on. The terms of this sequence are called (s,t)-Pell-Lucas numbers and we denoted the n^{th} (s,t)-Pell-Lucas numbers by $Q_n(s,t)$. The (s,t)-Pell-Lucas numbers for negative subscripts are defined as $Q_{-n}(s,t)=\dfrac{Q_n(s,t)}{(-t)^n}$, for all $n\geq 1$. It can be seen that $Q_n(s,t)=2sP_n(s,t)+2tP_{n-1}(s,t)$ and $Q_n(s,t)=P_{n+1}(s,t)+tP_{n-1}(s,t)$ for all $n\in\mathbb{Z}$ For more detailed information about (s,t)-Pell and (s,t)-Pell-Lucas numbers can be found in [1].

From the definitions of (s,t)-Pell and (s,t)-Pell-Lucas numbers, we have that the characteristic equation of (1) and (2) are in the form

$$x^2=2sx+t \tag{3}$$

and the root of equation (3) are $\alpha=s+\sqrt{s^2+t}$ and $\beta=s-\sqrt{s^2+t}$. We note that $\alpha+\beta=2s$, $\alpha-\beta=2\sqrt{s^2+t}$ and $\alpha\beta=-t$. Also, from the definitions of (s,t)-Pell and (s,t)-Pell-Lucas numbers, we have that if $s=\frac{1}{2}$, $t=1$, then the classical Fibonacci and Lucas sequence are obtained, and if $s=1$, $t=1$, then the classical Pell and Pell-Lucas sequence are obtained. It is well known that the Fibonacci, Lucas, Pell and Pell-Lucas sequences are the famous recursive sequences that have been studied in the literatures by many authors for over several years, because they are extensively used in various research areas such as Engineering, Architecture, Nature and Art (for examples see: [2-7]).

In this paper, we will establish some identities for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using matrix methods. Moreover, we present the solution of some Diophantine equations by applying these identities. In the rest of this paper, for convenience we will use the symbol P_n and Q_n instead of $P_n(s,t)$ and $Q_n(s,t)$ respectively.

2. Main Results

In this section, we will establish some identities for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using the square matrix X which satisfy the property $X^2=2sX+tI$. Now, we begin with the following three Lemmas.

Lemma 2.1. If X is a square matrix with $X^2=2sX+tI$, then $X^n=P_nX+tP_{n-1}I$ for all $n\in\mathbb{Z}$.

Proof. If $n=0$, then the proof is obvious. It can be shown by induction that $X^n=P_nX+tP_{n-1}I$ for all $n\in\mathbb{N}$. Now, we will show that $X^{-n}=P_{-n}X+tP_{-n-1}I$ for all $n\in\mathbb{N}$. Let $Y=2sI-X=-tX^{-1}$. Then we have

$$Y^2=(2sI-X)^2=2s(2sI-X)+tI=2sY+tI.$$

It implies that $Y^n=P_nY+tP_{n-1}I$. That is $(-tX^{-1})^n=P_n(2sI-X)+tP_{n-1}I$. Thus

$$\begin{aligned}(-t)^nX^{-n}&=2sP_nI-P_nX+tP_{n-1}I\\&=-P_nX+(2sP_n+tP_{n-1})I\\&=-P_nX+P_{n+1}I.\end{aligned}$$

Therefore,

$$X^{-n} = -\frac{P_n}{(-t)^n}X + \frac{P_{n+1}}{(-t)^n}I = P_{-n}X + tP_{-(n+1)}I = P_{-n}X + tP_{-n-1}I.$$

This complete the proof.

Lemma 2.2. Let $W = \begin{bmatrix} s & 2(s^2+t) \\ \frac{1}{2} & s \end{bmatrix}$, then $W^n = \begin{bmatrix} \frac{1}{2}Q_n & 2(s^2+t)P_n \\ \frac{1}{2}P_n & \frac{1}{2}Q_n \end{bmatrix}$ for all $n \in \mathbb{Z}$.

Proof. Since $W^2 = 2sW + tI$, the proof follows from Lemma 2.1 and using $Q_n = 2sP_n + 2tP_{n-1}$.

Lemma 2.3. $Q_n^{\,2} - 4(s^2+t)P_n^{\,2} = 4(-t)^n$ for all $n \in \mathbb{Z}$.

Proof. Since $\det(W^n) = (\det(W))^n = (-t)^n$ and $\det(W^n) = \frac{1}{4}Q_n^2 - (s^2+t)P_n^2$, we get

$Q_n^2 - 4(s^2+t)P_n^2 = 4(-t)^n$.

Lemma 2.4. $2Q_{m+n} = Q_mQ_n + 4(s^2+t)P_mP_n$ for all $m, n \in \mathbb{Z}$.

Proof. Since $W^{m+n} = W^mW^n$, we get the result.

Lemma 2.5. $\alpha^n = \alpha P_n + tP_{n-1}$ and $\beta^n = \beta P_n + tP_{n-1}$ for all $n \in \mathbb{Z}$.

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, then $X^2 = 2sX + tI$. By Lemma 2.1, we have $X^n = P_nX + tP_{n-1}I$. It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha P_n + tP_{n-1} & 0 \\ 0 & \beta P_n + tP_{n-1} \end{bmatrix}.$$

This implies that $\alpha^n = \alpha P_n + tP_{n-1}$ and $\beta^n = \beta P_n + tP_{n-1}$.

By using Lemma 2.1 and Lemma 2.5, we get the following Theorem.

Theorem 2.6. *Let* $A = \begin{bmatrix} \alpha & 0 \\ t & \beta \end{bmatrix}$, *then* $A^n = \begin{bmatrix} \alpha^n & 0 \\ tP_n & \beta^n \end{bmatrix}$ *for all* $n \in \mathbb{Z}$

Proof. Since $A^2 = \begin{bmatrix} 2s\alpha + t & 0 \\ t(\alpha+\beta) & 2s\beta + t \end{bmatrix} = 2sA + tI$, by Lemma 2.1 and Lemma 2.5, we get that

$A^n = P_n A + tP_{n-1} I = \begin{bmatrix} \alpha^n & 0 \\ tP_n & \beta^n \end{bmatrix}$. Thus, we get the result.

By using Theorem 2.6, we get the following Theorem.

Theorem 2.7. *Let* $m, n \in \mathbb{Z}$. *Then*

$$4(s^2+t)(-t)^m P_n^2 + 4(s^2+t)(-t)^n P_m^2 - Q_{m+n}^2 = -4(s^2+t)P_m P_n Q_{m+n} - 4(-t)^{m+n}.$$

Proof. Let a matrix A as in Theorem 2.6. It can be seen that

$$\frac{1}{t}A^{n+1} + A^{n-1} = \begin{bmatrix} \dfrac{2\sqrt{s^2+t}}{t}\alpha^n & 0 \\ Q_n & -\dfrac{2\sqrt{s^2+t}}{t}\beta^n \end{bmatrix}.$$

Since $\left(\frac{1}{t}A^{n+1} + A^{n-1}\right)\left(\frac{1}{t}A^{m+1} + A^{m-1}\right) = \frac{1}{t^2}A^{m+n+2} + \frac{2}{t}A^{m+n} + A^{m+n-2}$, we get that

$$2\sqrt{s^2+t}P_{m+n} = \alpha^m Q_n - \beta^n Q_m.$$

Thus,

$$\begin{aligned} 4(s^2+t)P_{m+n}^2 &= (2\sqrt{s^2+t}P_{m+n})(2\sqrt{s^2+t}P_{m+n}) \\ &= (\alpha^m Q_n - \beta^n Q_m)(\alpha^n Q_m - \beta^m Q_n). \end{aligned}$$

Since $4(s^2+t)P_{m+n}^2 = Q_{m+n}^2 - 4(-t)^{m+n}$, we obtain

$$(-t)^m Q_n^2 + (-t)^n Q_m^2 + Q_{m+n}^2 = Q_m Q_n Q_{m+n} + 4(-t)^{m+n}. \tag{4}$$

Since $Q_n^2 = 4(s^2+t)P_n^2 + 4(-t)^m$ and $Q_m Q_n = 2Q_{m+n} - 4(s^2+t)P_m P_n$, we get that

$$4(s^2+t)(-t)^m P_n^2 + 4(s^2+t)(-t)^n P_m^2 - Q_{m+n}^2 = -4(s^2+t)P_m P_n Q_{m+n} - 4(-t)^{m+n}, \tag{5}$$

and so the proof is completed.

Example 2.8. Let $m = 1$ and $n = 2$. Then

$$4(s^2+t)(-t)^1 P_2^2 + 4(s^2+t)(-t)^2 P_1^2 - Q_3^2 = -4(s^2+t)P_1 P_2 Q_3 - 4(-t)^3.$$

Proof. Consider,

$$4(s^2+t)(-t)^1P_2^2+4(s^2+t)(-t)^2P_1^2-Q_3^2=4(s^2+t)(-t)(2s)^2+4(s^2+t)t^2(1)^2-(8s^3+6st)^2$$
$$=-16s^4t-16s^2t^2+4s^2t^2+4t^3-64s^6-96s^4t-36s^2t^2$$
$$=-64s^6-112s^4t-48s^2t^2+4t^3,$$

and

$$-4(s^2+t)P_1P_2Q_3-4(-t)^3=-4(s^2+t)(1)(2s)(8s^3+6st)+4t^3$$
$$=-64s^6-112s^4t-48s^2t^2+4t.^3$$

Thus, $4(s^2+t)(-t)^1P_2^2+4(s^2+t)(-t)^2P_1^2-Q_3^2=-4(s^2+t)P_1P_2Q_3-4(-t)^3$.

Theorem 2.9. Let $m,n\in\mathbb{Z}$. Then

$$(-t)^mQ_n^2-4(s^2+t)(-t)^nP_m^2-4(s^2+t)P_{m+n}^2=-4(s^2+t)Q_nP_mP_{m+n}+4(-t)^{m+n}.$$

Proof. By using a similar argument as in Theorem 2.7 and the property

$$\left(\frac{1}{t}A^{n+1}+A^{n-1}\right)A^m=A^m\left(\frac{1}{t}A^{n+1}+A^{n-1}\right)=\frac{1}{t}A^{m+n+1}+A^{m+n-1},$$

we get that

$Q_{m+n}=\alpha^mQ_n-2\sqrt{s^2+t}\beta^nP_m$ and $Q_{m+n}=2\sqrt{s^2+t}\alpha^nP_m+\beta^mQ_n$.

It follows that

$$Q_{m+n}^2=(\alpha^mQ_n-2\sqrt{s^2+t}\beta^nP_m)(2\sqrt{s^2+t}\alpha^nP_m+\beta^mQ_n)$$
$$=4(s^2+t)Q_nP_mP_{m+n}+(-t)^mQ_n^2-4(s^2+t)(-t)^nP_m^2.$$

Since $Q_{m+n}^2=4(s^2+t)P_{m+n}^2+4(-t)^{m+n}$, we have

$$(-t)^mQ_n^2-4(s^2+t)(-t)^nP_m^2-4(s^2+t)P_{m+n}^2=-4(s^2+t)Q_nP_mP_{m+n}+4(-t)^{m+n}. \quad (6)$$

This completed the proof.

Example 2.10. Let $m=2$ and $n=0$. Then

$$(-t)^2Q_0^2-4(s^2+t)(-t)^0P_2^2-4(s^2+t)P_2^2=-4(s^2+t)Q_0P_2P_2+4(-t)^2.$$

Proof. Consider,

$$(-t)^2Q_0^2-4(s^2+t)(-t)^0P_2^2-4(s^2+t)P_2^2=t^2(2)^2-4(s^2+t)(1)(2s)^2-4(s^2+t)(2s)^2$$
$$=-32s^4-32s^2t+4t^2,$$

and

$$-4(s^2+t)Q_0P_2P_2+4(-t)^2=-4(s^2+t)(2)(2s)(2s)+4t^2$$
$$=-32s^4-32s^2t+4t^2.$$

Thus, $(-t)^2 Q_0^2 - 4(s^2+t)(-t)^0 P_2^2 - 4(s^2+t)P_2^2 = -4(s^2+t)Q_0 P_2 P_2 + 4(-t)^2$.

3. Applications

In this section, by applying Theorem 2.7 and Theorem 2.9, we give the solutions of some Diophantine equations. We will investigate in two cases:

Case 1: If $s \in \mathbb{Z}^+$ and $t=1$, then we get the following Theorems

Theorem 3.1. If m and n are even integers, then the integer solutions of the equation $z^2 - 4(s^2+1)x^2 - 4(s^2+1)y^2 = 4(s^2+1)xyz + 4$ are given by $(x,y,z) = (P_m(s,1), P_n(s,1), Q_{m+n}(s,1))$. If m and n are odd integers, then the integer solutions of the equation $z^2 + 4(s^2+1)x^2 + 4(s^2+1)y^2 = 4(s^2+1)xyz + 4$ are given by $(x,y,z) = (P_m(s,1), P_n(s,1), Q_{m+n}(s,1))$ and if m is an odd integer and n is an even integer, then the integer solutions of the equation $z^2 - 4(s^2+1)x^2 + 4(s^2+1)y^2 = 4(s^2+1)xyz - 4$ are given by $(x,y,z) = (P_m(s,1), P_n(s,1), Q_{m+n}(s,1))$.

Proof. The result follows immediately from Theorem 2.7.

Theorem 3.2. If m and n are even integers, then the integer solutions of the equation $z^2 + x^2 + y^2 = xyz + 4$ are given by $(x,y,z) = (Q_m(s,1), Q_n(s,1), Q_{m+n}(s,1))$ If m and n are odd integers, then the integer solutions of the equation $z^2 - x^2 - y^2 = xyz + 4$ are given by $(x,y,z) = (Q_m(s,1), Q_n(s,1), Q_{m+n}(s,1))$ and if m is an odd integer and n is an even integer, then the integer solutions of the equation $z^2 + x^2 - y^2 = xyz - 4$ are given by $(x,y,z) = (Q_m(s,1), Q_n(s,1), Q_{m+n}(s,1))$.

Proof. The result follows directly from (4).

Theorem 3.3. If m and n are even integers, then the integer solutions of the equation $x^2 - 4(s^2+1)y^2 - 4(s^2+1)z^2 = -4(s^2+1)xyz + 4$ are given by $(x,y,z) = (Q_n(s,1), P_m(s,1), P_{m+n}(s,1))$. If m and n are odd integers, then the integer solutions of the equation $x^2 - 4(s^2+1)y^2 + 4(s^2+1)z^2 = 4(s^2+1)xyz - 4$ are given by $(x,y,z) = (Q_n(s,1), P_m(s,1), P_{m+n}(s,1))$ and if m is an odd integer and n is an even integer, then the integer solutions of the equation $x^2 + 4(s^2+1)y^2 + 4(s^2+1)z^2 = 4(s^2+1)xyz + 4$ are given by $(x,y,z) = (Q_n(s,1), P_m(s,1), P_{m+n}(s,1))$.

Proof. The result follows immediately from Theorem 2.9.

Case 2: If $s \in \mathbb{Z}^+$ and $t = -1$, then we get the following Theorems

Theorem 3.4. The integer solutions of the equation $z^2 - 4(s^2 - 1)x^2 - 4(s^2 - 1)y^2 = 4(s^2 + 1)xyz + 4$ are given by $(x, y, z) = (P_m(s, -1), P_n(s, -1), Q_{m+n}(s, -1))$.

Proof. The result follows immediately from Theorem 2.7.

Theorem 3.5. The integer solutions of the equation $z^2 + x^2 + y^2 = xyz + 4$ are given by $(x, y, z) = (Q_m(s, -1), Q_n(s, -1), Q_{m+n}(s, -1))$.

Proof. The result follows directly from (4).

Theorem 3.6. The integer solutions of the equation $4(s^2 - 1)z^2 - x^2 + 4(s^2 - 1)y^2 = 4(s^2 - 1)xyz - 4$ are given by $(x, y, z) = (Q_n(s, -1), P_m(s, -1), P_{m+n}(s, -1))$.

Proof. The result follows immediately from Theorem 2.8.

4. Conclusion

Nowadays, many mathematicians are interested in solving Diophantine equations. We think it is a little hard and interesting to give all integer (positive integer) solutions of the Diophantine equations.

$$z^2 - 4(s^2 + 1)x^2 - 4(s^2 + 1)y^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 + 4(s^2 + 1)x^2 + 4(s^2 + 1)y^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 - 4(s^2 + 1)x^2 + 4(s^2 + 1)y^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 + x^2 + y^2 = xyz + 4$$
$$z^2 - x^2 - y^2 = xyz + 4$$
$$z^2 + x^2 - y^2 = xyz - 4$$
$$x^2 - 4(s^2 + 1)y^2 - 4(s^2 + 1)z^2 = -4(s^2 + 1)xyz + 4$$
$$x^2 - 4(s^2 + 1)y^2 + 4(s^2 + 1)z^2 = 4(s^2 + 1)xyz - 4$$
$$x^2 + 4(s^2 + 1)y^2 + 4(s^2 + 1)z^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 - 4(s^2 - 1)x^2 - 4(s^2 - 1)y^2 = 4(s^2 + 1)xyz + 4$$

and

$$4(s^2 - 1)z^2 - x^2 + 4(s^2 - 1)y^2 = 4(s^2 - 1)xyz - 4.$$

Although they have infinite many integer solutions by the above Theorems.

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On the (s,t)-Pell and (s,t)-Pell-Lucas numbers by matrix methods*

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Abstract

In this paper, we investigate some generalization of Pell and Pell-Lucas numbers, which is called (s,t)-Pell and (s,t)-Pell-Lucas numbers, and we define the 2×2 matrix W, which satisfy the relation $W^2 = 2sW + tI$. After that, we establish some identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers and some sum formulas for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using this matrix.

Keywords: Fibonacci number; Lucas number; Pell number; Pell-Lucas number; (s,t)-Pell number; (s,t)-Pell-Lucas number.

MSC: 11B37; 15A15.

1. Introduction

For over several years, there are many recursive sequences that have been studied in the literatures. The famous examples of these sequences are Fibonacci, Lucas, Pell and Pell-Lucas, because they are extensively used in various research areas such as Engineering, Architecture, Nature and Art (for examples see: [2, 3, 4, 5, 6, 7]). For $n \geq 2$, the classical Fibonacci $\{F_n\}$, Lucas $\{L_n\}$, Pell $\{P_n\}$ and Pell-Lucas $\{Q_n\}$

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sequences are defined by $F_n = F_{n-1}+F_{n-2}$, $L_n = L_{n-1}+L_{n-2}$, $P_n = 2P_{n-1}+P_{n-2}$, and $Q_n = 2Q_{n-1} + Q_{n-2}$, with the initial conditions $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $P_0 = 0$, $P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively. For more detialed information about Fibonacci, Lucas, Pell, Pell-Lucas sequences can be found in [2, 3].

Recently, Fibonacci, Lucas, Pell and Pell-Lucas were generalized and studied by many authors in the different ways to derive many identities. In 2012, Gulec and Taskara [1] introduced a new generalization of Pell and Pell-Lucas sequence which is called (s,t)-Pell and (s,t)-Pell-Lucas sequences as in the definition 1.1 and by considering these sequences, they introduced the matrix sequences which have elements of (s,t)-Pell and (s,t)-Pell-Lucas sequences. Further, they obtained some properties of (s,t)-Pell and (s,t)-Pell-Lucas matrices sequences by using elementary matrix algebra.

Definition 1.1. [1] Let s,t be any real number with $s^2 + t > 0$, $s > 0$ and $t \neq 0$. Then the (s,t)-Pell sequences $\{\mathcal{P}_n(s,t)\}_{n\in\mathbb{N}}$ and the (s,t)-Pell-Lucas sequences $\{\mathcal{Q}_n(s,t)\}_{n\in\mathbb{N}}$ are defined respectively by

$$\mathcal{P}_n(s,t) = 2s\mathcal{P}_{n-1}(s,t) + t\mathcal{P}_{n-2}(s,t), \text{ for } n \geq 2, \tag{1.1}$$

$$\mathcal{Q}_n(s,t) = 2s\mathcal{Q}_{n-1}(s,t) + t\mathcal{Q}_{n-2}(s,t), \text{ for } n \geq 2, \tag{1.2}$$

with initial conditions $\mathcal{P}_0(s,t) = 0$, $\mathcal{P}_1(s,t) = 1$ and $\mathcal{Q}_0(s,t) = 2$, $\mathcal{Q}_1(s,t) = 2s$.

In particular, if $s = \frac{1}{2}, t = 1$, then the classical Fibonacci and Lucas sequence are obtained, and if $s = t = 1$, then the classical Pell and Pell-Lucas sequences are obtained. From the definition 1.1, we have that the characteristic equation of (1.1) and (1.2) are in the form

$$x^2 = 2sx + t \tag{1.3}$$

and the root of equation (1.3) are $\alpha = s + \sqrt{s^2+t}$ and $\beta = s - \sqrt{s^2+t}$. Note that $\alpha + \beta = 2s$, $\alpha - \beta = 2\sqrt{s^2+t}$ and $\alpha\beta = -t$. Moreover, it can be seen that [1]

$$\mathcal{Q}_n(s,t) = 2s\mathcal{P}_n(s,t) + 2t\mathcal{P}_{n-1}(s,t), \text{ for all } n \geq 0. \tag{1.4}$$

In this paper, we introduce the 2×2 matrix W which satisfy the relation $W^2 = 2sW + tI$. After that, we establish some identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers and some sum formulas for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using this matrix. Now, we first define (s,t)-Pell and (s,t)-Pell-Lucas numbers for negative subscript as follows:

$$\mathcal{P}_{-n}(s,t) = \frac{-\mathcal{P}_n(s,t)}{(-t)^n}, \text{ and } \mathcal{Q}_{-n}(s,t) = \frac{\mathcal{Q}_n(s,t)}{(-t)^n}, \tag{1.5}$$

for all $n \geq 1$. In the rest of this paper, for convenience we will use the symbol $\mathcal{P}_n$ and $\mathcal{Q}_n$ instead of $\mathcal{P}_n(s,t)$ and $\mathcal{Q}_n(s,t)$ respectively.

2. Main results

We begin this section with the following Lemma.

Lemma 2.1. *If X is a square matrix with $X^2 = 2sX + tI$, then*

$$X^n = \mathcal{P}_n X + t\mathcal{P}_{n-1} I \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $X^n = \mathcal{P}_n X + t\mathcal{P}_{n-1} I$ for all $n \in \mathbb{N}$. Now, we will show that $X^{-n} = \mathcal{P}_{-n} X + t\mathcal{P}_{-n-1} I$ for all $n \in \mathbb{N}$. Let $Y = 2sI - X = -tX^{-1}$. Then we have

$$Y^2 = (2sI - X)^2 = 2s(2sI - X) + tI = 2sY + tI.$$

It implies that $Y^n = \mathcal{P}_n Y + t\mathcal{P}_{n-1} I$. That is $(-tX^{-1})^n = \mathcal{P}_n(2sI - X) + t\mathcal{P}_{n-1} I$. Thus

$$\begin{aligned}(-t)^n X^{-n} &= 2s\mathcal{P}_n I - \mathcal{P}_n X + t\mathcal{P}_{n-1} I \\ &= -\mathcal{P}_n X + (2s\mathcal{P}_n + t\mathcal{P}_{n-1}) I \\ &= -\mathcal{P}_n X + \mathcal{P}_{n+1} I.\end{aligned}$$

Therefore, $X^{-n} = -\frac{\mathcal{P}_n}{(-t)^n} X + \frac{\mathcal{P}_{n+1}}{(-t)^n} I = \mathcal{P}_{-n} X + t\mathcal{P}_{-(n+1)} I = \mathcal{P}_{-n} X + t\mathcal{P}_{-n-1} I$.

This complete the proof. □

By using Lemma 2.1, we obtain the Binet's formula for (s,t)-Pell and (s,t)-Pell-Lucas numbers.

Corollary 2.2 (Binet's formula)**.** *The n^{th} (s,t)-Pell and (s,t)-Pell-Lucas number are given by*

$$\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \textit{and} \quad \mathcal{Q}_n = \alpha^n + \beta^n, \quad \textit{for all} \quad n \in \mathbb{Z},$$

where $\alpha = s + \sqrt{s^2 + t}$ and $\beta = s - \sqrt{s^2 + t}$ are the roots of the characteristic equation (1.3)*.*

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, then $X^2 = 2sX + tI$. By Lemma 2.1, we have $X^n = \mathcal{P}_n X + t\mathcal{P}_{n-1} I$. It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha\mathcal{P}_n + t\mathcal{P}_{n-1} & 0 \\ 0 & \beta\mathcal{P}_n + t\mathcal{P}_{n-1} \end{bmatrix}.$$

Thus, $\alpha^n = \alpha\mathcal{P}_n + t\mathcal{P}_{n-1}$ and $\beta^n = \beta\mathcal{P}_n + t\mathcal{P}_{n-1}$, which implies that

$$\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathcal{Q}_n = \alpha^n + \beta^n, \quad \text{for all} \quad n \in \mathbb{Z}.$$ □

Let us define the 2×2 matrix W as follows:

$$W = \begin{bmatrix} s & 2(s^2 + t) \\ \frac{1}{2} & s \end{bmatrix}. \tag{2.1}$$

Then it easy to see that $W^2 = 2sW + tI$. From this fact and Lemma 2.1, we get the following Lemma.

Lemma 2.3. *Let W be a matrix as in (2.1). Then* $W^n = \begin{bmatrix} \frac{1}{2}\mathcal{Q}_n & 2(s^2+t)\mathcal{P}_n \\ \frac{1}{2}\mathcal{P}_n & \frac{1}{2}\mathcal{Q}_n \end{bmatrix}$ *for all $n \in \mathbb{Z}$.*

Proof. Since $W^2 = 2sW + tI$, the proof follows from Lemma 2.1 and using $\mathcal{Q}_n = 2s\mathcal{P}_n + 2t\mathcal{P}_{n-1}$. □

Now, by using the matrix W, we obtain some identities of (s,t)-Pell and (s,t)-Pell-Lucas numbers.

Lemma 2.4. *Let m, n be any integers. Then the following results hold.*

(i) $\mathcal{Q}_n^2 - 4(s^2+t)\mathcal{P}_n^2 = 4(-t)^n$,

(ii) $2\mathcal{Q}_{m+n} = \mathcal{Q}_m\mathcal{Q}_n + 4(s^2+t)\mathcal{P}_m\mathcal{P}_n$,

(iii) $2\mathcal{P}_{m+n} = \mathcal{P}_m\mathcal{Q}_n + \mathcal{Q}_m\mathcal{P}_n$,

(iv) $2(-t)^n\mathcal{Q}_{m-n} = \mathcal{Q}_m\mathcal{Q}_n - 4(s^2+t)\mathcal{P}_m\mathcal{P}_n$,

(v) $2(-t)^n\mathcal{P}_{m-n} = \mathcal{P}_m\mathcal{Q}_n - \mathcal{Q}_m\mathcal{P}_n$,

(vi) $\mathcal{Q}_m\mathcal{Q}_n = \mathcal{Q}_{m+n} + (-t)^n\mathcal{Q}_{m-n}$,

(vii) $\mathcal{P}_m\mathcal{Q}_n = \mathcal{P}_{m+n} + (-t)^n\mathcal{P}_{m-n}$.

Proof. Since $\det(W^n) = (\det(W))^n = (-t)^n$ and $\det(W^n) = \frac{1}{4}\mathcal{Q}_n^2 - (s^2+t)\mathcal{P}_n^2$, we get that $\mathcal{Q}_n^2 - 4(s^2+t)\mathcal{P}_n^2 = 4(-t)^n$ and then (i) immediately seen. Since $W^{m+n} = W^mW^n$, we obtain

$$\begin{bmatrix} \frac{1}{2}\mathcal{Q}_{m+n} & 2(s^2+t)\mathcal{P}_{m+n} \\ \frac{1}{2}\mathcal{P}_{m+n} & \frac{1}{2}\mathcal{Q}_{m+n} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\big(\mathcal{Q}_m\mathcal{Q}_n + 4(s^2+t)\mathcal{P}_m\mathcal{P}_n\big) & (s^2+t)(\mathcal{Q}_m\mathcal{P}_n + \mathcal{P}_m\mathcal{Q}_n) \\ \frac{1}{4}(\mathcal{P}_m\mathcal{Q}_n + \mathcal{Q}_m\mathcal{P}_n) & \frac{1}{4}\big(4(s^2+t)\mathcal{P}_m\mathcal{P}_n + \mathcal{Q}_m\mathcal{Q}_n\big) \end{bmatrix}.$$

Thus, identities (ii) and (iii) are easily seen. Next, we note that $W^{m-n} = W^m(W^{-n}) = W^m(W^n)^{-1}$. Thus, we get that

$$\begin{bmatrix} \frac{1}{2}\mathcal{Q}_{m-n} & 2(s^2+t)\mathcal{P}_{m-n} \\ \frac{1}{2}\mathcal{P}_{m-n} & \frac{1}{2}\mathcal{Q}_{m-n} \end{bmatrix} = \frac{1}{(-t)^n}\begin{bmatrix} \frac{1}{4}\big(\mathcal{Q}_m\mathcal{Q}_n - 4(s^2+t)\mathcal{P}_m\mathcal{P}_n\big) & (s^2+t)(-\mathcal{Q}_m\mathcal{P}_n + \mathcal{P}_m\mathcal{Q}_n) \\ \frac{1}{4}(\mathcal{P}_m\mathcal{Q}_n - \mathcal{Q}_m\mathcal{P}_n) & \frac{1}{4}\big(-4(s^2+t)\mathcal{P}_m\mathcal{P}_n + \mathcal{Q}_m\mathcal{Q}_n\big) \end{bmatrix}.$$

Therefore, the identities (iv) and (v) can be derived directly. The proof of (vi) and (vii) goes on in the same fashion as above by using the property
$W^{m+n} + (-t)^nW^{m-n} = W^m(W^n + (-t)^nW^{-n})$. □

Next, we give the following Lemma for using in the next Theorems.

Lemma 2.5. *Let W be a matrix as in (2.1). Then*

$$H = W + tW^{-1} = \begin{bmatrix} 0 & 4(s^2+t) \\ 1 & 0 \end{bmatrix}.$$

Proof. Since $\det(W) = -t$, we get that $W^{-1} = -\frac{1}{t}\begin{bmatrix} s & -2(s^2+t) \\ -\frac{1}{2} & s \end{bmatrix}$. Thus, $H = \begin{bmatrix} 0 & 4(s^2+t) \\ 1 & 0 \end{bmatrix}$. □

Finally, by using matrices W and H, we obtain some sum formulas for (s,t)-Pell and (s,t)-Pell-Lucas numbers.

Theorem 2.6. *Let $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$ with $(-t)^m - \mathcal{Q}_m \neq -1$. Then*

$$\sum_{j=0}^{n} \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k - \mathcal{Q}_{mn+m+k} + (-t)^m\big(\mathcal{Q}_{mn+k} - \mathcal{Q}_{k-m}\big)}{1 + (-t)^m - \mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n} \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k - \mathcal{P}_{mn+m+k} + (-t)^m\big(\mathcal{P}_{mn+k} - \mathcal{P}_{k-m}\big)}{1 + (-t)^m - \mathcal{Q}_m}$$

Proof. It is know that

$$I - (W^m)^{n+1} = (I - W^m)\sum_{j=0}^{n}(W^m)^j.$$

By Lemma 2.4 (i), we have

$$\det(I - W^m) = (1 - \frac{1}{2}\mathcal{Q}_m)^2 - (s^2+t)\mathcal{P}_m^2 = 1 + (-t)^m - \mathcal{Q}_m.$$

Since $\det(I - W^m) \neq 0$, we can write

$$\begin{aligned} (I - W^m)^{-1}\big(I - (W^m)^{n+1}\big)W^k &= \sum_{j=0}^{n} W^{mj+k} \\ &= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{n}\mathcal{Q}_{mj+k} & 2(s^2+t)\sum_{j=0}^{n}\mathcal{P}_{mj+k} \\ \frac{1}{2}\sum_{j=0}^{n}\mathcal{P}_{mj+k} & \frac{1}{2}\sum_{j=0}^{n}\mathcal{Q}_{mj+k} \end{bmatrix}. \end{aligned} \quad (2.2)$$

Since

$$(I - W^m)^{-1} = \frac{1}{1 + (-t)^m - \mathcal{Q}_m}\begin{bmatrix} 1 - \frac{1}{2}\mathcal{Q}_m & 2(s^2+t)\mathcal{P}_m \\ \frac{1}{2}\mathcal{P}_m & 1 - \frac{1}{2}\mathcal{Q}_m \end{bmatrix}$$

$$= \frac{1}{1+(-t)^m - \mathcal{Q}_m}\Big((1-\frac{1}{2}\mathcal{Q}_m)I + \frac{1}{2}\mathcal{P}_m H\Big),$$

we have

$$\begin{aligned}
&(I-W^m)^{-1}\big(I-(W^m)^{n+1}\big)W^k \\
&= \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)I + \frac{1}{2}\mathcal{P}_m H\Big)(W^k - W^{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \\
&= \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)(W^k - W^{mn+m+k}) + \frac{1}{2}\mathcal{P}_m H(W^k - W^{mn+m+k})\Big)}{1+(-t)^m - \mathcal{Q}_m} \\
&= (1-\frac{1}{2}\mathcal{Q}_m)\begin{bmatrix} \frac{\frac{1}{2}(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{2(s^2+t)(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \\ \frac{\frac{1}{2}(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{\frac{1}{2}(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \end{bmatrix} \\
&\quad + \frac{1}{2}\mathcal{P}_m \begin{bmatrix} \frac{2(s^2+t)(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{2(s^2+t)(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \\ \frac{\frac{1}{2}(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} & \frac{2(s^2+t)(\mathcal{P}_k - \mathcal{P}_{mn+m+k})}{1+(-t)^m - \mathcal{Q}_m} \end{bmatrix} \qquad (2.3)
\end{aligned}$$

Using (2.2) and (2.3), we obtain

$$\begin{aligned}
&\sum_{j=0}^{n} \mathcal{Q}_{mj+k} \\
&= \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k}) + 2(s^2+t)\mathcal{P}_m(\mathcal{P}_k - \mathcal{P}_{mn+m+k})\Big)}{1+(-t)^m - \mathcal{Q}_m}. \qquad (2.4)
\end{aligned}$$

By Lemma 2.4 (*iv*), (2.4) becomes

$$\sum_{j=0}^{n} \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k - \mathcal{Q}_{mn+m+k} + (-t)^m\big(\mathcal{Q}_{mn+k} - \mathcal{Q}_{k-m}\big)}{1+(-t)^m - \mathcal{Q}_m}.$$

On the other hand, using (2.2) and (2.3) we get

$$\sum_{j=0}^{n} \mathcal{P}_{mj+k} = \frac{\Big((1-\frac{1}{2}\mathcal{Q}_m)(\mathcal{P}_k - \mathcal{P}_{mn+m+k}) + \frac{1}{2}\mathcal{P}_m(\mathcal{Q}_k - \mathcal{Q}_{mn+m+k})\Big)}{1+(-t)^m - \mathcal{Q}_m}. \qquad (2.5)$$

By Lemma 2.4 (*v*), (2.5) becomes

$$\sum_{j=0}^{n} \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k - \mathcal{P}_{mn+m+k} + (-t)^m\big(\mathcal{P}_{mn+k} - \mathcal{P}_{k-m}\big)}{1+(-t)^m - \mathcal{Q}_m}. \qquad \square$$

Theorem 2.7. *Let $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$ with $(-t)^m + \mathcal{Q}_m \neq -1$. If n is even, then*

$$\sum_{j=0}^{n}(-1)^j \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k + \mathcal{Q}_{mn+m+k} + (-t)^m\big(\mathcal{Q}_{mn+k} + \mathcal{Q}_{k-m}\big)}{1 + (-t)^m + \mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n}(-1)^j \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k + \mathcal{P}_{mn+m+k} + (-t)^m\big(\mathcal{P}_{mn+k} + \mathcal{P}_{k-m}\big)}{1 + (-t)^m + \mathcal{Q}_m}$$

Proof. Let n is an even natural number. Then we have

$$I + (W^m)^{n+1} = (I + W^m)\sum_{j=0}^{n}(-1)^j (W^m)^j.$$

By Lemma 2.4 (i), we have

$$\det(I + W^m) = (1 + \frac{1}{2}\mathcal{Q}_m)^2 - (s^2 + t)\mathcal{P}_m^2 = 1 + \mathcal{Q}_m + (-t)^m.$$

Since $\det(I + W^m) \neq 0$, we can write

$$\begin{aligned}
&(I + W^m)^{-1}\big(I + (W^m)^{n+1}\big)W^k\\
&= \sum_{j=0}^{n}(-1)^j W^{mj+k}\\
&= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{n}(-1)^j \mathcal{Q}_{mj+k} & 2(s^2+t)\sum_{j=0}^{n}(-1)^j \mathcal{P}_{mj+k} \\ \frac{1}{2}\sum_{j=0}^{n}(-1)^j \mathcal{P}_{mj+k} & \frac{1}{2}\sum_{j=0}^{n}(-1)^j \mathcal{Q}_{mj+k} \end{bmatrix}.
\end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}
(I + W^m)^{-1} &= \frac{1}{1 + \mathcal{Q}_m + (-t)^m}\begin{bmatrix} 1 + \frac{1}{2}\mathcal{Q}_m & -2(s^2+t)\mathcal{P}_m \\ -\frac{1}{2}\mathcal{P}_m & 1 + \frac{1}{2}\mathcal{Q}_m \end{bmatrix}\\
&= \frac{1}{1 + \mathcal{Q}_m + (-t)^m}\Big((1 + \frac{1}{2}\mathcal{Q}_m)I - \frac{1}{2}\mathcal{P}_m H\Big),
\end{aligned}$$

we have

$$\begin{aligned}
&(I + W^m)^{-1}\big(I + (W^m)^{n+1}\big)W^k\\
&= \frac{\Big((1 + \frac{1}{2}\mathcal{Q}_m)I - \frac{1}{2}\mathcal{P}_m H\Big)(W^k + W^{mn+m+k})}{1 + \mathcal{Q}_m + (-t)^m}\\
&= \frac{\Big((1 + \frac{1}{2}\mathcal{Q}_m)(W^k + W^{mn+m+k}) - \frac{1}{2}\mathcal{P}_m H(W^k + W^{mn+m+k})\Big)}{1 + \mathcal{Q}_m + (-t)^m}
\end{aligned}$$

$$
\begin{aligned}
&= (1+\frac{1}{2}\mathcal{Q}_m)\begin{bmatrix} \frac{\frac{1}{2}(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{2(s^2+t)(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \\ \frac{\frac{1}{2}(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{\frac{1}{2}(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \end{bmatrix} \\
&- \frac{1}{2}\mathcal{P}_m\begin{bmatrix} \frac{2(s^2+t)(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{2(s^2+t)(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \\ \frac{\frac{1}{2}(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} & \frac{2(s^2+t)(\mathcal{P}_k+\mathcal{P}_{mn+m+k})}{1+\mathcal{Q}_m+(-t)^m} \end{bmatrix}. \qquad (2.7)
\end{aligned}
$$

Using (2.6) and (2.7), we obtain

$$
\begin{aligned}
&\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} \\
&= \frac{\Big((1+\frac{1}{2}\mathcal{Q}_m)(\mathcal{Q}_k+\mathcal{Q}_{mn+m+k})-2(s^2+t)\mathcal{P}_m(\mathcal{P}_k+\mathcal{P}_{mn+m+k})\Big)}{1+\mathcal{Q}_m+(-t)^m}. \qquad (2.8)
\end{aligned}
$$

By Lemma 2.4 (*iv*), (2.8) becomes

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k+\mathcal{Q}_{mn+m+k}+(-t)^m\big(\mathcal{Q}_{k-m}+\mathcal{Q}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}.$$

Similarly it can be easily seen that

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k} = \frac{\mathcal{P}_k+\mathcal{P}_{mn+m+k}+(-t)^m\big(\mathcal{P}_{k-m}+\mathcal{P}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}. \qquad \square$$

Theorem 2.8. *Let $n\in\mathbb{N}$ and $m,k\in\mathbb{Z}$ with $(-t)^m+\mathcal{Q}_m\neq -1$. If n is odd, then*

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k-\mathcal{Q}_{mn+m+k}+(-t)^m\big(\mathcal{Q}_{k-m}-\mathcal{Q}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n}(-1)^j\mathcal{P}_{mj+k} = \frac{\mathcal{P}_k-\mathcal{P}_{mn+m+k}+(-t)^m\big(\mathcal{P}_{k-m}-\mathcal{P}_{mn+k}\big)}{1+(-t)^m+\mathcal{Q}_m}$$

Proof. Let n is an odd natural number. Then we get

$$\sum_{j=0}^{n}(-1)^j\mathcal{Q}_{mj+k} = \sum_{j=0}^{n-1}(-1)^j\mathcal{Q}_{mj+k} - \mathcal{Q}_{mn+k}.$$

Since n is an odd natural number then $n-1$ is even. By Thorem 2.7, we have

$$\sum_{j=0}^{n-1}(-1)^j\mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k+\mathcal{Q}_{mn+k}+(-t)^m\big(\mathcal{Q}_{mn+k-m}+\mathcal{Q}_{k-m}\big)}{1+(-t)^m+\mathcal{Q}_m}$$

and

$$\sum_{j=0}^{n}(-1)^j \mathcal{Q}_{mj+k}$$
$$= \frac{\mathcal{Q}_k + (-t)^m\big(\mathcal{Q}_{mn+k-m} + \mathcal{Q}_{k-m}\big) - (-t)^m \mathcal{Q}_{mn+k} - \mathcal{Q}_{mn+k}\mathcal{Q}_m}{1 + (-t)^m + \mathcal{Q}_m}. \tag{2.9}$$

Using Lemma 2.4 (*vi*) in (2.9), we get

$$\sum_{j=0}^{n}(-1)^j \mathcal{Q}_{mj+k} = \frac{\mathcal{Q}_k - \mathcal{Q}_{mn+m+k} + (-t)^m\big(\mathcal{Q}_{k-m} - \mathcal{Q}_{mn+k}\big)}{1 + (-t)^m + \mathcal{Q}_m}.$$

In a similar way, it can be seen that

$$\sum_{j=0}^{n}(-1)^j \mathcal{P}_{mj+k} = \sum_{j=0}^{n-1}(-1)^j \mathcal{P}_{mj+k} - \mathcal{P}_{mn+k}.$$

By Theorem 2.7, it follows that

$$\sum_{j=0}^{n}(-1)^j \mathcal{P}_{mj+k}$$
$$= \frac{\mathcal{P}_k + (-t)^m\big(\mathcal{P}_{mn+k-m} + \mathcal{P}_{k-m}\big) - (-t)^m \mathcal{P}_{mn+k} - \mathcal{P}_{mn+k}\mathcal{Q}_m}{1 + (-t)^m + \mathcal{Q}_m}. \tag{2.10}$$

Using Lemma 2.4 (*vii*) in (2.10), we obtain

$$\sum_{j=0}^{n}(-1)^j \mathcal{P}_{mj+k} = \frac{\mathcal{P}_k - \mathcal{P}_{mn+m+k} + (-t)^m\big(\mathcal{P}_{k-m} - \mathcal{P}_{mn+k}\big)}{1 + (-t)^m + \mathcal{Q}_m}. \qquad \square$$

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Some Pell and Pell-Lucas identities by matrix methods and their applications

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Abstract

In this paper, we establish some identities involving Pell and Pell-Lucas numbers by using matrix methods. Moreover, we present the solution of some Diophantine equations by applying these identities.

Keywords: Pell numbers, Pell-Lucas numbers, Diophantine equations.

1. Introduction

The Pell sequence $\{P_n\}$ is defined by $P_0 = 0, P_1 = 1,$ and $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$. The first few terms of $\{P_n\}$ are $0, 1, 2, 5, 12$ and so on. The terms of this sequence are called Pell numbers and we denoted the n^{th} Pell numbers by P_n. The Pell numbers for negative subscripts are defined as $P_{-n} = (-1)^{n+1} P_n$, for $n \geq 1$. Then it is known that $P_n = 2P_{n-1} + P_{n-2}$, for $n \in \mathbb{Z}$. Also, the Pell-Lucas sequence $\{Q_n\}$ is defined by $Q_0 = 2, Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$. The first few terms of $\{Q_n\}$ are $2, 2, 6, 14, 34$ and so on. The terms of this sequence are called Pell-Lucas numbers and we denoted the n^{th} Pell-Lucas numbers by Q_n. The Pell-Lucas numbers for negative subscripts are defined as $Q_{-n} = (-1)^n Q_n$, for $n \geq 1$. It can be seen that $Q_n = 2P_n + 2P_{n-1}$ and $Q_n = P_{n+1} + P_{n-1}$ for all $n \in \mathbb{Z}$. The Binet's formula for $\{P_n\}$ and $\{Q_n\}$ are $P_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, and $Q_n = \alpha^n + \beta^n$, for $n \geq 0$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 = 2x + 1$. It is well-known that many identities of Pell and Pell-Lucas

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numbers are proved by using Binet's formula, induction or metrics (see [1-3]).

In this paper, we establish some identities for Pell and Pell-Lucas numbers by using matrix methods. Moreover, we present the solutions of some Diophantine equations by applying these identities.

2. Main results

In this section, we establish some identities for Pell and Pell-Lucas numbers by using matrix methods and we begin with the following Lemma.

Lemma 2.1 *If* X *is a square matrix with* $X^2 = 2X + I,$ *then* $X^n = P_n X + P_{n-1} I,$ *for all* $n \in \mathbb{Z}$.

Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $X^n = P_n X + P_{n-1} I$, for all $n \in \mathbb{N}$. Now, we will show that

$$X^{-n} = P_{-n} X + P_{-n-1} I \text{, for all } n \in \mathbb{N}.$$

Let $Y = 2I - X = -X^{-1}$. Then we have

$$\begin{aligned} Y^2 &= 4I - 4X + X^2 \\ &= 2(2I - X) + I \\ &= 2Y + I. \end{aligned}$$

It implies that $Y^n = P_n Y + P_{n-1} I$.

That is $(-X^{-1})^n = P_n(2I - X) + P_{n-1} I.$

Thus, $$\begin{aligned} (-1)^n X^{-n} &= 2P_n I - P_n X + P_{n-1} I \\ &= -P_n X + (2P_n + P_{n-1}) I \\ &= -P_n X + P_{n+1} I. \end{aligned}$$

Therefore, $$\begin{aligned} X^{-n} &= (-1)^{n+1} P_n X + (-1)^n P_{n+1} I \\ &= P_{-n} X + P_{-(n+1)} I \\ &= P_{-n} X + P_{-n-1} I. \end{aligned}$$

This complete the proof. □

Form Lemma 2.1, we can give Corollary 2.2 easily. Also one can consult [1] for more information about the matrices M.

Corollary 2.2 *Let* $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. *Then*

$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, for *all* $n \in \mathbb{Z}$.

Next, let us define the matrix W as in the following Lemma and by using this matrix, we obtain some identities for Pell and Pell-Lucas numbers.

Lemma 2.3 *Let* $W = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. *Then*

$W^n = \begin{bmatrix} \frac{1}{2}Q_n & P_n \\ 2P_n & \frac{1}{2}Q_n \end{bmatrix}$, for *all* $n \in \mathbb{Z}$.

Proof. Note that $W^2 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = 2W + I$.

By Lemma 2.1, we have $W^n = P_n W + P_{n-1} I$.

It follows that

$$\begin{aligned} W^n &= \begin{bmatrix} P_n + P_{n-1} & P_n \\ 2P_n & P_n + P_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}Q_n & P_n \\ 2P_n & \frac{1}{2}Q_n \end{bmatrix}. \end{aligned}$$

Therefore, we get the result. □

Now, by using the matrix W, we get Lemma 2.4 and Lemma 2.5, respectively.

Lemma 2.4 *Let* n *be any integer. Then the following equality holds:*

$$Q_n^2 - 8P_n^2 = 4(-1)^n.$$

Proof. Since $\det(W) = -1$ and $\det(W^n) = \frac{1}{4}Q_n^2 - 2P_n^2$, it follows that $Q_n^2 - 8P_n^2 = 4(-1)^n$. □

Lemma 2.5 *For any integers m and n, the following equality holds:*

$$2Q_{m+n} = Q_m Q_n + 8P_m P_n.$$

Proof. Using $W^{m+n} = W^m W^n$, we get the result. □

Lemma 2.6 *Let n be any integer. Then following equalities hold:*

(i) $\alpha^n = \alpha P_n + P_{n-1}$,

(ii) $\beta^n = \beta P_n + P_{n-1}$.

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, then $X^2 = 2X + I$.

By Lemma 2.1, we have $X^n = P_n X + P_{n-1} I$.

It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha P_n + P_{n-1} & 0 \\ 0 & P_n + P_{n-1} \end{bmatrix}.$$

This implies that

$\alpha^n = \alpha P_n + P_{n-1}$ and $\beta^n = \beta P_n + P_{n-1}$. □

By Lemma 2.1 and Lemma 2.6, we get the following Theorem.

Theorem 2.7 *Let* $A = \begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}$, *then*

$A^n = \begin{bmatrix} \alpha^n & 0 \\ P_n & \beta^n \end{bmatrix}$, *for all* $n \in \mathbb{Z}$.

Proof. Since

$$\begin{aligned} A^2 &= \begin{bmatrix} \alpha^2 & 0 \\ \alpha+\beta & \beta^2 \end{bmatrix} \\ &= \begin{bmatrix} 2\alpha+1 & 0 \\ 2 & 2\beta+1 \end{bmatrix} \\ &= 2A + I, \end{aligned}$$

it follows from Lemma 2.1 and Lemma 2.6 that

$$\begin{aligned} A^n &= P_n A + P_{n-1} I \\ &= \begin{bmatrix} \alpha P_n + P_{n-1} & 0 \\ P_n & \beta P_n + P_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \alpha^n & 0 \\ P_n & \beta^n \end{bmatrix}. \end{aligned}$$

Therefore, the result is proved. □

By using Theorem 2.7, we get the following theorem.

Theorem 2.8 *For any integers m and n, the following equality holds:*

$$(-1)^{m+n} Q_{m+n}^2 - 8(-1)^m P_m^2 - 8(-1)^n P_n^2 = 8(-1)^{m+n} P_m P_n Q_{m+n} + 4.$$

Proof. Let a matrix A as in Theorem 2.7.

Then we have

$$A^{n+1} + A^{n-1} = \begin{bmatrix} 2\sqrt{2}\,\alpha^n & 0 \\ Q_n & -2\sqrt{2}\,\beta^n \end{bmatrix}$$

Since

$$(A^{m+1} + A^{m-1})(A^{n+1} + A^{n-1}) = A^{m+n+2} + 2A^{m+n} + A^{m+n-2},$$

we get that $2\sqrt{2}\,P_{m+n} = \alpha^n Q_m - \beta^m Q_n$. Thus,

$$\begin{aligned} 8P_{m+n}^2 &= (2\sqrt{2}\,P_{m+n})(2\sqrt{2}\,P_{n+m}) \\ &= (\alpha^n Q_m - \beta^m Q_n)(\alpha^m Q_n - \beta^n Q_m) \\ &= Q_m Q_n Q_{m+n} - (-1)^n Q_m^2 - (-1)^m Q_n^2. \end{aligned} \quad (2.1)$$

From Lemma 2.4 and (2.1), we obtain

$$(-1)^{m+n} Q_{m+n}^2 + (-1)^m Q_m^2 + (-1)^n Q_n^2 = (-1)^{m+n} Q_m Q_n Q_{m+n} + 4 \quad (2.2)$$

By Lemma 2.4, Lemma 2.5 and (2.2), we get that

$$(-1)^{m+n} Q_{m+n}^2 - 8(-1)^m P_m^2 - 8(-1)^n P_n^2 = 8(-1)^{m+n} P_m P_n Q_{m+n} + 4. \quad (2.3)$$

This complete the proof. □

Theorem 2.9 *For any integers m and n, the following equality holds:*

$$8(-1)^{m+n} P_{m+n}^2 - (-1)^n Q_n^2 + 8(-1)^m P_m^2 = 8(-1)^{m+n} Q_n P_m P_{m+n} - 4.$$

Proof. By similar argument as in Theorem 2.8 and using the properties

$$\begin{aligned} (A^{n+1} + A^{n-1})A^m &= A^m (A^{n+1} + A^{n-1}) \\ &= A^{m+n+1} + A^{m+n-1}, \end{aligned}$$

we get that $Q_{m+n} = \alpha^m Q_n - 2\sqrt{2}\beta^n P_m$

and $Q_{m+n} = 2\sqrt{2}\alpha^n P_m + \beta^m Q_n$.

Therefore, we have

$$Q_{m+n}^2 = (\alpha^m Q_n - 2\sqrt{2}\beta^n P_m)(2\sqrt{2}\alpha^n P_m + \beta^m Q_n)$$
$$= 8Q_n P_m P_{m+n} + (-1)^m Q_n^2 - 8(-1)^n P_m^2. \qquad (2.4)$$

By Lemma 2.4 and (2.4), we obtain

$$8(-1)^{m+n} P_{m+n}^2 - (-1)^n Q_n^2 + 8(-1)^m P_m^2$$
$$= 8(-1)^{m+n} Q_n P_m P_{m+n} - 4. \qquad (2.5)$$

Therefore, the proof is completed. □

3. Applications

In this section we give the solutions of some Diophantine equations by applying Theorem 2.8 and Theorem 2.9.

Theorem 3.1 *If m and n are even integers, then $(x,y,z) = (P_m, P_n, Q_{m+n})$ is a solution of the equation $z^2 - 8x^2 - 8y^2 = 8xyz + 4$. If m and n are odd integers, then $(x,y,z) = (P_m, P_n, Q_{m+n})$ is a solution of the equation $z^2 + 8x^2 + 8y^2 = 8xyz + 4$, and if m is an odd integer and n is an even integer, then $(x,y,z) = (P_m, P_n, Q_{m+n})$ is a solution of the equation $z^2 - 8x^2 + 8y^2 = 8xyz - 4$.*

Proof. The result follows immediately from Theorem 2.8. □

Theorem 3.2 *If m and n are even integers, then $(x,y,z) = (Q_m, Q_n, Q_{m+n})$ is a solution of the equation $z^2 + x^2 + y^2 = xyz + 4$. If m and n are odd integers, then $(x,y,z) = (Q_m, Q_n, Q_{m+n})$ is a solution of the equation $z^2 - x^2 - y^2 = xyz + 4$, and if m is an odd integer and n is an even integer, then $(x,y,z) = (Q_m, Q_n, Q_{m+n})$ is a solution of the equation $z^2 + x^2 - y^2 = xyz - 4$.*

Proof. The result follows directly from (2.1). □

Theorem 3.3 *If m and n are even integers, then $(x,y,z) = (Q_n, P_m, P_{m+n})$ is a solution of the equation $8z^2 - x^2 + 8y^2 = 8xyz - 4$. If m and n are odd integers, then $(x,y,z) = (Q_n, P_m, P_{m+n})$ is a solution of the equation $8z^2 + x^2 - 8y^2 = 8xyz - 4$, and if m is an odd integer and n is an even integer, then $(x,y,z) = (Q_n, P_m, P_{m+n})$ is a solution of the equation $8z^2 + x^2 + 8y^2 = 8xyz + 4$*

Proof. The result follows immediately from Theorem 2.9. □

4. Conclusion

In this paper, some identities for Pell and Pell-Lucas numbers are established by using matrix methods and the solutions of some Diophantine equations are presented by applying these identities.

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Mann-type algorithms for solving the monotone inclusion problem and the fixed point problem in reflexive Banach spaces

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Abstract

In this paper, we introduce two algorithms for finding a common solution of the monotone inclusion problem and the fixed point problem for a relatively nonexpansive mapping in reflexive Banach spaces. The weak convergence results for both algorithms are established without the prior knowledge of the Lipschitz constant of the mapping. An application to the variational inequality problem is considered. Finally, some numerical experiments of the proposed algorithms including comparisons with other algorithms are provided.

Keywords Maximal monotone operator · Banach space · Weak convergence · Fixed point problem

Mathematics Subject Classification 47H09 · 47H10 · 47J25 · 47J05

1 Introduction

Let E be a real Banach space with its dual space E^*. Let $A : E \to E^*$ be a monotone operator and $B : E \to 2^{E^*}$ be a maximal monotone operator. The *monotone inclusion problem* is to find an element $x^* \in E$ such that

$$0 \in (A + B)x^*. \tag{1.1}$$

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We denote by $(A+B)^{-1}0$ the set of solutions of (1.1). The monotone inclusion problem has wide applications in many fields such as machine learning, statistical regression, image processing and signal recovery [17,18,20,38,56]. Moreover, this problem includes the core of many mathematical problems, as special cases, such as: variational inequalities, split feasibility problem, minimization problem, Nash equilibrium problem in noncooperative games and so on [12,27,42].

A simple and efficient method for solving (1.1) is the *forward-backward splitting algorithm* introduced by Lions and Mercier [26] in a Hilbert space H. This method is of the following recursive scheme

$$x_{n+1} = J^B_{\lambda_n}(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{1.2}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$ denotes the resolvent of B, I denotes the identity mapping on H and $\{\lambda_n\}$ is a positive real sequence. It is known that this method converges weakly to an element in $(A+B)^{-1}0$ under the assumption that A is α-inverse strongly monotone, that is,

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in H, \tag{1.3}$$

where $\alpha > 0$. Note that the inverse strong monotonicity of A is a strict assumption. To avoid this assumption, Tseng [57] introduced the following algorithm which is known as the *Tseng's splitting algorithm* for solving (1.1):

$$\begin{cases} x_1 \in H, \\ y_n = J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $A : H \to H$ is monotone and L-Lipschitz continuous and $\{\lambda_n\}$ is the sequence of suitable stepsize in $\left(0, \frac{1}{L}\right)$. He proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element in $(A+B)^{-1}0$. It is remarked that the stepsize of Tseng's splitting method requires the prior knowledge of the Lipschitz constant of the mapping. However, from a practical point of view, the Lipschitz constant is very difficult to approximate. In recent years, modifications of Tseng's splitting method have been received great attention by many authors, see, for instance, [21,22,54].

Recently, in 2019, Shehu [50] extended Tseng's result to Banach spaces. He proposed the following iterative process for approximating a solution of (1.1) in a 2-uniformly convex Banach space E which is also uniformly smooth:

$$\begin{cases} x_1 \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = Jy_n - \lambda_n(Ay_n - Ax_n), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $A : E \to E^*$ is monotone and L-Lipschitz continuous, $J^B_{\lambda_n} := (J + \lambda_n B)^{-1} J$ is the resolvent of B and J is the duality mapping from E into E^*. He obtained weak convergence theorem to the solution of (1.1) provided the stepsize λ_n is chosen in $\left(0, \frac{1}{\sqrt{2\mu\kappa L}}\right)$, where μ is the 2-uniform convexity constant of E and κ is the 2-uniform smoothness constant of E^*. In addition, he also proposed a variant of (1.5) with a linesearch for solving (1.1).

On the other hand, the fixed point problem is to find an element $x^* \in H$ such that

$$x^* = Tx^*, \tag{1.6}$$

where $T : H \to H$ is a nonlinear mapping. The set of fixed points of T is denoted by $F(T)$. Numerous problems in optimization, such as convex minimization problem, variational inequality problem, minimax problem, and equilibrium problem can be formulated as a fixed point equation (1.6). Several types of iterative method have been constructed for solving the fixed point problem in various settings. One classical method for studying the fixed point problem is the *Mann's iteration* [29] (see also [10,42]) which is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where T is a self-mapping on H and $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. It was proved that if $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.7) converges weakly to an element in $F(T)$.

Consider the problem of finding a common solution of the monotone inclusion problem (1.1) and the fixed point problem (1.6), that is, find $x^* \in H$ such that

$$0 \in (A + B)x^* \text{ and } x^* = Tx^*. \tag{1.8}$$

In order to find and approximate a solution of this problem, when $A : H \to H$ is α-inverse strongly monotone, Manaka and Takahashi [28] introduced the following algorithm:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)TJ^B_{\lambda_n}(x_n - \lambda_n Ax_n), \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where $T : H \to H$ is a nonspreading mapping, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to an element in $(A + B)^{-1}0 \cap F(T)$ under certain assumptions.

In recent years, several methods for solving the monotone inclusion problem and the fixed point problem have been studied extensively by many authors, see, for instance, [37,51–53,55].

In this paper, motivated by Manaka and Takahashi [28] and Shehu [50], we propose two algorithms for finding a common solution of the monotone inclusion problem with the sum of two monotone mappings and the fixed point problem for a relatively nonexpansive mapping. The stepsize of the first algorithm is established by using Armijo linesearch and the second one using self-adaptive stepsize. We prove weak convergence theorems for the proposed algorithms under suitable conditions in reflexive Banach spaces. The major advantage of both algorithms is that they do not require the knowledge of the Lipschitz constant of the mapping.

The rest of this paper is organized as follows. We first recall some definitions and useful results in Sect. 2. The weak convergence results of our two algorithms are then established in Sect. 3. In Sect. 4, we provide an application of our results and finally, in Sect. 5, we provide some preliminary numerical results including comparisons to other algorithms.

2 Preliminaries

Throughout this paper, let E be a real reflexive Banach space with its dual E^* and $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. We denote by $\mathrm{dom} f = \{x \in E : f(x) < \infty\}$ the domain of f. We use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong convergence and weak convergence of the sequence $\{x_n\} \subset E$ to x, respectively. We also denote by $\langle x, j\rangle$ the value of functional $j \in E^*$ at $x \in E$. The *subdifferential* of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \leq f(y),\ \forall y \in E\},\ x \in E.$$

The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(x^*) = \sup_{x \in E}\{\langle x, x^*\rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x, x^*\rangle$ (see [47, Theorem 23.5]). The function f on E is said to be *cofinite* if $\mathrm{dom} f^* = E^*$ and f is said to be *strongly coercive* if $\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$.

For any $x \in \mathrm{int}(\mathrm{dom} f)$ and $y \in E$, the *directional derivative* of f at x in the direction $y \in E$ is given by

$$f'(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteaux differentiable* at x if the limit as $t \to 0$ in (2.1) exists for each y. In this case, the *gradient* of f at x is the linear function $\nabla f(x) : E \to E^*$ defined by $\langle y, \nabla f(x)\rangle = f'(x, y)$ for any $y \in E$. For more details about gradient, we recommend [8, Remark 3.32]. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \mathrm{int}(\mathrm{dom} f)$. It is known that if f is continuous at x and $\partial f(x)$ is single valued, then f is Gâteaux differentiable at x and $\nabla f(x) = \partial f(x)$ (see [5, Proposition 2.40]). The mapping ∇f is said to be *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ implies that $\nabla f(x_n) \rightharpoonup^* \nabla f(x)$. The function f is said to be *Fréchet differentiable* at x if the limit (2.1) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly*

Fréchet differentiable on a subset C of E if the limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that every Fréchet differentiable function is Gâteaux differentiable and if f is Fréchet differentiable, then it is continuous but if f is Gâteaux differentiable, then it is not necessary that f is continuous (see [35, p. 142]).

The function $f : E \to (-\infty, \infty]$ is said to be *Legendre* [45, p. 25] if and only if it satisfies the following two conditions:

(L1) $\text{int}(\text{dom} f) \neq \emptyset$ and f is Gâteaux differentiable with $\text{dom}\nabla f = \text{int}(\text{dom} f)$;
(L2) $\text{int}(\text{dom} f^*) \neq \emptyset$ and f^* is Gâteaux differentiable with $\text{dom}\nabla f^* = \text{int}(\text{dom} f^*)$.

Several examples of Legendre and related functions are presented in [39].

In a reflexive Banach space, we always obtain $(\partial f)^{-1} = \partial f^*$ (see [9, p. 83]). This fact, when combined with conditions (L1) and (L2), implies the following facts:

(i) ∇f is a bijection with $\nabla f = (\nabla f^*)^{-1}$ (see [6, Theorem 5.10]);
(i) $\text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom} f^*)$ and $\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom} f)$ (see [46, p. 123]),

where $\text{ran}\nabla f$ denotes the range of ∇f.

Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable function. The function $D_f : \text{dom} f \times \text{int}(\text{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

is called the *Bregman distance* with respect to f [11]. The Bregman distance has the following two important properties called the *two-point identity* and the *three-point identity*, respectively: for any $x, y \in \text{int}(\text{dom} f)$

$$D_f(x, y) + D_f(y, x) = \langle x - y, \nabla f(x) - \nabla f(y)\rangle \tag{2.2}$$

and for any $x \in \text{dom} f$ and $y, z \in \text{int}(\text{dom} f)$

$$D_f(x, y) = D_f(x, z) - D_f(y, z) + \langle x - y, \nabla f(z) - \nabla f(y)\rangle. \tag{2.3}$$

If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then $\nabla f(x) = Jx$, where J is the normalized duality mapping defined by $Jx = \{j \in E^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2\}$. So, we obtain

$$\begin{aligned} D_f(x, y) &= \frac{1}{2}\big(\|x\|^2 - \|y\|^2 - 2\langle x - y, Jy\rangle\big) \\ &= \frac{1}{2}\big(\|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2\big) \\ &= \frac{1}{2}\phi(x, y), \end{aligned}$$

where ϕ is called the *Lyapunov functional* which was studied in [3,43]. For a 2-uniformly convex and smooth Banach space E, the Lyapunov functional satisfies the following inequality:

$$\phi(x, y) \geq c\|x - y\|^2, \tag{2.4}$$

where $c > 0$ is the 2-uniformly convexity constant of E (see [31, Lemma 2.3]). For a real Hilbert space, it is well known that $\phi(x, y) = \|x - y\|^2$ and $c = 1$.

Define the *negative entropy function* $f(x) = \sum_{i=1}^{m} x_i \ln(x_i)$ over the nonnegative octant $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_i \geq 0\}$, then we have the *Kullback-Leibler distance* given by

$$D_f(x, y) = \sum_{i=1}^{m} \left(x_i \ln\left(\frac{x_i}{y_i}\right) + y_i - x_i \right).$$

For more examples of Bregman distances, see for instance, [25,33,34,40]. In general, the Bregman distance is not a metric due to the fact that it is not symmetric and it does not satisfy the triangle inequality. Note that $D_f(x, x) = 0$, but $D_f(x, y) = 0$ may not imply $x = y$. In our case when f is Legendre this indeed holds (see [6, Theorem 7.3(vi), p. 642]).

A Gâteaux differentiable function f is said to be σ-*strongly convex* if there exists a constant $\sigma > 0$ such that

$$f(x) \geq f(y) + \langle x - y, \nabla f(y) \rangle + \frac{\sigma}{2}\|x - y\|^2, \ \forall x \in \operatorname{dom} f, \ \ y \in \operatorname{int}(\operatorname{dom} f).$$

From the definition of Bregman distance, we have

$$D_f(x, y) \geq \frac{\sigma}{2}\|x - y\|^2. \tag{2.5}$$

The function f is said to be *totally convex at* x if $\upsilon_f(x, t) > 0$, whenever $t > 0$ and is called *totally convex* if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$, where *modulus of total convexity* of f at $x \in \operatorname{dom} f$ is the function $\upsilon_f(x, \cdot) : [0, \infty) \to [0, \infty]$ defined by

$$\upsilon_f(x, t) = \inf\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\}.$$

It is well known that if f is totally convex and Fréchet differentiable, then f is cofinite (see [45, Proposition 2.3, p. 39]). The function f is said to be *totally convex on bounded sets* if $\upsilon_f(X, t) > 0$ for any nonempty bounded subset X of E and $t > 0$, where the *modulus of total convexity* of the function f on the set X is the function $\upsilon_f : \operatorname{int}(\operatorname{dom} f) \times [0, \infty) \to [0, \infty]$ defined by

$$\upsilon_f(X, t) = \inf\{\upsilon_f(x, t) : x \in X \cap \operatorname{dom} f\}.$$

Several examples of totally convex functions can be found in [14]. Let $B_r = \{x \in E : \|x\| \leq r\}$ for all $r > 0$ and $S_E = \{x \in E : \|x\| = 1\}$. Then a function $f : E \to \mathbb{R}$

is said to be *uniformly convex* on bounded subsets of E if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. The function ρ_r is called the *gauge of the uniform convexity* of f. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [13, Theorem 2.10]). The function f is also said to be *uniformly smooth* on bounded subsets of E if $\lim_{t \to 0^+} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \frac{\alpha f(x + (1-\alpha) t y) + (1-\alpha) f(x - \alpha t y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$.

If f is uniformly convex, then we know the following lemma.

Lemma 2.1 [32, Lemma 2.3] *Let E be a Banach space, let $r > 0$ be a constant and $f : E \to \mathbb{R}$ be a uniformly convex on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{m} \lambda_k x_k\right) \leq \sum_{k=0}^{m} \lambda_k f(x_k) - \lambda_i \lambda_j \rho_r(\|x_i - y_j\|)$$

for all $i, j \in \{0, 1, 2, \ldots, m\}$, $x_k \in B_r$, $\lambda_k \in (0, 1)$ for $k = 0, 1, 2, \ldots, m$ with $\sum_{k=0}^{m} \lambda_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. A point $p \in C$ is called a *fixed point* of T if $p = Tp$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. The mapping $T : C \to \mathrm{int}(\mathrm{dom} f)$ is said to be *relatively nonexpansive* [15] if it satisfies the following conditions:

(i) $F(T) \neq \emptyset$;
(ii) $D_f(p, Tx) \leq D_f(p, x)$ for all $p \in F(T)$ and $x \in C$;
(iii) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, it follows that $p \in F(T)$.

Remark 2.2 If T satisfies (i) and (ii), then T is called *Bregman quasi-nonexpansive*. From above definition, it is known that if E is a Hilbert space and $f(x) = \frac{1}{2}\|x\|^2$, then T is quasi-nonexpansive with $I - T$ is demi-closed at zero.

Lemma 2.3 [44, Lemma 15.5] *Let $f : E \to (-\infty, \infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of int(dom f) and $T : C \to C$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Recall that the *Bregman projection with respect to f* of $x \in \mathrm{int}(\mathrm{dom} f)$ onto the nonempty, closed and convex set $C \subset \mathrm{dom} f$ is the unique $P_C^f(x) \subset C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 2.4 If E is a uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then P_C^f coincides the generalized projection Π_C (see [2, Definition 7.2]) and if E is a Hilbert space, then P_C^f is the metric projection P_C.

Lemma 2.5 [13, Corollary 4.4] *Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and $x \in E$. Then the following statements hold:*

(i) $z = \Pi_C^f(x)$ *if and only if* $\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leq 0, \forall y \in C$.
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre function. We define the function $V_f : E \times E^* \to [0, \infty)$ associated with f by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \ \forall x \in E, \ x^* \in E^*.$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \ \forall x \in E, \ x^* \in E^*.$$

Moreover, if $f : E \to (-\infty, \infty]$ is a proper and convex function, then $f^* : E \to (-\infty, \infty]$ is a proper lower semicontinuous and convex function (see [36, p. 42]). Hence V_f is convex in the second variable. Thus for all $x \in E$,

$$D_f\Big(x, \nabla f^*\Big(\sum_{i=1}^{N} t_i \nabla f(x_i)\Big)\Big) \leq \sum_{i=1}^{N} t_i D_f(x, x_i), \tag{2.6}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Let C be a nonempty subset of E. Then a mapping $A : C \to E^*$ is said to be:

(i) *monotone* if $\langle x - y, Ax - Ay\rangle \geq 0$ for all $x, y \in C$.
(ii) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$.
(iii) *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*.

For a set-valued operator $A : E \to 2^{E^*}$, we define its domain, range and graph as follows: $\mathrm{dom}A = \{x \in E : Ax \neq \emptyset\}$, $\mathrm{ran}A = \bigcup\{Ax : x \in \mathrm{dom}A\}$ and $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$, respectively. An operator A is said to be *monotone* if for each $(x, x^*), (y, y^*) \in G(A)$, we have $\langle x - y, x^* - y^*\rangle \geq 0$. A monotone operator A is said to be *maximal*, if its graph is not contained in the graph of any other monotone operator on E. It is known that if $f : E \to \mathbb{R}$ is Gâteaux differentiable, strictly convex and cofinite, then A is maximal monotone if and only if $\mathrm{ran}(\nabla f + \lambda A) = E^*$ for $\lambda > 0$ (see [7, Corollary 2.4]).

Let $f : E \to (-\infty, \infty]$ be a Fréchet differentiable function which is bounded on bounded subsets of E and A be a maximal monotone operator, then the *resolvent* of A for $\lambda > 0$ defined by

$$J_\lambda^A(x) = (\nabla f + \lambda A)^{-1} \nabla f(x), \quad \forall x \in E$$

is a single-valued Bregman quasi-nonexpansive mapping from E onto dom$A$ with $F(J_\lambda^A) = A^{-1}0$.

Lemma 2.6 [4, Corollary 2.1] *Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded operator, and $B : E \to 2^{E^*}$ be a maximal monotone operator. Then $A + B$ is maximal monotone.*

Lemma 2.7 [45, Lemma 3.1] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose that $x \in E$, if $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.8 [32, Lemma 2.4] *Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then* $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ *if and only if* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.9 [30, Proposition 9] *Let $f : E \to \mathbb{R}$ be a Legendre function such that ∇f is weakly sequentially continuous. Suppose that the sequence $\{x_n\}$ is bounded and that* $\lim_{n\to\infty} D_f(u, x_n)$ *exists for any weak subsequential limit u of $\{x_n\}$. Then $\{x_n\}$ converges weakly to u.*

3 Main results

In this section, we introduce two algorithms without the knowledge of the Lipschitz constant of the mapping for solving the monotone inclusion problem and the fixed point problem. From now on, let E be a real reflexive Banach space, $f : E \to \mathbb{R}$ be a σ-strongly convex, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, $B : E \to 2^{E^*}$ be a maximal monotone mapping, and $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap (A + B)^{-1}0 \neq \emptyset$.

Algorithm 1: Mann-type splitting algorithm with linesearch

Step 0. Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.5). Let $x_1 \in E$ be arbitrary. Set $n = 1$.
Step 1. Compute

$$y_n = J^B_{\lambda_n} \nabla f^*(\nabla f(x_n) - \lambda_n A x_n), \tag{3.1}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\lambda_n \|Ax_n - Ay_n\| \le \mu \|x_n - y_n\|. \tag{3.2}$$

Step 2. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)). \tag{3.3}$$

Step 3. Compute

$$x_{n+1} = \nabla f^*((1 - \alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n))). \tag{3.4}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.1 *Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$. *If* $y_n = x_n = x_{n+1}$ *for some* n, *then* $x_n \in \Omega$.

Proof If $y_n = x_n$, then $x_n = J^B_{\lambda_n} \nabla f^*(\nabla f(x_n) - \lambda_n A x_n)$. It follows that $x_n = (\nabla f + \lambda_n B)^{-1} \nabla f \circ \nabla f^*(\nabla f - \lambda_n A)x_n$, that is, $\nabla f(x_n) - \lambda_n A x_n \in \nabla f(x_n) + \lambda_n B x_n$, which implies that $0 \in (A + B)x_n$. Hence $x_n \in (A + B)^{-1}0$. Again since $y_n = x_n$ and ∇f is one-to-one, we have from (3.3) that $z_n = x_n$. On the other hand, if $x_{n+1} = x_n$, then by $x_{n+1} = \nabla f^*((1 - \alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n)))$, we have $\nabla f(x_n) = (1 - \alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n)$ and hence $x_n = Tx_n$. This implies that $x_n \in F(T)$. Therefore, $x_n \in \Omega := F(T) \cap (A + B)^{-1}0$. □

Lemma 3.2 *The Armijo linesearch rule defined by* (3.2) *is well defined and*

$$\min\{\gamma, \frac{\mu l}{L}\} \le \lambda_n \le \gamma.$$

Proof Since A is L-Lipschitz continuous on E, we have

$$\begin{aligned} &\|Ax_n - A(J^B_{\gamma l^{m_n}} \nabla f^*(\nabla f(x_n) - \gamma l^{m_n} Ax_n)) \\ &\quad \| \le L \|x_n - J^B_{\gamma l^{m_n}} \nabla f^*(\nabla f(x_n) - \gamma l^{m_n} Ax_n)\|. \end{aligned}$$

Using the fact that $L > 0$ and $\mu > 0$, we get

$$\begin{aligned} \frac{\mu}{L}\|Ax_n - A(J^B_{\gamma l^{m_n}} \nabla f^*(\nabla f(x_n) - \gamma l^{m_n} Ax_n))\| \le \mu \|x_n - J^B_{\gamma l^{m_n}} \nabla f^*(\nabla f(x_n) \\ -\gamma l^{m_n} Ax_n)\|. \end{aligned}$$

This implies that (3.2) holds for all $\gamma l^{m_n} \leq \frac{\mu}{L}$ and so λ_n is well defined. Obviously, $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$, then the lemma is proved. Otherwise, if $\lambda_n < \gamma$, then we have from (3.2) that

$$\|Ax_n - A\big(J^B_{\frac{\lambda_n}{l}} \nabla f^*\big(\nabla f(x_n) - \frac{\lambda_n}{l} Ax_n\big)\big)\| > \frac{\mu}{\frac{\lambda_n}{l}} \|x_n - J^B_{\frac{\lambda_n}{l}} \nabla f^*\big(\nabla f(x_n) - \frac{\lambda_n}{l} Ax_n\big)\|.$$

Again by the L-Lipschitz continuity of A, we obtain $\lambda_n > \frac{\mu l}{L}$. This completes the proof □

Lemma 3.3 *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then*

$$D_f(p, z_n) \leq D_f(p, x_n) - \Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, x_n) - \Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, z_n),$$
$$\forall p \in (A + B)^{-1}0.$$

Proof Let $p \in (A + B)^{-1}0$. By the definition of Bregman distance, we have

$$\begin{aligned} D_f(p, z_n) &= D_f(p, \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n))) \\ &= f(p) - f(z_n) - \langle p - z_n, \nabla f(y_n) - \lambda_n(Ay_n - Ax_n)\rangle \\ &= f(p) - f(z_n) - \langle p - z_n, \nabla f(y_n)\rangle + \lambda_n \langle p - z_n, Ay_n - Au_n\rangle \\ &= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle + \langle p - z_n, \nabla f(y_n)\rangle \\ &\quad + f(y_n) - f(z_n) - \langle p - z_n, \nabla f(y_n)\rangle \\ &\quad + \lambda_n \langle p - z_n, Ay_n - Au_n\rangle \\ &= f(p) - f(y_n) - \langle p - y_n, \nabla f(y_n)\rangle - f(z_n) + f(y_n) \\ &\quad + \langle z_n - y_n, \nabla f(y_n)\rangle + \lambda_n \langle p - z_n, Ay_n - Ax_n\rangle \\ &= D_f(p, y_n) - D_f(z_n, y_n) + \lambda_n \langle p - z_n, Ay_n - Ax_n\rangle. \end{aligned} \tag{3.5}$$

From (2.3), we have

$$D_f(p, y_n) = D_f(p, x_n) - D_f(y_n, x_n) + \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\begin{aligned} D_f(p, z_n) &= D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \\ &\quad + \lambda_n \langle p - z_n, Ay_n - Ax_n\rangle \\ &= D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \langle p - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \\ &\quad + \lambda_n \langle y_n - z_n, Ay_n - Ax_n\rangle - \lambda_n \langle y_n - p, Ay_n - Ax_n\rangle \\ &= D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \lambda_n \langle y_n - z_n, Ay_n - Ax_n\rangle \\ &\quad - \langle y_n - p, \nabla f(x_n) - \nabla f(y_n) - \lambda_n(Ax_n - Ay_n)\rangle. \end{aligned} \tag{3.7}$$

By the definition of y_n, we have $\nabla f(x_n) - \lambda_n Ax_n \in \nabla f(y_n) + \lambda_n By_n$. Since B is maximal monotone, there exists $v_n \in By_n$ such that $\nabla f(x_n) - \lambda_n Ax_n = \nabla f(y_n) +$

$\lambda_n v_n$, it follows that

$$v_n = \frac{1}{\lambda_n}\big(\nabla f(x_n) - \nabla f(y_n) - \lambda_n A x_n\big). \tag{3.8}$$

Since $0 \in (A+B)p$ and $Ay_n + v_n \in (A+B)y_n$, it follows from Lemma 2.6 that $A+B$ is maximal monotone. Hence

$$\langle y_n - p, Ay_n + v_n\rangle \geq 0. \tag{3.9}$$

Substituting (3.8) into (3.9), we have

$$\frac{1}{\lambda_n}\langle y_n - p, \nabla f(x_n) - \nabla f(y_n) - \lambda_n A x_n + \lambda_n A y_n\rangle \geq 0.$$

That is

$$\langle y_n - p, \nabla f(x_n) - \nabla f(y_n) - \lambda_n (A x_n - A y_n)\rangle \geq 0. \tag{3.10}$$

Combining (3.7) and (3.10), we have

$$\begin{aligned} D_f(p, z_n) &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \lambda_n \langle y_n - z_n, Ay_n - Ax_n\rangle \\ &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \lambda_n \|y_n - z_n\| \|Ay_n - Ax_n\| \\ &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \mu \|y_n - z_n\| \|y_n - x_n\| \\ &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \frac{\mu}{2}\Big(\|y_n - z_n\|^2 + \|y_n - x_n\|^2\Big). \end{aligned} \tag{3.11}$$

Thus from (2.5), we can rewrite (3.11) as follows:

$$D_f(p, z_n) \leq D_f(p, x_n) - \Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, x_n) - \Big(1 - \frac{\mu}{\sigma}\Big) D_f(y_n, z_n).$$

□

Theorem 3.4 *Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\{\alpha_n\} \subset [a,b] \subset (0,1)$ for some $a, b > 0$. Suppose, in addition, that ∇f is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to an element in Ω.*

Proof First, we show that $\{x_n\}$ is bounded. Let $z \in F(T) \cap (A+B)^{-1}0$. Since $\mu \in (0, 2\sigma)$, it follows from Lemma 3.3 that

$$D_f(z, z_n) \leq D_f(z, x_n). \tag{3.12}$$

Consequently,

$$
\begin{aligned}
D_f(z, x_{n+1}) &\leq (1-\alpha_n)D_f(z, z_n) + \alpha_n D_f(z, Tz_n) \\
&\leq (1-\alpha_n)D_f(z, z_n) + \alpha_n D_f(z, z_n) \\
&\leq D_f(z, x_n).
\end{aligned}
\tag{3.13}
$$

This implies that $\lim_{n\to\infty} D_f(z, x_n)$ exists and hence $\{D_f(z, x_n)\}$ is bounded. By Lemma 2.7, we have $\{x_n\}$ is bounded. By our assumption that f is Fr$\acute{e}$chet differentiable, we have f is uniformly smooth. From this, we also have f^* is uniformly convex (see [59, Theorem 3.5.5]). Then by the property of D_f and Lemmas 2.1, 3.3, we have

$$
\begin{aligned}
D_f(z, x_{n+1}) &= V_f(z, (1-\alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n)) \\
&= f(z) - \langle z, (1-\alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n)\rangle + f^*((1-\alpha_n)\nabla f(z_n) \\
&\quad + \alpha_n \nabla f(Tz_n)) \\
&\leq (1-\alpha_n) f(z) + \alpha_n f(z) - (1-\alpha_n)\langle z, \nabla f(z_n)\rangle - \alpha_n \langle z, \nabla f(Tz_n)\rangle \\
&\quad + (1-\alpha_n) f^*(\nabla f(z_n)) + \alpha_n f^*(\nabla f(Tz_n)) - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) \\
&\quad - \nabla f(Tz_n)\|) \\
&= (1-\alpha_n)(f(z) - \langle z, \nabla f(z_n)\rangle + f^*(\nabla f(z_n))) + \alpha_n (f(z) - \langle z, \nabla f(Tz_n)\rangle \\
&\quad + f^*(\nabla f(Tz_n))) \\
&\quad - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) - \nabla f(Tz_n)\|) \\
&= (1-\alpha_n)V_f(z, \nabla f(z_n)) + \alpha_n V_f(z, \nabla f(Tz_n)) - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) \\
&\quad - \nabla f(Tz_n)\|) \\
&= (1-\alpha_n)D_f(z, z_n) + \alpha_n D_f(z, Tz_n) - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) - \nabla f(Tz_n)\|) \\
&\leq D_f(z, z_n) - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) - \nabla f(Tz_n)\|) \\
&\leq D_f(z, x_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, x_n) - \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, z_n) \\
&\quad - \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) - \nabla f(Tz_n)\|).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, x_n) + \left(1 - \frac{\mu}{\sigma}\right) D_f(y_n, z_n) + \alpha_n(1-\alpha_n)\rho_r^*(\|\nabla f(z_n) \\
&\quad - \nabla f(Tz_n)\|) \\
&\leq D_f(z, x_n) - D_f(z, x_{n+1}).
\end{aligned}
\tag{3.14}
$$

By Lemma 2.8 and the property of ρ_r^*, we have

$$
\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0 \tag{3.15}
$$

and

$$\lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(Tz_n)\| = 0. \tag{3.16}$$

Since f is bounded and uniformly smooth on bounded sets of E, it follows that ∇f is uniformly continuous on bounded subsets of E (see [59, Proposition 3.6.3]). Thus we have

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(x_n)\| = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(z_n)\| = 0. \tag{3.18}$$

Combining (3.17) and (3.18), we also have

$$\begin{aligned} \|\nabla f(x_n) - \nabla f(z_n)\| &\leq \|\nabla f(x_n) - \nabla f(y_n)\| + \|\nabla f(y_n) - \nabla f(z_n)\| \\ &\to 0. \end{aligned}$$

Since f is uniformly convex on bounded subsets of E, it follows that ∇f^* is uniformly continuous on bounded subsets of E^* (see [59, Theorem 3.5.10]) and hence

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

and

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.20}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in E$. From (3.19), we also have $z_{n_k} \rightharpoonup x^*$. Since $\|z_n - Tz_n\| \to 0$ and $I - T$ is demi-closed at zero, we have $x^* \in F(T)$. We next show that $x^* \in (A+B)^{-1}0$. Let $(v, w) \in G(A+B)$, we have $w - Av \in Bv$. From the definition of y_{n_k}, we note that

$$\nabla f(x_{n_k}) - \lambda_{n_k} Ax_{n_k} \in \nabla f(y_{n_k}) + \lambda_{n_k} By_{n_k},$$

or equivalently,

$$\frac{1}{\lambda_{n_k}}\big(\nabla f(x_{n_k}) - \nabla f(y_{n_k}) - \lambda_{n_k} Ax_{n_k}\big) \in By_{n_k}.$$

By the maximal monotonicity of B, we have

$$\left\langle v - y_{n_k}, w - Av + \frac{1}{\lambda_{n_k}}\big(\nabla f(x_{n_k}) - \nabla f(y_{n_k}) - \lambda_{n_k} Ax_{n_k}\big)\right\rangle \geq 0.$$

Moreover, by the monotonicity of A, we have

$$
\begin{aligned}
\langle v - y_{n_k}, w\rangle &\geq \left\langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}\big(\nabla f(x_{n_k}) - \nabla f(y_{n_k}) - \lambda_{n_k} Ax_{n_k}\big)\right\rangle \\
&= \langle v - y_{n_k}, Av - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, \nabla f(x_{n_k}) - \nabla f(y_{n_k})\rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k}\rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle \\
&\quad + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, \nabla f(x_{n_k}) - \nabla f(y_{n_k})\rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, \nabla f(x_{n_k}) - \nabla f(y_{n_k})\rangle.
\end{aligned}
\tag{3.21}
$$

Since A is Lipschitz continuous and $y_{n_k} \rightharpoonup x^*$, it follows from (3.15) and (3.17) that

$$\langle v - x^*, w\rangle \geq 0.$$

By the monotonicity of $A + B$, we get $0 \in (A + B)x^*$, that is, $x^* \in (A + B)^{-1}0$. Hence $x^* \in F(T) \cap (A + B)^{-1}0$. From Lemma 2.9, we conclude that the sequence $\{x_n\}$ converges weakly to x^*. This completes the proof. □

If E is a 2-uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have the following result.

Corollary 3.5 *Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, $B : E \to 2^{E^*}$ be a maximal monotone mapping, and $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap (A + B)^{-1}0 \neq \emptyset$. Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, c)$, where c is a constant given by (2.4). For any $x_1 \in E$, let $\{x_n\}$ be defined by*

$$
\begin{cases}
y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \\
z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\
x_{n+1} = J^{-1}((1 - \alpha_n)Jz_n + \alpha_n J(Tz_n)), \quad \forall n \geq 1,
\end{cases}
$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n\|Ax_n - Ay_n\| \leq \mu\|x_n - y_n\|.$$

Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Suppose, in addition, that J is weakly sequentially continuous on E. Then $\{x_n\}$ converges weakly to an element in Ω.

Next, we propose another Mann-type splitting algorithm with self-adaptive stepsize for solving the variational inclusion problem and the fixed point problem.

Algorithm 2: Man-type splitting algorithm with self-adaptive stepsize

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sigma)$, where σ is a constant given by (2.5). Let $x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^B \nabla f^*(\nabla f(x_n) - \lambda_n A x_n). \tag{3.22}$$

Step 2. Compute

$$z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)). \tag{3.23}$$

Step 3. Compute

$$x_{n+1} = \nabla f^*((1 - \alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n))), \tag{3.24}$$

where the stepsize is adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.25}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.6 *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then the sequence $\{\lambda_n\}$ is nonincreasing and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}.$$

Moreover,

$$\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|x_n - y_n\|, \ \forall n \geq 1.$$

Proof It is obvious from (3.25) that $\lambda_{n+1} \leq \lambda_n$ for all $n \geq 1$. Since A is L-Lipschitz continuous, in the case $Ax_n - Ay_n \neq 0$, we have

$$\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

Clearly,

$$\lambda_{n+1} \geq \min\big\{\frac{\mu}{L}, \lambda_n\big\}.$$

By induction, we obtain immediately that the sequence $\{\lambda_n\}$ is bounded from below by $\min\{\frac{\mu}{L}, \lambda_1\}$. Thus there exists $\lambda := \lim_{n\to\infty} \lambda_n \geq \min\{\frac{\mu}{L}, \lambda_1\}$.

On the other hand, by the definition of $\{\lambda_n\}$, we have

$$\lambda_{n+1} = \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\right\} \leq \frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}.$$

This implies that

$$\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|x_n - y_n\|, \ \forall n \geq 1.$$

□

Lemma 3.7 *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Then*

$$D_f(p, z_n) \leq D_f(p, x_n) - \Big(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(y_n, z_n) - \Big(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(y_n, x_n),$$
$$\forall p \in (A + B)^{-1}0.$$

Proof Let $p \in (A + B)^{-1}0$. By using the same arguments as in the proof of Lemma 3.3, we can deduce that

$$D_f(p, z_n) \leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \lambda_n\langle y_n - z_n, Ay_n - Ax_n\rangle.$$

Thus from Lemma 3.6, we have

$$\begin{aligned} D_f(p, z_n) &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \lambda_n\|y_n - z_n\|\|Ay_n - Ax_n\| \\ &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) + \mu\frac{\lambda_n}{\lambda_{n+1}}\|y_n - z_n\|\|y_n - x_n\| \\ &\leq D_f(p, x_n) - D_f(y_n, x_n) - D_f(z_n, y_n) \\ &\quad + \frac{\mu}{2}\frac{\lambda_n}{\lambda_{n+1}}\Big(\|y_n - z_n\|^2 + \|y_n - x_n\|^2\Big). \end{aligned} \tag{3.26}$$

From (2.5), we can rewrite (3.26) as follows:

$$D_f(p, z_n) \leq D_f(p, x_n) - \Big(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(y_n, z_n) - \Big(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\Big)D_f(y_n, x_n).$$

□

Theorem 3.8 *Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Suppose, in addition, that ∇f is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to an element in Ω.*

Proof We only show that $\{x_n\}$ is bounded. Since $\lim_{n\to\infty} \lambda_n$ exists and $\mu \in (0, \sigma)$, we have $\lim_{n\to\infty}\left(1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \frac{\mu}{\sigma} > 0$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu}{\sigma}\frac{\lambda_n}{\lambda_{n+1}} > 0, \quad \forall n \geq n_0.$$

This implies by Lemma 3.7 that

$$D_f(p, z_n) \leq D_f(p, x_n), \quad \forall n \geq n_0.$$

Consequently,

$$\begin{aligned} D_f(z, x_{n+1}) &\leq (1-\alpha_n)D_f(z, z_n) + \alpha_n D_f(z, Tz_n) \\ &\leq (1-\alpha_n)D_f(z, z_n) + \alpha_n D_f(z, z_n) \\ &\leq D_f(z, x_n). \end{aligned}$$

This shows that $\lim_{n\to\infty} D_f(p, x_n)$ exists and hence $\{D_f(p, x_n)\}$ is bounded. Moreover, we also have $\{x_n\}$ is bounded. By using the same arguments and techniques as those of Theorem 3.4, we can show that the sequence $\{x_n\}$ converges weakly to an element in $\Omega := F(T) \cap (A+B)^{-1}0$. Then the proof is completed. □

If E is a 2-uniformly convex and uniformly smooth Banach space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have the following result.

Corollary 3.9 *Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, $B : E \to 2^{E^*}$ be a maximal monotone mapping, and $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap (A+B)^{-1}0 \neq \emptyset$. Given $\lambda_1 > 0$ and $\mu \in (0, c)$, where c is a constant given by (2.4). For any $x_1 \in E$, let $\{x_n\}$ be defined by*

$$\begin{cases} y_n = J_{\lambda_n}^B J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = J^{-1}((1-\alpha_n)Jz_n + \alpha_n J(Tz_n)), \quad \forall n \geq 1, \end{cases}$$

where the stepsize is adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Suppose, in addition, that J is weakly sequentially continuous on E. Then $\{x_n\}$ converges weakly to an element in Ω.

Remark 3.10 Our main results improve and generalize the main results of Shehu [50] in the following ways:

(i) For the structure of Banach spaces, we extend the duality mapping to more general case, σ-strongly convex, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets.
(ii) We extend the main results of Shehu [50] from the problem of finding a solution of the monotone inclusion problem in a 2-uniformly convex and uniformly smooth Banach space to the problem of finding a common solution of the monotone inclusion problem and the fixed point problem in a reflexive Banach space. However, the result of Theorem 3.4 contains the result of Theorem 3.10 in [50] as a special case.
(iii) The sequence of stepsizes of our algorithms is chosen without the prior knowledge of the Lipschitz constant and the uniform smoothness constant of the mapping, while the sequence of stepsize of Theorem 3.6 in [50] requires the knowledge of them.

Remark 3.11 Our main results improve and generalize the main result of Manaka and Takahashi [28] in the following ways:

(i) We extend Theorem 3.1 in [28] from a Hilbert space to a reflexive Banach space.
(ii) We relax the strict assumption of the mapping A with the weaker assumption that A is a monotone and L-Lipschitz continuous mapping.

4 Application

In this section, we apply our results to the problem of finding a common solution of the variational inequality problem and the fixed point problem in Banach spaces. Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $f : E \to (-\infty, \infty]$ be a Legendre and totally convex function and $A : C \to E^*$ be a monotone mapping. The *variational inequality problem* is to find an element $x^* \in C$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $VI(C, A)$. In particular, if A is a continuous and hemicontinuous mapping, then $VI(C, A)$ is closed and convex (see [31, Proposition 2.6]). Recall that the *indicator function* of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is known that i_C is proper convex, lower semicontinuous and convex function with its subdifferential ∂i_C is maximal monotone (see [48, Theorem A]). Moreover, from [1, Proposition 2.5.13], we know that

$$\partial i_C(v) = N_C(v) = \{u \in E^* : \langle y - v, u \rangle \leq 0, \ \forall y \in C\},$$

where N_C is the normal cone for C at a point v.

Theorem 4.1 *Let $f : E \to \mathbb{R}$ be a σ-strongly convex, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, and $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap VI(C, A) \neq \emptyset$. Given $\gamma > 0$, $l \in (0, 1)$ and $\mu \in (0, \sigma)$. For any $x_1 \in E$, let $\{x_n\}$ be defined by*

$$\begin{cases} y_n = P_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A x_n), \\ z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n))), \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|.$$

Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Suppose, in addition, that ∇f is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by (4.2) converges weakly to an element in Ω.

Proof In this case, we set $B = N_C$ in Theorem 3.4 and from [58, p. 18], we know that $J_\lambda^{\partial i_C} = P_C^f$. Moreover, as shown in [24, Theorem 2.9], we have $(A + N_C)^{-1}0 = VI(C, A)$. Therefore, we get the desired result from Theorem 3.4. □

Theorem 4.2 *Let $f : E \to \mathbb{R}$ be a σ-strongly convex, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping, and $T : E \to E$ be a relatively nonexpansive mapping. Assume that the solution set $\Omega := F(T) \cap VI(C, A) \neq \emptyset$. Given $\lambda_1 > 0$ and $\mu \in (0, \sigma)$. For any $x_1 \in E$, let $\{x_n\}$ be defined by*

$$\begin{cases} y_n = P_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A x_n), \\ z_n = \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Ax_n)), \\ x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n))), \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where the stepsize is adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n \right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\{\alpha_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$. Suppose, in addition, that ∇f is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by (4.3) converges weakly to an element in Ω.

Proof The proof is quite similar to that of Theorem 4.1. So we omit it.

Table 1 Numerical results of Algorithm 1 and Algorithm 2 for Example 5.1

		Algorithm 1	Algorithm 2
$m = 10$	No. of Iter.	7	8
	cpu (Time)	0.0019	2.0470e-04
$m = 50$	No. of Iter.	5	5
	cpu (Time)	3.2780e-04	1.6950e-04
$m = 100$	No. of Iter.	4	5
	cpu (Time)	0.0020	2.9260e-04
$m = 500$	No. of Iter.	4	4
	cpu (Time)	0.0045	0.0024

5 Numerical examples

5.1 Numerical behavior of Algorithm 1 and Algorithm 2

In this subsection, we provide the numerical experiments to illustrate the convergence of the proposed algorithms.

Example 5.1 Let $E = \mathbb{R}^m$ $(m = 2k)$ with $C = \{x = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m : u_i \geq 0,\ \sum_{i=1}^m u_i = 1\}$. Let $f : C \to \mathbb{R}$ be define by $f(x) = \sum_{i=1}^m u_i \ln(u_i)$ (it is strongly convex with $\sigma = 1$ respect to ℓ_1-norm on C). Then we have $\nabla f(x) = (1 + \ln(u_1), 1 + \ln(u_2), \ldots, 1 + \ln(u_m))$ and $\nabla f^*(x) = (e^{u_1 - 1}, e^{u_2 - 1}, \ldots, e^{u_m - 1})$. Let $Ax = (2u_1, 0, 2u_3, 0, 2u_5, 0, \ldots, 0, 2u_{2k-1}, 0)$ and $Bx = N_C(x)$. We see that A is monotone and Lipschitz continuous with $L = 2$, and B is maximal monotone. Hence $J_\lambda^{\partial i_C}(x) = P_C^f(x)$. In this case, we define the Bregman projection onto C (see [19, Remark 4]) by

$$P_C^f(a) = \left(\frac{u_1 e^{a_1}}{\sum_{i=1}^m u_i e^{a_i}}, \frac{u_2 e^{a_2}}{\sum_{i=1}^m u_i e^{a_i}}, \ldots, \frac{u_m e^{a_m}}{\sum_{i=1}^m u_i e^{a_i}} \right), \quad a \in \mathbb{R}^m \text{ and } x \in \operatorname{int}(C).$$

Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a mapping defined by $Tx = x$ for all $x \in \mathbb{R}^m$. We perform the numerical tests of Algorithms 1 and 2 with four different cases of m $(m = 10, 50, 100, 500)$. We take $l = 0.2$, $\gamma = 5$, $\mu = 0.9$ in Algorithm 1 and take $\lambda_1 = 0.8$ and $\mu = 0.9$ in Algorithm 2. The starting point x_1 is randomly generated in $\mathbb{R}^m$. We use the stopping criterion

$$E_n = ||x_{n+1} - x_n\| \leq 10^{-5}. \tag{5.1}$$

The numerical results are presented in Table 1 and Fig. 1.

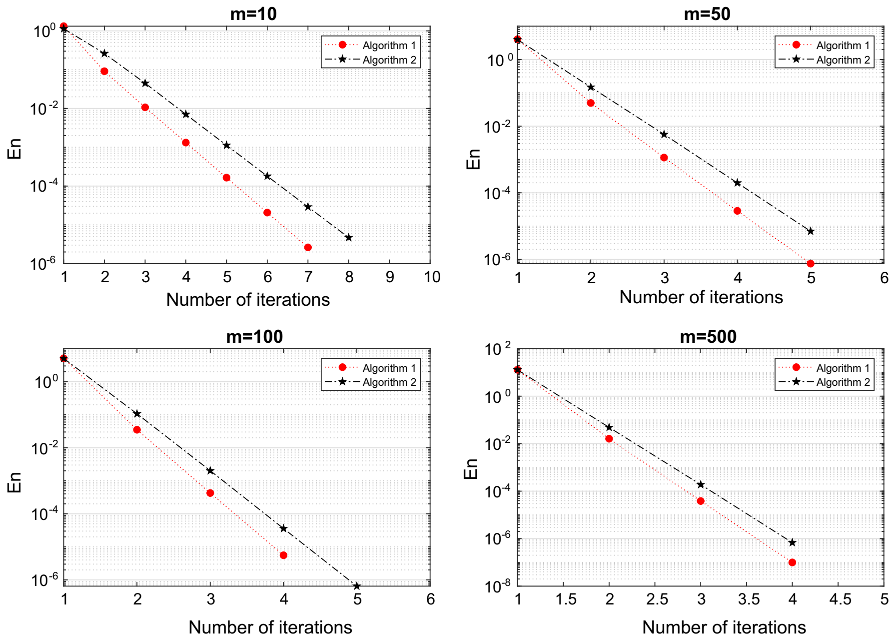

Fig. 1 The error ploting of Algorithm 1 and Algorithm 2 in Example 5.1

5.2 Comparison of Algorithm 1 and Algorithm 2 with other algorithms

In this subsection, we provide the numerical experiments of Algorithms 1 and 2 in both finite and infinite-dimensional spaces. Moreover, we compare the proposed algorithms with some existing methods. In what follows, let us define $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$.

Example 5.2 Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be an operator defined by $Ax = Mx + q$ with $q \in \mathbb{R}^m$ and

$$M = NN^t + S + D,$$

where $N, D \in \mathbb{R}^{m\times m}$ and S is an $m \times m$ skew-symmetric matrix (hence the operator does not arise from an optimization problem), D is a positive definite diagonal matrix (hence the variational inequality has a unique solution) and q is equal to the zero vector. It is clear that A is monotone and Lipschitz continuous with $L = \|M\|$ (see [23]). Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a mapping defined by $Tx = \frac{x}{2}$ for all $x \in \mathbb{R}^m$. It is easy to see that T is nonexpansive (hence it is quasi-nonexpansive). The feasible set C is described by linear inequality constraints $Nx \le b$ for some random matrix $N \in \mathbb{R}^{k\times m}$ and a random vector $b \in \mathbb{R}^k$ with nonnegative entries. Hence the zero vector is feasible and therefore the unique solution of the corresponding variational inequality. These projections are computed by solving a quadratic optimization problem using the MATLAB solver quadprog.

Table 2 Numerical results of Algorithm 1, Algorithm 2, HFBA and VMEM for Example 5.2

		Algorithm 1	Algorithm 2	VMEM	HFBA
$m = 50$	No. of Iter.	3	9	21	476
	cpu (Time)	0.0547	0.0468	0.3284	02.7282
$m = 100$	No. of Iter.	4	10	21	551
	cpu (Time)	0.1654	0.1397	00.7207	8.4411
$m = 500$	No. of Iter.	4	11	22	797
	cpu (Time)	6.7415	4.6021	22.0554	375.0762
$m = 1000$	No. of Iter.	3	12	23	953
	cpu (Time)	41.6583	32.3860	122.3584	2.9860e+03

In these experiments, we compare our Algorithm 1 and Algorithm 2 with a Halpern-type forward backward algorithm (HFBA) proposed in ([53, Theorem 3.1]) and a viscosity-type modified extragradient method (VMEM) proposed in ([52, Algorithm 3.1]). We see that the operators A of HFBA and VMEM are α-inverse strongly monotone, then A is Lipschitz continuous. We perform the numerical experiments with four different cases of m ($m = 50, 100, 500, 1000$). We take $l = 0.001$, $\gamma = 0.002$ and $\mu = 0.03$ in Algorithm 1 and take $\lambda_1 = 4$ and $\mu = 0.03$ in Algorithm 2. For both Algorithm 1 and Algorithm 2, we take $\alpha_n = \frac{n}{2(n+1)}$. For HFBA, we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{x}{2(x+1)}$, $\lambda_n = \frac{1}{2\|M\|}$ and for VMEM, we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = r_n = \frac{1}{2}$, $\gamma_n = \frac{1}{2} - \frac{1}{n+1}$, $\lambda_n = \frac{1}{2\|M\|}$, and $f(x) = 0.2x$.

The starting point x_1, u are randomly generated in $\mathbb{R}^m$. We use stopping criterion $\|x_{n+1} - x_n\| \leq 10^{-5}$. The numerical results are presented in Table 2 and Fig. 2.

Example 5.3 In this example, we apply our proposed algorithms to solve the fixed point problem and the split feasibility problem in the infinite dimensional Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\mathcal{A} : H_1 \to H_2$ be a bounded linear operator with its adjoint $\mathcal{A}^*$. The split feasibility problem (SFP) is to find an element

$$x^* \in C \text{ such that } \mathcal{A}x^* \in Q. \tag{5.2}$$

We will use Γ to denote the solution set of SFP (5.2). Let $H_1 = H_2 = L_2([0, 1])$ with norm

$$\|x\|_2 = \left(\int_0^1 |x(t)|^2 dt \right)^{1/2}$$

and inner product

$$\langle x, y \rangle = \int_0^1 x(t) y(t) dt$$

for all $x, y \in L_2([0, 1])$.

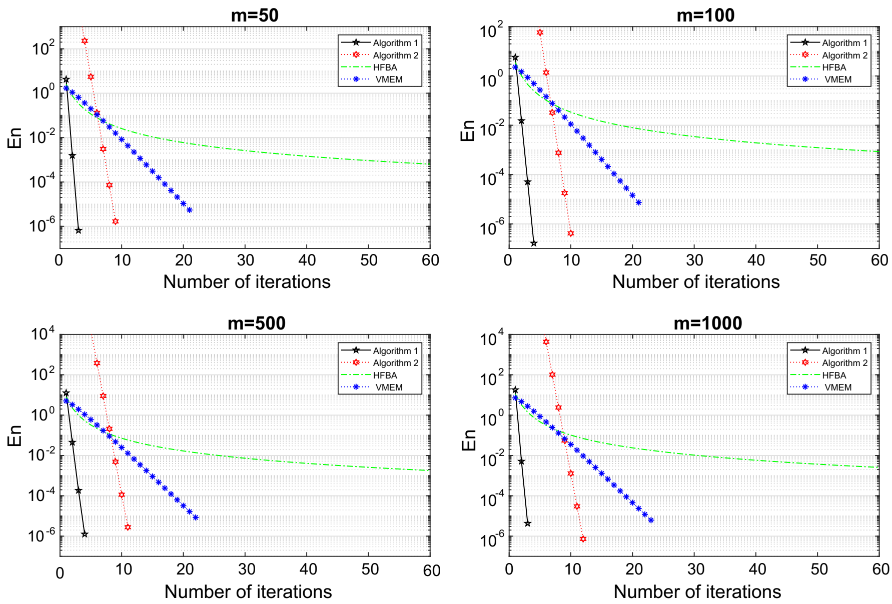

Fig. 2 The error ploting of Algorithm 1, Algorithm 2, HFBA and VMEM for Example 5.2

Now, let

$$C = \{x \in L_2([0,1]) : \|x\| \leq 1\} \text{ and } Q = \{x \in L_2([0,1]) : \left\langle \frac{t}{2}, x \right\rangle = 0\}.$$

Let $\mathcal{A} : L_2([0,1]) \to L_2([0,1])$ be a mapping defined by $(\mathcal{A}x)(t) = \frac{x(t)}{2}$ for all $x \in L_2([0,1])$. Then we have $(\mathcal{A}^*x)(t) = \frac{x(t)}{2}$ and $\|\mathcal{A}\| = \frac{1}{2}$. We see that the solution set of SFP is nonempty because of $x^*(t) = 0$ is a solution. Let $T : L_2([0,1]) \to L_2([0,1])$ be a mapping defined by

$$(Tx)(t) = \int_0^1 tx(s)ds, \quad t \in [0,1].$$

It is not hard to show that T is nonexpansive (hence it is quasi-nonexpansive) with a fixed point $x^*(t) = 0$. We aim to find an element $x^* \in C$ such that

$$x^* \in \Omega := F(T) \cap \Gamma. \tag{5.3}$$

Hence $\Omega = \{0\}$. Following ([21, Example 1]), we define $Ax = \nabla\left(\frac{1}{2}\|\mathcal{A}x - P_Q\mathcal{A}x\|^2\right) = \mathcal{A}^*(I - P_Q)\mathcal{A}x$ and $Bx = N_C(x)$. Clearly A is L-Lipschitz continuous with $L = \|\mathcal{A}\| = \frac{1}{2}$. In this experiment, we compare our Algorithm 1 and Algorithm 2 with a Halpern-type iteration for the split feasibility problem and the fixed point problem (HSFP) proposed in ([16, Theorem 3.1]), Halpern-type forward

Table 3 Numerical results of Algorithm 1, Algorithm 2, HSFP, HFBA and VMEM for Example 5.3

		Algorithm 1	Algorithm 2	HSFP	HFBA	VMEM
$x_1 = t^2 + 1$	No. of Iter.	2	3	8	12	13
$u = t$	cpu (Time)	0.4613	0.8929	3.3371	1.5090	2.5327
$x_1 = t^2 + t$	No. of Iter.	2	3	8	9	10
$u = t$	cpu (Time)	0.5497	1.0524	7.9545	1.3856	2.3888
$x_1 = 2t^3 + 3t$	No. of Iter.	2	3	8	9	15
$u = t$	cpu (Time)	0.5349	0.9868	8.3606	1.2968	3.1761

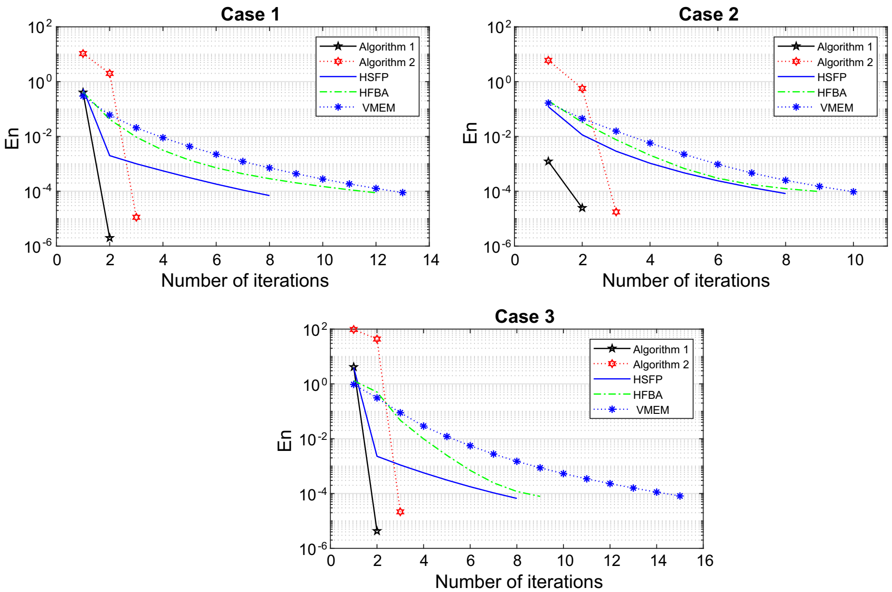

Fig. 3 The error ploting of Algorithm 1, Algorithm 2, HSFP, HFBA and VMEM in Example 5.3

backward algorithm (HFBA) proposed in ([53, Theorem 3.1]) and a viscosity-type modified extragradient method (VMEM) proposed in ([52, Algorithm 3.1]).

We take $\gamma = 0.002$, $l = 0.0001$ and $\mu = 0.03$ in Algorithm 1 and take $\lambda_1 = 3.5$ and $\mu = 0.03$ in Algorithm 2. For both Algorithm 1 and Algorithm 2, we take $\alpha_n = \frac{n}{75(n+1)}$. For HFBA and VMEM, we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$, $r_n = \frac{1}{5}$, $f(x) = 0.2x$ and $\lambda_n = 4$. For HSFP, we take $\lambda_n = 0.001$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$. We use stopping criterion $\|x_{n+1} - x_n\| \leq 10^{-4}$. We perform the numerical experiments with the following three cases of starting point x_1:

Case 1 : $x_1 = t^2 + 1$;
Case 2 : $x_1 = t^2 + t$;
Case 3 : $x_1 = 2t^3 + 3t$.

The numerical results are presented in Table 3 and Fig. 3.

6 Conclusions

In this paper, we have proposed two algorithms with different stepsizes by using Armijo linesearch and self-adaptive stepsize for solving the monotone inclusion problem and the fixed point problem for a relatively nonexpansive mapping in reflexive Banach spaces. The weak convergence theorems of the algorithms have been proved without the computation of the Lipschitz constant of the mapping. An application related to the obtained results has been provided. Finally, several numerical experiments have been performed to illustrate the convergence of the algorithms and compare them with some known algorithms.

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Research article

Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces

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Abstract: In this work, we introduce two modified Tseng's splitting algorithms with a new non-monotone adaptive step size for solving monotone inclusion problem in the framework of Banach spaces. Under some mild assumptions, we establish the weak and strong convergence results of the proposed algorithms. Moreover, we also apply our results to variational inequality problem, convex minimization problem and signal recovery, and provide several numerical experiments including comparisons with other related algorithms.

Keywords: maximal monotone operator; Banach space; strong convergence; self adaptive method
Mathematics Subject Classification: 47H09, 47H10, 47J25

1. Introduction

Let E be a real Banach space with its dual space E^*. In this paper, we study the so-called *monotone inclusion problem*:

$$\text{find } z \in E \text{ such that } 0 \in (A + B)z, \tag{1.1}$$

where $A : E \to E^*$ is a single mapping and $B : E \to 2^{E^*}$ is a multi-valued mapping. The set of solutions of the problem (1.1) is denoted by $(A + B)^{-1}0 := \{x \in E : 0 \in (A + B)x\}$. This problem draws much attention since it stands at the core of many mathematical problems, such as: variational inequalities, split feasibility problem and minimization problem with applications in machine learning, statistical regression, image processing and signal recovery (see [17, 33, 44]).

A classical method for solving the problem (1.1) in Hilbert space H, is known as *forward-backward splitting algorithm* (FBSA) [15, 29] which generates iterative sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = J_\lambda^B (I - \lambda A) x_n, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $J_\lambda^B := (I + \lambda B)^{-1}$ is the resolvent operator of an operator B. Here, I denotes the identity operator on H. It was proved that the sequence generated by (1.2) converges weakly to an element in $(A+B)^{-1}0$ under the assumption of the α-cocoercivity of the operator A, that is,

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in H$$

and λ is chosen in $(0, 2\alpha)$. In fact, FBSA includes, as special cases, the proximal point algorithm (when $A = 0$) [11, 20, 34] and the gradient method [18].

In order to get strong convergence result, Takashashi et al. [41] introduced the following algorithm:

$$\begin{cases} x_1, u \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n), \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where A is an α-cocoercive mapping on H. It was shown that if $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following assumptions:

$$0 < a \leq \lambda_n \leq b < 2\alpha, \ \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence $\{x_n\}$ defined by (1.3) converges strongly to an element in $(A + B)^{-1}0$.

In 2016, Cholamjiak [12] introduced the following FBSA in a uniformly convex and q-uniformly smooth Banach space E:

$$\begin{cases} x_1, u \in E, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{\lambda_n}^B (x_n - \lambda_n A x_n), \ \forall n \geq 1, \end{cases} \tag{1.4}$$

where $J_{\lambda_n}^B := (I + \lambda_n B)^{-1}$ is the resolvent operator of an m-accretive operator B and A is an α-cocoercive mapping. He proved that the sequence generated by (1.4) converges strongly to a solution of the problem (1.1) under the following assumptions:

$$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1) \text{ with } \alpha_n + \beta_n + \gamma_n = 1,$$

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n \to \infty} \gamma_n > 0,$$

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}},$$

where κ_q is the q-uniform smoothness coefficient of E.

In recent years, the FBSA for solving the monotone inclusion problem (1.1), when A is α-cocoercive, was studied and modified by many authors in various settings (see, *e.g.*, [1, 9, 10, 13, 26, 27, 32, 37, 38, 46]). It is important to remark that the α-cocoercivity of the operator A is a strong assumption. To relax this assumption, Tseng [45] introduced the following so-called *Tseng's splitting method*:

$$\begin{cases} x_1 \in H, \\ y_n = J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \quad \forall n \ge 1, \end{cases} \tag{1.5}$$

where A is monotone and L-Lipschitz continuous with $L > 0$. It was proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly to an element in $(A + B)^{-1}0$ provided the step size λ_n is chosen in $\left(0, \frac{1}{L}\right)$. It is worth noting that Tseng's splitting method is a requirement to know Lipschitz constant of the mapping. Unfortunately, Lipschitz constants are often unknown or difficult to approximate.

Very recently, Shehu [37] extended Tseng's result to Banach spaces. He proposed the following iterative process for approximating a solution of the problem (1.1) in a 2-uniformly convex Banach space E which is also uniformly smooth:

$$\begin{cases} x_1 \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = Jy_n - \lambda_n(Ay_n - Ax_n), \quad \forall n \ge 1, \end{cases} \tag{1.6}$$

where $A : E \to E^*$ is monotone and L-Lipschitz continuous, $J^B_{\lambda_n} := (J + \lambda_n B)^{-1} J$ is the resolvent of B and J is the duality mapping from E into E^*. He obtain weak convergence theorem to the solution of the problem (1.1) provided the step size λ_n is chosen in $\left(0, \frac{1}{\sqrt{2\mu\kappa L}}\right)$, where μ is the 2-uniform convexity constant of E and κ is the 2-uniform smoothness constant of E^*. At the same time, he also proposed a variant of (1.6) with a linesearch for solving the problem (1.1). It is known that any algorithm with a linesearch needs an inner loop with some stopping criterion over iteration.

In this paper, motivated by Shehu [37], we propose two modifications of Tseng's splitting method with non-monotone adaptive step sizes for solving the problem (1.1) in the framework of Banach spaces. The step size of our methods does not require the prior knowledge of the Lipschitz constant of operator and without any linesearch procedure. The remainder of this paper is organized as follows: We recall some definitions and lemmas in Section 2. Our methods are presented and analyzed in Section 3. Theoretical applications to variational inequality problem and convex minimization problem are considered in Section 4 and finally, in Section 5, we provide some numerical experiments to illustrate the behaviour of our methods.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real numbers and the set of positive integers, respectively. Let E be a real Banach space with its dual space E^*. We denote $\langle x, f\rangle$ by the value of a functional f in E^* at x in E, that is, $\langle x, f\rangle = f(x)$. For a sequence $\{x_n\}$ in E, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $S_E = \{x \in E : \|x\| = 1\}$. The space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. The space E is said to be *uniformly smooth* if the limit (2.1) converges uniformly in $x, y \in S_E$. It is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. The space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S_E, \|x - y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. Let $p \geq 2$. The space E is said to be *p-uniformly convex* if there is a $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Let $1 < q \leq 2$. The space E is said to be *q-uniformly smooth* if there exists a $\kappa > 0$ such that $\rho_E(t) \leq \kappa t^q$ for all $t > 0$, where ρ_E is the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in S_E\right\}$$

for all $t \geq 0$. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is observed that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [2]). If E is uniformly convex then E is reflexive and strictly convex and if E is uniformly smooth then E is reflexive and smooth (see [14]). Moreover, we know that for every $p > 1$, L_p and ℓ_p are $\min\{p, 2\}$-uniformly smooth and $\max\{p, 2\}$-uniformly convex, while Hilbert space is 2-uniformly smooth and 2-uniformly convex (see [4, 23, 47] for more details).

Definition 2.1. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*.

If E is a Hilbert space, then $J = I$ is the identity mapping on E. It is known that E is smooth if and only if J is single-valued from E into E^* and if E is a reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E. Moreover, if E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded subsets of E (see [2, 14] for more details). A duality mapping J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup^* Jx$.

Lemma 2.2. [39] *Let E be a smooth Banach space and J be the duality mapping on E. Then $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Definition 2.3. A mapping $A : E \to E^*$ is said to be:

• *α-cocoercive* if there exists a constant $\alpha > 0$ such that $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in E$;

• *monotone* if $\langle x - y, Ax - Ay\rangle \geq 0$ for all $x, y \in E$;

• *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $||Ax - Ay|| \leq L||x - y||$ for all $x, y \in E$;

• *hemicontinuous* if for each $x, y \in E$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) = A(tx+(1-t)y)$ is continuous with respect to the weak* topology of E^*.

Remark 2.4. It is easy to see that if A is cocoercive, then A is monotone and Lipschitz continuous but the converse is not true in general.

The next lemma can be found in [49] (see also [47]).

Lemma 2.5. (i) *Let E be a 2-uniformly smooth Banach space. Then there exists a constant $\kappa > 0$ such that*

$$||x - y||^2 \leq ||x||^2 - 2\langle y, Jx\rangle + \kappa||y||^2, \ \forall x, y \in E.$$

(ii) *Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that*

$$||x - y||^2 \geq ||x||^2 - 2\langle y, Jx\rangle + c||y||^2, \ \forall x, y \in E.$$

Remark 2.6. It is well-known that $\kappa = c = 1$ whenever E is a Hilbert space. Hence these inequalities reduce to the following well-known polarization identity:

$$||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2.$$

Moreover, we refer to [49] for the exact values of constants κ and c.

Next, we recall the following Lyapunov function which was introduced in [3]:

Definition 2.7. Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \forall x, y \in E. \tag{2.2}$$

If E is a Hilbert space, then $\phi(x, y) = ||x - y||^2$ for all $x, y \in E$. In addition, the Lyapunov function ϕ has the following properties:

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E. \tag{2.3}$$

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z), \ \forall x, y, z \in E, \ \alpha \in [0, 1]. \tag{2.4}$$

$$\phi(x, y) = \phi(x, z) - \phi(y, z) + 2\langle y - x, Jy - Jz\rangle, \ \forall x, y, z \in E. \tag{2.5}$$

Lemma 2.8. [6] *Let E be a 2-uniformly convex Banach space, then there exists a constant $c > 0$ such that*

$$c||x - y||^2 \leq \phi(x, y),$$

where c is a constant in Lemma 2.5 (ii).

We make use of the following functional $V : E \times E^* \to \mathbb{R}$ studied in [3]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,\ \forall x \in E,\ x^* \in E^*. \tag{2.6}$$

Obviously, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.9. [3] *Let E be a reflexive, strictly convex and smooth Banach space. Then the following statement holds:*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*),\ \forall x \in E,\ x^*, y^* \in E^*.$$

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed and convex subset of E. Then for any $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

Such a mapping $\Pi_C : E \to C$ defined by $z = \Pi_C(x)$ is called the *generalized projection* of E onto C. If E is a Hilbert space, then Π_C is coincident with the metric projection denoted by P_C.

Lemma 2.10. [3] *Let E be a reflexive, strictly convex and smooth Banach space and C be a closed and convex subset of E. Let $x \in E$ and $z \in C$. Then the following statements hold:*

(i) $z = \Pi_C(x)$ *if and only if* $\langle y - z, Jx - Jz\rangle \le 0$, $\forall y \in C$.
(ii) $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \le \phi(y, x)$, $\forall y \in C$.

Lemma 2.11. [25] *Let C be a closed and convex subset of a smooth and uniformly convex Banach space E. Let $\{x_n\}$ be a sequence in E such that $\phi(p, x_{n+1}) \le \phi(p, x_n)$ for all $p \in C$ and $n \ge 1$. Then the sequence $\{\Pi_C(x_n)\}$ converges strongly to some element $x^* \in C$.*

Let $B : E \to 2^{E^*}$ be a multi-valued mapping. The effective domain of B is denoted by $D(B) = \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $R(B) = \bigcup\{Bx : x \in D(B)\}$. The set of zeros of B is denoted by $B^{-1}0 = \{x \in D(B) : 0 \in Bx\}$. A multi-valued mapping B is said to be *monotone* if

$$\langle x - y, u - v\rangle \ge 0,\ \ \forall x, y \in D(B),\ u \in Bx \text{ and } v \in By.$$

A monotone operator B on E is said to be *maximal* if its graph $G(B) = \{(x, y) \in E \times E^* : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone operator on E. In other words, the maximality of B is equivalent to $R(J + \lambda B) = E^*$ for all $\lambda > 0$ (see [5, Theorem 1.2]). It is known that if B is maximal monotone, then $B^{-1}0$ is closed and convex (see [39]).

For a maximal monotone operator B, we define the resolvent of B by $J_\lambda^B(x) = (J + \lambda B)^{-1}Jx$ for $x \in E$ and $\lambda > 0$. It is also known that $B^{-1}0 = F(J_\lambda^B)$.

Lemma 2.12. [5] *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded operator and $B : E \to 2^{E^*}$ be a maximal monotone operator. Then $A + B$ is maximal monotone.*

Lemma 2.13. ([48]) *Assume that $\{a_n\}$ is a sequence of nonnegative real sequences such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n,\ \forall n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence of real sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14. ([30]) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $\ell \in \mathbb{N}$*. Define the sequence* $\{\sigma(n)\}$ *of integers as follows:*

$$\sigma(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

for all $n \geq n_0$ *(for some* n_0 *large enough). Then* $\{\sigma(n)\}_{n\geq n_0}$ *is a non-decreasing sequence such that* $\lim_{n\to\infty} \sigma(n) = \infty$*, and it holds that*

$$\Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\sigma(n)+1}.$$

Lemma 2.15. ([42]) *Assume that* $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \theta_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \theta_n < \infty$*, then* $\lim\limits_{n\to\infty} \lambda_n$ *exists.*

3. Main results

In this section, we introduce two modified Tseng's splitting algorithms for solving the monotone inclusion problem in Banach spaces. In order to prove the convergence results of these algorithms, we need make the following assumptions:

Assumption 3.1. *(A1) The Banach space* E *is a real* 2*-uniformly convex and uniformly smooth.*
(A2) The mappings $A : E \to E^*$ *is monotone and L-Lipschitz continuous, and* $B : E \to 2^{E^*}$ *is maximal monotone.*
(A3) The solution set of the problem (1.1) *is nonempty, that is,* $(A + B)^{-1}0 \neq \emptyset$.

Algorithm 1: Tseng type splitting algorithm for monotone inclusion problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n). \tag{3.1}$$

If $x_n = y_n$, then stop and x_n is a solution of the problem (1.1). Otherwise, go to **Step 2**.

Step 2. Compute

$$x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \tag{3.2}$$

where the sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.2. *Assume that Assumption 3.1 holds. Let $\{x_n\}$, $\{y_n\}$ and $\{\lambda_n\}$ be sequences generated by Algorithm 1. Then the following statements hold:*

(*i*) *If $x_n = y_n$ for all $n \in \mathbb{N}$, then $x_n \in (A + B)^{-1}0$.*

(*ii*) $\lim\limits_{n\to\infty} \lambda_n = \lambda \in \left[\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \theta\right]$, *where $\theta = \sum_{n=1}^{\infty} \theta_n$. Moreover*

$$\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|x_n - y_n\|, \ \forall n \geq 1.$$

Proof. (*i*) If $x_n = y_n$, then $x_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n)$. It follows that $x_n = (J + \lambda_n B)^{-1} J \circ J^{-1}(J - \lambda_n A)x_n$, that is, $Jx_n - \lambda_n Ax_n \in Jx_n + \lambda_n Bx_n$, which implies that $0 \in (A + B)x_n$. Hence $x_n \in (A + B)^{-1}0$.

(*ii*) In the case $Ax_n - Ay_n \neq 0$, using the Lipschitz continuity of A, we have

$$\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

From (3.3) and mathematical induction, we have the sequence $\{\lambda_n\}$ has upper bound $\lambda_1 + \theta$ and lower bound $\min\{\frac{\mu}{L}, \lambda_1\}$. From Lemma 2.15, we have $\lim\limits_{n\to\infty} \lambda_n$ exists and we denote $\lambda = \lim\limits_{n\to\infty} \lambda_n$. It is obvious that $\lambda \in \left[\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \theta\right]$. By the definition of λ_n, we have

$$\lambda_{n+1} = \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} \leq \frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}.$$

This implies that

$$\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|x_n - y_n\|, \ \forall n \geq 1. \tag{3.4}$$

$\square$

Lemma 3.3. *Assume that Assumption 3.1 holds. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Hence*

$$\phi(z, x_{n+1}) \leq \phi(z, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, x_n), \quad \forall z \in (A + B)^{-1}0, \tag{3.5}$$

where c and κ are constants in Lemma 2.5.

Proof. Let $z \in (A + B)^{-1}0$. From Lemma 2.5 (i) and (2.5), we have

$$\begin{aligned}
\phi(z, x_{n+1}) &= \phi(z, J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n))) \\
&= V(z, Jy_n - \lambda_n(Ay_n - Ax_n)) \\
&= \|z\|^2 - 2\langle z, Jy_n - \lambda_n(Ay_n - Ax_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Ax_n)\|^2 \\
&\leq \|z\|^2 - 2\langle z, Jy_n\rangle + 2\lambda_n\langle z, Ay_n - Ax_n\rangle + \|Jy_n\|^2 - 2\lambda_n\langle y_n, Ay_n - Ax_n\rangle + \kappa\|\lambda_n(Ay_n - Ax_n)\|^2 \\
&= \|z\|^2 - 2\langle z, Jy_n\rangle + \|y_n\|^2 - 2\lambda_n\langle y_n - z, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(z, y_n) - 2\lambda_n\langle y_n - z, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(z, x_n) - \phi(y_n, x_n) + 2\langle y_n - z, Jy_n - Jx_n\rangle - 2\lambda_n\langle y_n - z, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(z, x_n) - \phi(y_n, x_n) + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 - 2\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.6)
\end{aligned}$$

Combining (3.4) and (3.6), we have

$$\begin{aligned}
\phi(z, x_{n+1}) \leq\ & \phi(z, x_n) - \phi(y_n, x_n) + \kappa\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - x_n\|^2 \\
& -2\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.7)
\end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned}
\phi(z, x_{n+1}) \leq\ & \phi(z, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, x_n) \\
& -2\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.8)
\end{aligned}$$

Now, we will show that

$$\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0.$$

From the definition of y_n, we note that $Jx_n - \lambda_n Ax_n \in Jy_n + \lambda_n By_n$. Since B is maximal monotone, there exists $v_n \in By_n$ such that $Jx_n - \lambda_n Ax_n = Jy_n + \lambda_n v_n$, we have

$$v_n = \frac{1}{\lambda_n}(Jx_n - Jy_n - \lambda_n Ax_n). \tag{3.9}$$

Since $0 \in (A + B)z$ and $Ay_n + v_n \in (A + B)y_n$, it follows from Lemma 2.12 that $A + B$ is maximal monotone. Hence

$$\langle y_n - z, Ay_n + v_n\rangle \geq 0. \tag{3.10}$$

Substituting (3.9) into (3.10), we have

$$\frac{1}{\lambda_n}\langle y_n - z, Jx_n - Jy_n - \lambda_n Ax_n + \lambda_n Ay_n\rangle \geq 0.$$

Hence

$$\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0. \tag{3.11}$$

Combining (3.8) and (3.11), thus this lemma is proved. □

Theorem 3.4. *Assume that Assumption 3.1 holds. Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to an element in $(A+B)^{-1}0$.*

Proof. Since $\lim_{n\to\infty}\lambda_n$ exists and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$, it follows that $\lim_{n\to\infty}\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \frac{\kappa\mu^2}{c} > 0$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \geq n_0. \tag{3.12}$$

Combining (3.5) and (3.12), we have

$$\phi(z, x_{n+1}) \leq \phi(z, x_n), \quad \forall n \geq n_0.$$

This show that $\lim_{n\to\infty}\phi(z, x_n)$ exists and hence $\{\phi(z, x_n)\}$ is bounded. Applying Lemma 2.8, we also have $\{x_n\}$ is bounded. From (3.5), we have

$$\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, x_n) \leq \phi(z, x_n) - \phi(z, x_{n+1}). \tag{3.13}$$

Thus we have

$$\lim_{n\to\infty}\phi(y_n, x_n) = 0.$$

Applying Lemma 2.8, we also have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.14}$$

Since J is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.15}$$

Using the fact that A is Lipschitz continuous, we have

$$\lim_{n\to\infty}\|Ax_n - Ay_n\| = 0. \tag{3.16}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in E$. From (3.14), we have $y_{n_k} \rightharpoonup x^*$. We will show that $x^* \in (A+B)^{-1}0$. Let $(v, w) \in G(A+B)$, we have $w - Av \in Bv$. From the definition of y_{n_k}, we note that

$$Jx_{n_k} - \lambda_{n_k} Ax_{n_k} \in Jy_{n_k} + \lambda_{n_k} By_{n_k},$$

which implies that

$$\frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \in By_{n_k}.$$

By the maximal monotonicity of B, we have

$$\left\langle v - y_{n_k}, w - Av - \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \right\rangle \geq 0$$

and by the monotonicity of A, we have

$$\begin{aligned}
\langle v - y_{n_k}, w \rangle &\geq \left\langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \right\rangle \\
&= \langle v - y_{n_k}, Av - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k} \rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle.
\end{aligned}$$

Since $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$ and $y_{n_k} \rightharpoonup x^*$, it follows from (3.15) and (3.16) that

$$\langle v - x^*, w \rangle \geq 0.$$

By the monotonicity of $A+B$, we get $0 \in (A+B)x^*$, that is, $x^* \in (A+B)^{-1}0$. Hence $x^* \in (A+B)^{-1}0$. Note that $(A+B)^{-1}0$ is closed and convex. Put $u_n = \Pi_{(A+B)^{-1}0}(x_n)$. It follows from Lemma 2.11 that there exists $x^* \in (A+B)^{-1}0$ such that $u_n \to x^*$. Finally, we show that $x_n \rightharpoonup x^*$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in (A+B)^{-1}0$. Then we have

$$\langle \hat{x} - u_{n_k}, Jx_{n_k} - Ju_{n_k} \rangle \leq 0$$

for all $k \in \mathbb{N}$. Since $u_n \to x^*$, $x_{n_k} \rightharpoonup \hat{x}$ and J is weakly sequentially continuous, we have

$$\langle \hat{x} - x^*, J\hat{x} - Jx^* \rangle \leq 0.$$

By the strict monotonicity of J, we obtain $x^* = \hat{x}$. In summary, we have shown that every subsequence of $\{x_n\}$ has a further subsequence which converges weakly to x^*. We conclude that $x_n \rightharpoonup x^* = \lim_{n\to\infty} \Pi_{(A+B)^{-1}0}(x_n)$. This completes the proof. □

Algorithm 2: Halpern-Tseng type splitting algorithm for monotone inclusion problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.
Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n). \tag{3.17}$$

If $x_n = y_n$, then stop and x_n is a solution of the problem (1.1). Otherwise, go to **Step 2**.
Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.18}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n), \tag{3.19}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{3.20}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 3.5. *Assume that Assumption 3.1 holds. If $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $x^* \in (A + B)^{-1}0$.*

Proof. We will show that $\{x_n\}$ is bounded. Let $z \in (A + B)^{-1}0$. Using the same arguments as in the proof of Lemma 3.3, we can show that

$$\phi(z, z_n) \leq \phi(z, x_n) - \left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, x_n). \tag{3.21}$$

Since $\lim_{n\to\infty} \lambda_n$ exists and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$, it follows that $\lim_{n\to\infty}\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \frac{\kappa\mu^2}{c} > 0$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \geq n_0. \tag{3.22}$$

Combining (3.21) and (3.22), we have

$$\phi(z, z_n) \leq \phi(z, x_n), \quad \forall n \geq n_0. \tag{3.23}$$

By (2.4), we have

$$\phi(z, x_{n+1}) \;=\; \phi(z, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n))$$

$$
\begin{aligned}
&\leq \alpha_n\phi(z,u) + (1-\alpha_n)\phi(z,z_n) \\
&\leq \alpha_n\phi(z,u) + (1-\alpha_n)\phi(z,x_n) \\
&\leq \max\{\phi(z,u), \phi(z,x_n)\} \\
&\vdots \\
&\leq \max\{\phi(z,u), \phi(z,x_{n_0})\}.
\end{aligned}
$$

This implies that $\{\phi(z,x_n)\}$ is bounded. Applying Lemma 2.8, we also have $\{x_n\}$ is bounded.

Let $x^* = \Pi_{(A+B)^{-1}0}(u)$. From (3.21), we have

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n)) \\
&\leq \alpha_n\phi(x^*,u) + (1-\alpha_n)\phi(x^*,z_n) \\
&\leq \alpha_n\phi(x^*,u) + (1-\alpha_n)\phi(x^*,x_n) - (1-\alpha_n)\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,x_n).
\end{aligned}
$$

This implies that

$$
(1-\alpha_n)\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,x_n) \leq \phi(x^*,x_n) - \phi(x^*,x_{n+1}) + \alpha_n K, \tag{3.24}
$$

where $K = \sup_{n\in\mathbb{N}}\{|\phi(x^*,u) - \phi(x^*,x_n)|\}$.

Now, we will divide the rest of the proof into two cases.

Case 1. Suppose that there exists $N \in \mathbb{N}$ such that $\phi(x^*,x_{n+1}) \leq \phi(x^*,x_n)$ for all $n \geq N$. Hence $\lim_{n\to\infty}\phi(x^*,x_n)$ exists. By our assumptions, we have from (3.24) that

$$
\lim_{n\to\infty}\phi(y_n,x_n) = 0
$$

and hence

$$
\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.25}
$$

Since J is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$
\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.26}
$$

Using the fact that A is Lipschitz continuous, we have

$$
\lim_{n\to\infty}\|Ax_n - Ay_n\| = 0.
$$

Then from (3.18), we have

$$
\|Jz_n - Jy_n\| = \lambda_n\|Ax_n - Ay_n\| \to 0. \tag{3.27}
$$

Moreover from (3.26) and (3.27), we obtain

$$
\|Jx_{n+1} - Jx_n\| \quad \leq \quad \|Jx_{n+1} - Jz_n\| + \|Jz_n - Jy_n\| + \|Jy_n - Jx_n\|
$$

$$\begin{aligned} &= \alpha_n\|Ju - Jz_n\| + \|Jz_n - Jy_n\| + \|Jy_n - Jx_n\| \\ &\to 0. \end{aligned}$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subset of E^*, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.28}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ and

$$\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle = \lim_{k\to\infty}\langle x_{n_k} - x^*, Ju - Jx^*\rangle,$$

where $x^* = \Pi_{(A+B)^{-1}0}(u)$. By a similar argument to that of Theorem 3.4, we can show that $\hat{x} \in (A+B)^{-1}0$. Thus we have

$$\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle = \langle \hat{x} - x^*, Ju - Jx^*\rangle \le 0.$$

From (3.28), we also have

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, Ju - Jx^*\rangle \le 0. \tag{3.29}$$

Finally, we show that $x_n \to x^*$. From Lemma 2.9, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n)) \\ &= V(x^*, \alpha_n Ju + (1-\alpha_n)Jz_n) \\ &\le V(x^*, \alpha_n Ju + (1-\alpha_n)Jz_n - \alpha_n(Ju - Jx^*)) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\ &= V(x^*, \alpha_n Jx^* + (1-\alpha_n)Jz_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\ &= \alpha_n\phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, z_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\ &\le (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle. \end{aligned} \tag{3.30}$$

This together with (3.29) and (3.30), so we can conclude by Lemma 2.13 that $\phi(x^*, x_n) \to 0$. Therefore $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}$$

for all $n \ge n_0$ (for some n_0 large enough). From Lemma 2.14, we have $\sigma(n)$ is non-decreasing such that $\lim_{n\to\infty}\sigma(n) = \infty$ and the following inequalities hold for all $n \ge n_0$:

$$\Gamma_{\sigma(n)} < \Gamma_{\sigma(n)+1} \ \text{ and } \ \Gamma_n \le \Gamma_{\sigma(n)+1}. \tag{3.31}$$

Put $\Gamma_n = \phi(x^*, x_n)$ for all $n \in \mathbb{N}$. As proved in the **Case 1**, we obtain

$$(1-\alpha_{\sigma(n)})\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda^2_{\sigma(n)}}{\lambda^2_{\sigma(n)+1}}\Big)\phi(y_{\sigma(n)}, x_{\sigma(n)}) \le \phi(x^*, x_{\sigma(n)}) - \phi(x^*, x_{\sigma(n)+1}) + \alpha_{\sigma(n)}K$$

$$\leq \alpha_{\sigma(n)} K,$$

where $K = \sup_{n\in\mathbb{N}}\{|\phi(x^*, u) - \phi(x^*, x_{\sigma(n)})|\}$. By our assumptions, we have

$$\lim_{n\to\infty} \phi(y_{\sigma(n)}, x_{\sigma(n)}) = 0$$

and hence

$$\lim_{n\to\infty} \|x_{\sigma(n)} - y_{\sigma(n)}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we can show that

$$\lim_{n\to\infty} \|x_{\sigma(n)+1} - x_{\sigma(n)}\| = 0$$

and

$$\limsup_{n\to\infty} \langle x_{\sigma(n)+1} - x^*, Ju - Jx^* \rangle \leq 0.$$

From (3.30) and (3.31), we have

$$\begin{aligned} \phi(x^*, x_{\sigma(n)+1}) &\leq (1 - \alpha_{\sigma(n)})\phi(x^*, x_{\sigma(n)}) + \alpha_{\sigma(n)}\langle x_{\sigma(n)+1} - x^*, Ju - Jx^* \rangle \\ &\leq (1 - \alpha_{\sigma(n)})\phi(x^*, x_{\sigma(n)+1}) + \alpha_{\sigma(n)}\langle x_{\sigma(n)+1} - x^*, Ju - Jx^* \rangle. \end{aligned}$$

This implies that

$$\phi(x^*, x_n) \leq \phi(x^*, x_{\sigma(n)+1}) \leq \langle x_{\sigma(n)+1} - x^*, Ju - Jx^* \rangle.$$

Hence $\limsup_{n\to\infty} \phi(x^*, x_n) = 0$ and so $\lim_{n\to\infty} \phi(x^*, x_n) = 0$. Therefore $x_n \to x^*$. This completes the proof. □

4. Theoretical applications

4.1. The case of variational inequality problem

Let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a mapping. The *variational inequality problem* is to find $x^* \in C$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of the problem (4.1) is denoted by $VI(C, A)$. In particular, if A is a continuous and monotone mapping, then $VI(C, A)$ is closed and convex (see [7, 24]). Recall that the indicator function of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is known that i_C is proper convex, lower semicontinuous and convex function with its subdifferential ∂i_C is maximal monotone (see [35]). From [2], we know that

$$\partial i_C(v) = N_C(v) = \{u \in E^* : \langle y - v, u \rangle \leq 0, \ \forall y \in C\},$$

where N_C is the normal cone for C at a point v. Thus we can define the resolvent of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (J + \lambda \partial i_C)^{-1} Jx, \quad \forall x \in E.$$

As shown in [40], for any $x \in E$ and $z \in C$, $z = J_\lambda^{\partial i_C}(x) \iff z = \Pi_C(x)$, where Π_C is the generalized projection from E onto C.

Lemma 4.1. [36] *Let C be a nonempty, closed convex subset of a Banach space E. Let $A : C \to E^*$ be a monotone and hemicontinuous operator and $T : E \to 2^{E^*}$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v) & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

If we set $B = \partial i_C$, then we obtain the following results regarding the problem (4.1).

Assumption 4.2. *(A1) The feasible set C is a nonempty, closed and convex subset of a real 2-uniformly convex and uniformly smooth Banach space E.*
(A2) The mapping $A : E \to E^*$ is monotone and L-Lipschitz continuous.
(A3) The solution set of the problem (4.1) is nonempty, that is, $VI(C, A) \neq \emptyset$.

Algorithm 3: Tseng type splitting algorithm for variational inequality problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $x_1 \in C$ be arbitrary. Set $n = 1$.
Step 1. Compute

$$y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n). \tag{4.2}$$

Step 2. Compute

$$x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \tag{4.3}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.4}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.3. *Assume that Assumption 4.2 holds. Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges weakly to an element in $(A + B)^{-1}0$.*

Algorithm 4: Halpern-Tseng type splitting algorithm for variational inequality problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $u, x_1 \in C$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n). \tag{4.5}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{4.6}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n), \tag{4.7}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.8}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.4. *Assume that Assumption 4.2 holds. If* $\{\alpha_n\} \subset (0, 1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by Algorithm 4 converges strongly to* $x^* \in VI(C, A)$.

4.2. *The case of convex minimization problem*

Let $f : E \to \mathbb{R}$ be a convex function and $g : E \to \mathbb{R}$ be a convex, lower semicontinuous and non-smooth function. We consider the following *convex minimization problem*:

$$\min_{x\in E} f(x) + g(x). \tag{4.9}$$

By Fermat's rule, we know that above problem is equivalent to the problem of finding $x \in E$ such that

$$0 \in \nabla f(x) + \partial g(x), \tag{4.10}$$

where ∇f is the gradient of f and ∂g is the subdifferential of g. In this situation, we assume that f is a convex and differentiable function with its gradient ∇f is L-Lipschitz continuous. Further, ∇f is cocoercive with a constant $1/L$ (see [31, Theorem 3.13]). This implies that ∇f is monotone and Lipschitz continuous. Moreover, ∂g is maximal monotone (see [35, Theorem A]). In this point of view, we set $A = \nabla f$ and $B = \partial g$, then we obtain the following results regarding the problem (4.9).

Assumption 4.5. *(A1) The Banach space E is real 2-uniformly convex and uniformly smooth Banach space.*

(A2) The functions $f : E \to \mathbb{R}$ is convex and differentiable and its gradient ∇f is L-Lipschitz continuous and $g : E \to \mathbb{R}$ is convex and lower semicontinuous which $f + g$ attains a minimizer.

Algorithm 5: Tseng type splitting algorithm for convex minimization problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^{\partial g} J^{-1}(Jx_n - \lambda_n \nabla f(x_n)). \tag{4.11}$$

Step 2. Compute

$$x_{n+1} = J^{-1}(Jy_n - \lambda_n(\nabla f(y_n) - \nabla f(x_n))), \tag{4.12}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|\nabla f(y_n) - \nabla f(x_n)\|}, \lambda_n + \theta_n\right\} & \text{if } \nabla f(y_n) - \nabla f(x_n) \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.13}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.6. *Assume that Assumption 4.5 holds. Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 5 converges weakly to a minimizer of $f + g$.*

Algorithm 6: Halpern-Tseng type splitting algorithm for convex minimization problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^{\partial g} J^{-1}(Jx_n - \lambda_n \nabla f(x_n)). \tag{4.14}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(\nabla f(y_n) - \nabla f(x_n))). \tag{4.15}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n), \tag{4.16}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|\nabla f(y_n) - \nabla f(x_n)\|}, \lambda_n + \theta_n\right\} & \text{if } \nabla f(y_n) - \nabla f(x_n) \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.17}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.7. *Assume that Assumption 4.5 holds. If* $\{\alpha_n\} \subset (0,1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by Algorithm 6 converges strongly to a minimizer of* $f + g$.

5. Numerical experiments

In this section, we provide some numerical experiments to illustrate the behaviour of our methods and compare them with some existing methods.

Example 5.1. *We consider the HpHard problem which is taken from [22]. Let* $A : \mathbb{R}^m \to \mathbb{R}^m$ *be an operator defined by* $Ax = Mx + q$ *with* $q \in \mathbb{R}^m$ *and*

$$M = NN^T + S + D,$$

where N *is an* $m \times m$ *matrix,* S *is an* $m \times m$ *skew-symmetric matrix and* D *is an* $m \times m$ *positive definite diagonal matrix. The feasible set is* $C = \mathbb{R}^+_m$. *It is clear that* A *is monotone and Lipschitz continuous with* $L = \|M\|$. *In this experiments, we compare our Algorithm 3 and Algorithm 4 with the extragradient method (EGM) proposed in [28] and the subgradient extragradient method (SEGM) proposed in [8]. The parameters are chosen as follows:*

- *Algorithm 3:* $\lambda_1 = 0.4/\|M\|$ *and* $\mu = 0.9$;
- *Algorithm 4:* $\lambda_1 = 0.4/\|M\|$, $\mu = 0.9$, $\alpha_n = \frac{1}{10000(n+2)}$ *and* $u = x_1$;
- *EGM and SEGM:* $\lambda = 0.4/\|M\|$.

All entries of N *and* S *are generated randomly in* $(-5, 5)$, *of* D *are in* $(0, 0.3)$, *of* q *uniformly generated from* $(-500, 0)$. *For every* m, *we have generated two random samples with different choices of* M *and* q. *We perform the numerical experiments with three different cases of* m ($m = 100, 500, 1000$). *We take the starting point* $x_1 = (1, 1, 1, \ldots, 1)^T \in \mathbb{R}^m$ *and use stopping criterion* $\|x_n - y_n\| \leq \varepsilon = 10^{-6}$. *The numerical results are reported in Table 1.*

Table 1. Numerical results for Example 5.1.

m	Algorithm 3 ($\theta_n = 0$)		Algorithm 3 ($\theta_n = 100/n^{1.1}$)		Algorithm 4 ($\theta_n = 0$)		Algorithm 4 ($\theta_n = 100/n^{1.1}$)		EGM		SEGM	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
100	2454	0.02	1162	0.01	35112	1.31	25204	0.65	2454	0.03	2454	0.04
	1920	0.04	917	0.02	35072	1.48	25203	0.66	1920	0.03	1920	0.05
500	2275	0.95	1104	0.29	35010	7.28	25201	5.12	2275	0.50	2275	0.65
	2291	0.93	1107	0.43	34989	7.20	25198	5.06	2291	0.47	2291	0.59
1000	2027	8.08	996	4.25	34993	113.2	25200	78.2	2027	7.83	2027	7.96
	2017	7.80	987	3.87	35003	109.8	25200	78.0	2017	7.01	2017	7.16

Example 5.2. *We consider the problem* (4.1) *in* $L_2([0, 2\pi])$ *with the inner product* $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt$ *and the norm* $\|x\| = \left(\int_0^{2\pi} x^2(t)dt\right)^{1/2}$ *for all* $x, y \in L_2([0, 2\pi])$. *Let* $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ *be an operator defined by*

$$(Ax)(t) = \frac{1}{2}\max\{0, x(t)\}$$

for all $x \in L_2([0,2\pi])$ and $t \in [0,2\pi]$. It can be easily verified that A is monotone and Lipschitz continuous with $L = 1$ (see [50, 51]). The feasible set is $C = \{x \in L_2([0,2\pi]) : \int_0^{2\pi}(t^2+1)x(t)dt \leq 1\}$. Observe that $0 \in VI(C,A)$ and so $VI(C,A) \neq \emptyset$. In this numerical experiment, we take all parameters α_n, λ_n and μ are the same as in Example 5.1. We perform numerical experiments with three different cases of starting point x_1 and use stopping criterion $\|x_n - y_n\| \leq \varepsilon = 10^{-3}$. The numerical results are reported in Table 2.

Table 2. Numerical results for Example 5.2.

x_1	Algorithm 3 ($\theta_n = 0$)		Algorithm 3 ($\theta_n = 0.001/(1.01)^n$)		Algorithm 4 ($\theta_n = 0$)		Algorithm 4 ($\theta_n = 0.001/(1.01)^n$)	
	iter.	time	iter.	time	iter.	time	iter.	time
$\frac{1}{100}\sin(t)$	7	9.9	7	8.9	7	9.8	7	10.1
$\frac{1}{3}t^2e^{-4t}$	5	0.4	5	0.3	5	0.3	5	0.3
$\frac{1}{70}(1-t^2)$	6	3.2	6	2.5	6	2.7	6	2.7

Example 5.3. *Consider the minimization problem:*

$$\min_{x\in\mathbb{R}^3} \|x\|_1 + 2\|x\|_2^2 + (-1,2,5)x + 1,$$

where $x = (w_1,w_2,w_3)^T \in \mathbb{R}^3$. Let $f(x) = 2\|x\|_2^2 + (-1,2,5)x + 1$ and $g(x) = \|x\|_1$. Thus we have $\nabla f(x) = 4x + (-1,2,5)^T$. It is easy to check that f is a convex and differentiable function and its gradient ∇f is Lipschitz continuous with $L = 4$. Moreover, g is a convex and lower semicontinuous function but not differentiable on $\mathbb{R}^3$. From [21], we know that

$$\begin{aligned} J_\lambda^{\partial g}(x) &= (I+\lambda\partial g)^{-1}(x) \\ &= \left(\max\{|w_1|-\lambda,0\}sgn(w_1), \max\{|w_2|-\lambda,0\}sgn(w_2), \max\{|w_3|-\lambda,0\}sgn(w_3)\right)^T \end{aligned}$$

for $\lambda > 0$. In this experiments, we compare our Algorithm 5 and Algorithm 6 with Algorithm (1.4) of Cholamjiak [12]. The parameters are chosen as follows:

- *Algorithm 5: $\lambda_1 = 0.1$ and $\mu = 0.9$;*
- *Algorithm 6: $\lambda_1 = 0.1$, $\mu = 0.9$, $\alpha_n = \frac{1}{10000(n+1)}$ and $u = x_1$;*
- *Algorithm (1.4): all parameters α_n, λ_n, δ_n, r_n and e_n are the same as Example 4.2 in [12], and $u = x_1$.*

We perform the numerical experiments with four different cases of starting point x_1 and use stopping criterion $\|x_{n+1} - x_n\| \leq \varepsilon = 10^{-12}$. The numerical results are reported in Table 3.

Table 3. Numerical results for Example 5.3.

x_1	Algorithm 5 ($\theta_n = 0$)		Algorithm 5 ($\theta_n = 100/n^{1.1}$)		Algorithm 6 ($\theta_n = 0$)		Algorithm 6 ($\theta_n = 100/n^{1.1}$)		Algorithm (1.4)	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
$(1, 2, 4)^T$	101	0.003	284	0.003	27818	0.10	25263	0.08	263957	0.33
$(1, -7, 3)^T$	103	0.002	288	0.003	27809	0.12	25264	0.08	314417	0.38
$(-100, 100, 50)^T$	111	0.004	315	0.004	27802	0.11	25252	0.09	1313442	1.58
$(-1000, -5000, -800)^T$	127	0.005	356	0.01	27787	0.11	25241	0.07	8004199	9.4

Example 5.4. *In signal processing, compressed sensing can be modeled as the following under-determinated linear equation system:*

$$y = Dx + \varepsilon, \tag{5.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $D : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator. It is known that to solve (5.1) *can be seen as solving the LASSO problem:*

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Dx - y\|_2^2 + \lambda\|x\|_1, \tag{5.2}$$

where $\lambda > 0$. Following [19], we define $Ax := \nabla\left(\frac{1}{2}\|Dx - y\|_2^2\right) = D^T(Dx - y)$ and $Bx := \partial(\lambda\|x\|_1)$. It is known that A is $\|D\|^2$-Lipschitz continuous and monotone. Moreover, B is maximal monotone (see [35]).

In this experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $D \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio (SNR)=40. The restoration accuracy is measured by the mean squared error (MSE) as follows:

$$E_n = \frac{1}{N}\|x_n - x\|_2^2 < 10^{-4}, \tag{5.3}$$

where x_n is an estimated signal of x.

We compare our proposed Algorithm 1 and Algorithm 2 with the forward-backward splitting algorithm (FBSA) (1.2), the Tseng's splitting algorithm (TSA) (1.5) and the contraction forward-backward splitting algorithm (CFBSA) proposed in ([43, Algorithm 3.1]). The parameters are chosen as follows:

- Algorithm 1: $\theta_n = 0$, $\lambda_1 = 0.0013$ and $\mu = 0.5$;
- Algorithm 2: $\theta_n = 0$, $\lambda_1 = 0.0013$, $\mu = 0.5$, $\alpha_n = \frac{1}{200n+5}$ and $u = (1, 1, \ldots, 1)^T$;
- CFBSA: $\alpha_n = \frac{1}{200n+5}$, $\mu = 0.5$, $\delta = 0.5$, $l = 0.5$, $\gamma = 0.45$ and $f(x) = \frac{x}{5}$;
- TSA: $\lambda_n = \frac{0.2}{\|D\|^2}$;
- FBSA: $\lambda = 2 \times 10^{-5}$.

The starting points x_1 of all methods are randomly chosen in $\mathbb{R}^N$. We perform the numerical test with the following three cases:

Case 1: $N = 512$, $M = 256$ and $m = 20$;
Case 2: $N = 1024$, $M = 512$ and $m = 30$;
Case 3: $N = 2048$, $M = 1024$ and $m = 60$;
The numerical results for all test are reported in Figures 1–6.

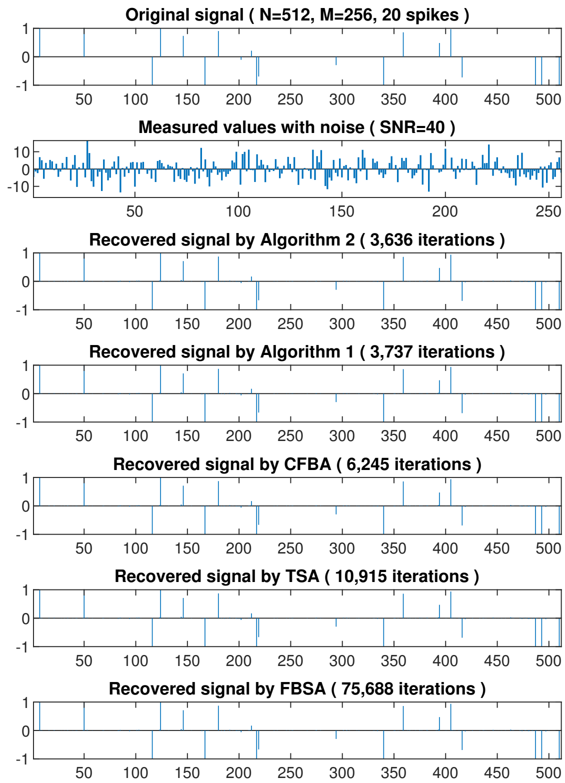

Figure 1. Comparison of recovered signal by using different algorithms in Case 1.

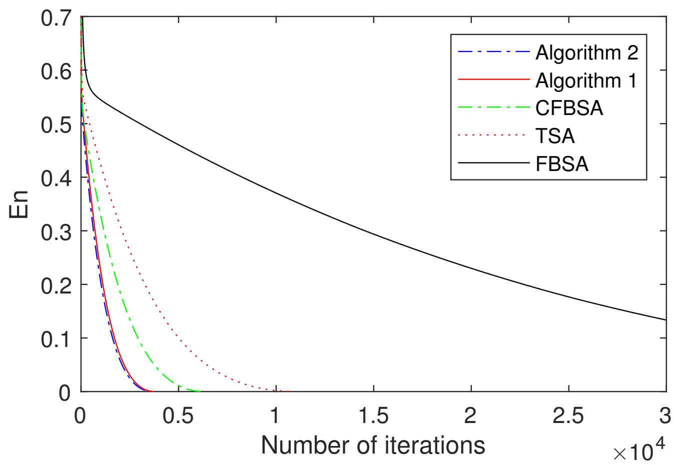

Figure 2. The plotting of MSE versus number of iterations in Case 1.

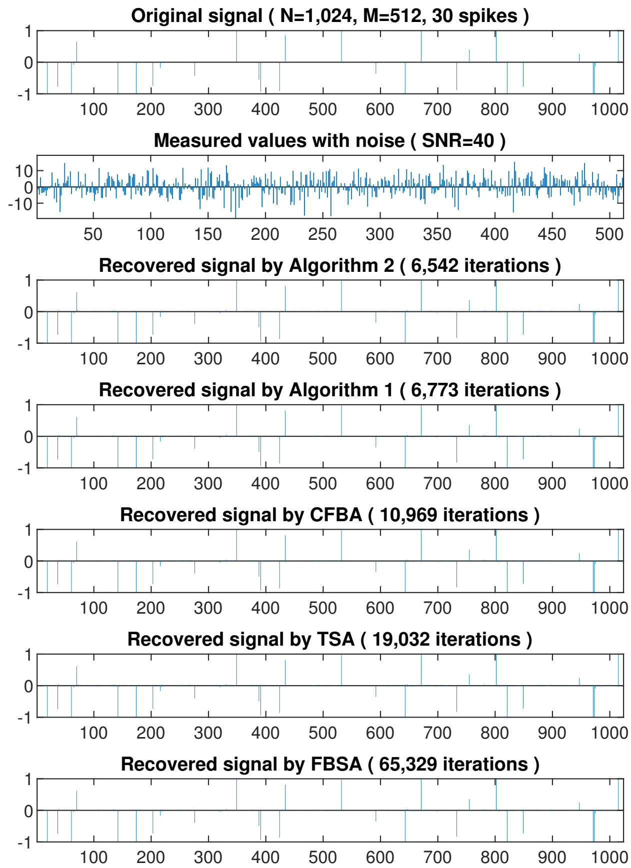

Figure 3. Comparison of recovered signal by using different algorithms in Case 2.

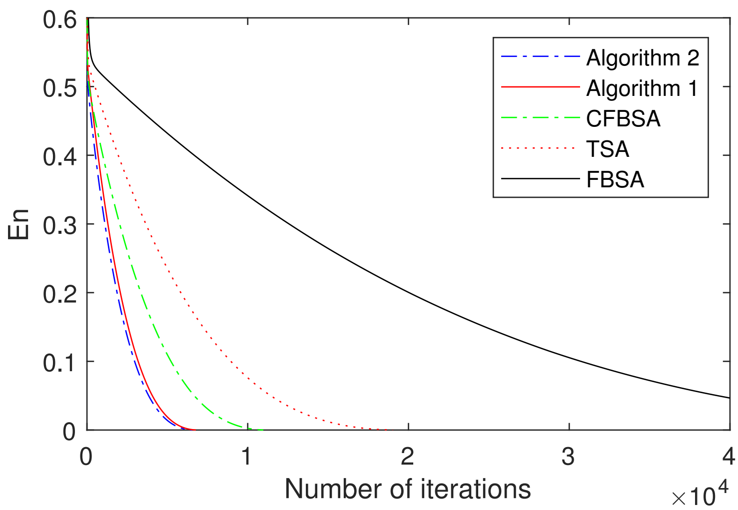

Figure 4. The plotting of MSE versus number of iterations in Case 2.

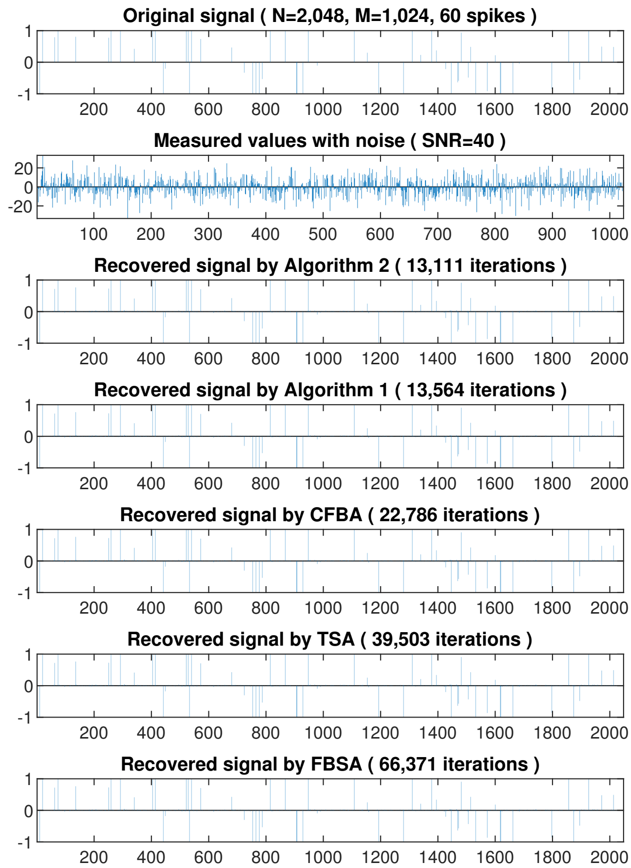

Figure 5. Comparison of recovered signal by using different algorithms in Case 3.

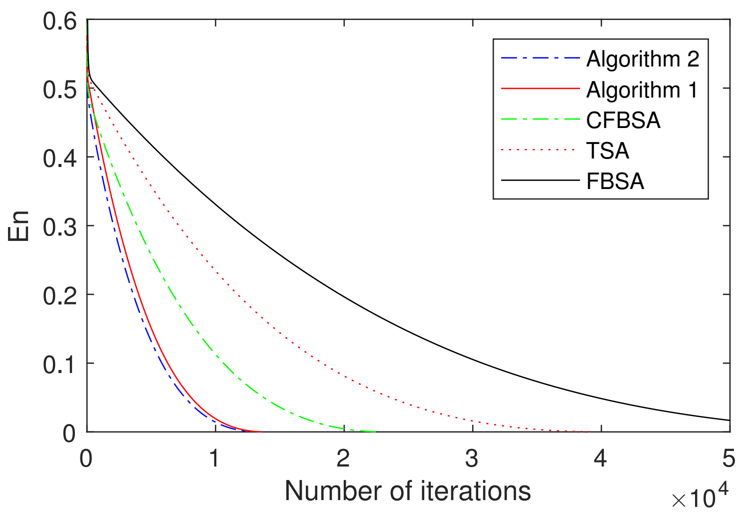

Figure 6. The plotting of MSE versus number of iterations in Case 3.

6. Conclusions

In this paper, we propose Tseng's splitting algorithms with non-monotone adaptive step sizes for finding zeros of the sum of two monotone operators in the framework of Banach space. Under some suitable conditions, we prove the weak and strong convergence results of the algorithms without the knowledge of the Lipschitz constant of the mapping. Some applications related to the obtained results are presented. Finally, several numerical experiments are performed to illustrate the convergence of our algorithms and compared with many known methods.

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Conflict of interest

The authors declare no conflict of interest.

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A MODIFIED POPOV'S SUBGRADIENT EXTRAGRADIENT METHOD FOR VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. In this paper, we propose a new modification of Popov's subgradient extragradient method for solving the variational inequality problem involving pseudo-monotone and Lipschitz-continuous mappings in the framework of Banach spaces. The weak convergence theorem of the proposed method is established without the knowledge of the Lipschitz constant of the Lipschitz continuous mapping. Finally, we provide several numerical experiments of the proposed method including comparisons with other related methods. Our result generalizes and extends many related results in the literature from Hilbert spaces to Banach spaces.

Keywords. Popov's method; Variational inequality problem; Pseudo-monotone mapping; Banach space.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote by $\langle x, f\rangle$ the value of $f \in E^*$ at $x \in E$, that is, $\langle x, f\rangle = f(x)$. Let C be a nonempty, closed and convex subset of E and let $A : C \to E^*$ be a mapping. The *variational inequality problem* (VIP) is to find an element $z \in C$ such that

$$\langle x - z, Az\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of the VIP is denoted by $VI(C,A)$. The VIP has been studied widely in many real-world problems, such as, artificial intelligence, computer science, control engineering, management science and operations research, and differential equations, fluid flow through

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porous media, contact problems in elasticity, transportation problems and economics equilibrium; see, e.g. [1, 2, 3, 4, 5, 6, 7] and the references therein.

A simple method for solving VIP in a real Hilbert space H is known as the *projected gradient method*, which is defined by the following scheme:

$$x_{n+1} = P_C(x_n - \lambda A x_n), \tag{1.2}$$

where P_C is the projection operator onto the convex and closed subset C of H and $\lambda > 0$ is a suitable stepsize. This method converges weakly to a solution of VIP under the following Assumption $(a1)$ or $(a2)$

$(a1)$ A is strongly monotone and Lipschitz continuous, and $\lambda \in (0, \frac{2\gamma}{L^2})$;
$(a2)$ A is inverse strongly monotone and $\lambda \in (0, 2\alpha)$,

where $\gamma > 0$ and $L > 0$ are strongly monotone and Lipschitz constants, respectively, and $\alpha > 0$ is the inverse strongly monotone constant of A.

We remark that Assumptions $(a1)$ and $(a2)$ are strong. Without the strong monotonicity, this method may diverges (see, e.g., [8]). In order to improve this drawback, Korpelevich [9] introduced the so-called *extragradient method* for solving the VIP in a finite-dimensional Euclidean space $\mathbb{R}^m$ as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.3}$$

where $C \subset \mathbb{R}^m$ is a nonempty, closed and convex set, $A : C \to \mathbb{R}^m$ is monotone and L-Lipschitz continuous, and $\lambda \in \left(0, \frac{1}{L}\right)$. It was proved that if $VI(C,A)$ is nonempty, then the sequences $\{x_n\}$ generated by (1.3) converges to an element in $VI(C,A)$. The idea of the extragradient method was successfully generalized and extended not only in Euclidean spaces but also in Hilbert and Banach spaces. In recent years, the extragradient method was further studied intensively and it has been extended in various ways by many authors; see, e.g., [10, 11, 12, 13, 14, 15, 16] and the references therein. Note that the extragradient method was based on a double-projection method onto the feasible set C. It needs to compute two projections onto C in each iteration step. In fact, in some cases, the structure of the set C is not explicit or complicated. As a result, the projection onto C might be difficult to compute. Moreover, in each iteration step of the extragradient method, one has to compute two values of the mapping A at points x_n and y_n.

To deal with the improvement of the extragradient method (1.3), Popov [17] proposed the so-called *Popov's extragradient method*, which only requires to compute one value of the mapping A at one point y_n per iteration. The Popov's extragradient method is of the form:

$$\begin{cases} x_{n+1} = P_C(x_n - \lambda A y_n), \\ y_{n+1} = P_C(x_{n+1} - \lambda A y_n), \end{cases} \tag{1.4}$$

where $\lambda \in \left(0, \frac{1}{3L}\right)$. In [17], the convergence of this method was proved in a finite dimensional Euclidean space.

In [18], Censor, Gibali and Reich proposed the so-called *subgradient extragradient method* for solving the VIP in a real Hilbert space H. They replaced the second projection onto C of the extragradient method (1.3) by a projection onto a half-space, which is easier to compute. The subgradient extragradient method is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{x \in H : \langle x_n - \lambda_n A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{1.5}$$

where $C \subset H$ is a nonempty, closed and convex set, $A : C \to H$ is monotone and L-Lipschitz continuous, and $\lambda \in \left(0, \frac{1}{L}\right)$. However, this method was still required to compute the value of the mapping A at two different points in each iteration step.

Recently, Malitsky and Semenov [19] combined the advantages of Popov's extragradient method (1.4) and subgradient extragradient method (1.5). They proposed the so-called *Popov's subgradient extragradient method* for solving VIP in a real Hilbert space H as follows:

$$\begin{cases} y_{n+1} = P_C(x_{n+1} - \lambda A y_n), \\ T_n = \{x \in H : \langle x_n - \lambda_n A y_{n-1} - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{1.6}$$

where $\lambda \in \left(0, \frac{1}{3L}\right)$. It was proved that the sequence $\{x_n\}$ generated by (1.6) converges weakly to a solution of the VIP provided that $VI(C,A)$ is nonempty. We remark that this method is not only requires to compute one projection onto the feasible set C but it also compute one value of the mapping A in each iteration step. In view of this, the Popov's subgradient extragradient method has received great attention in various ways; see, e.g., [20, 21, 22] and the references therein. However, the Popov's subgradient extragradient method (1.6) requires to know the Lipschitz constant or at least to know some estimation of it.

Motivated and inspired by the previous works, in this paper, we extend the Popov's subgradient extragradient method (1.6) from Hilbert spaces to Banach spaces, which are 2-uniformly convex and uniformly smooth. We prove the weak convergence result of the proposed algorithm to a solution of th VIP when A is pseudo-monotone and Lipschitz continuous. The advantage of our algorithm is that the stepsize does not requires to know the Lipschitz constant of the Lipschitz continuous mapping.

The outline of this paper is organized as follows: In Section 2, some preliminaries and facts are presented. In Section 3, we prove the weak convergence result of the proposed algorithm. Finally, in Section 4, we perform several numerical experiments to show the efficiency and advantages of the proposed algorithm. The result in this paper generalizes and extends many known results in the literature.

2. Preliminaries

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a real Banach space and E^* the dual space of E. For a sequence $\{x_n\} \subset E$, we denote $x_n \to x$ and $x_n \rightharpoonup x$ by the strong convergence and the weak convergence of $\{x_n\}$ to x, respectively. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E.

Definition 2.1. [23, 24, 25] Let E be a Banach space.

(1) The *modulus of convexity* $\delta_E : [0,2] \to [0,1]$ is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in U, \|x-y\| \geq \varepsilon\right\}.$$

(2) The *modulus of smoothness* $\rho_E : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(t) = \sup\left\{\frac{\|x+ty\|+\|x-ty\|}{2} - 1 : x,y \in U\right\}.$$

Definition 2.2. [23, 24, 25] A Banach space E is said to be:

(1) *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x,y \in U$ and $x \neq y$;
(2) *smooth* if $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,y \in U$;
(3) *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all$\varepsilon \in (0,2]$;
(4) *uniformly smooth* if $\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0$;
(5) *p-uniformly convex* if there exist $c > 0$ and $p \geq 2$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in (0,2]$;
(6) *q-uniformly smooth* if there exist $\kappa > 0$ and $1 < q \leq 2$ such that $\rho_E(t) \leq \kappa t^q$ for all $t > 0$.

Lemma 2.3. [23, 24, 25] *Let E be a Banach space. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then the following statements hold:*

(i) If E is p-uniformly convex (q-uniformly smooth), then E is uniformly convex (uniformly smooth.
(ii) E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex).
(iii) If E is uniformly convex (uniformly smooth), then E is strictly convex and reflexive (reflexive and smooth).

Remark 2.4. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p and ℓ_p, where $p > 1$. More precisely, L_p and ℓ_p are $\max\{p,2\}$-uniformly convex and $\min\{p,2\}$-uniformly smooth, while Hilbert spaces are 2-uniformly convex and 2-uniformly smooth (see [26] for more details).

Definition 2.5. [23, 24, 25] Let E be a Banach space.

(1) The *duality mapping* $J : E \to 2^{E^*}$ is defined by $Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$ for all $x \in E$.
(2) The duality mapping J from E into E^* is said to be *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup^* Jx$.

Lemma 2.6. [23, 24, 25] *Let E be a Banach space and let J be the duality mapping on E. The following facts are well-known:*

(i) If E is smooth, then J is single-valued and monotone, that is,

$$\langle x-y, Jx-Jy\rangle \geq 0,\ \forall x,y \in E.$$

Further, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.
(ii) If E is strictly convex, then J is one-to-one.
(iii) If E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded subsets of E.
(iv) If E is reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E.
(v) If E is a Hilbert space, then J is the identity mapping.

Definition 2.7. Let C be a nonempty subset of a Banach space E. A mapping $A : C \to E^*$ is said to be:

(1) *monotone* if
$$\langle x-y, Ax-Ay\rangle \geq 0, \ \forall x,y \in C;$$
(2) *pseudo-monotone* if
$$\langle y-x, Ax\rangle \geq 0 \Longrightarrow \langle y-x, Ay\rangle \geq 0, \ \forall x,y \in C;$$
(3) *Lipschitz continuous* if there exists a constant $L > 0$ such that
$$\|Ax-Ay\| \leq L\|x-y\|, \ \forall x,y \in C;$$
(4) *hemicontinuous* if for each $x,y \in C$, the mapping $f : [0,1] \to E^*$ defined by
$$f(t) = A(tx+(1-t)y)$$
is continuous with respect to the weak* topology of E^*.

Remark 2.8. Every monotone mapping is a pseudo-monotone mapping but converse is not true in general. The example of a pseudo-monotone mapping but is not monotone can be found in [27].

Definition 2.9. [28] Let E be a smooth Banach space. The *Lyapunov function* $\phi : E \times E \to \mathbb{R}$ is defined by
$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x,y \in E.$$

If E is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$ for all $x,y \in E$. From the definition of ϕ, it is clear that
$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \ \forall x,y \in E. \tag{2.1}$$
From (2.1), we can see that $\phi(x,y) \geq 0$ and if E is additionally assumed to be strictly convex, then
$$\phi(x,y) = 0 \iff x = y. \tag{2.2}$$
Furthermore, the function ϕ has the following two important properties:
$$\phi(x,y) + \phi(y,x) = 2\langle x-y, Jx-Jy\rangle, \ \forall x,y \in E \tag{2.3}$$
and
$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz-Jy\rangle, \ \forall x,y,z \in E. \tag{2.4}$$
Following [28], we have the functional $V : E \times E^* \to \mathbb{R}$, which is defined by
$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \ \forall x \in E, \ x^* \in E^*.$$
Then V is nonnegative and $V(x,x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Definition 2.10. [28] Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed convex subset of E. The *generalized projection mapping* is a mapping $\Pi_C : E \to C$ that assigns an arbitrary element $x \in E$ to the minimum element of the function $\phi(y,x)$, that is, $\Pi_C x = z$, where z is the solution to the following minimization problem:
$$\phi(z,x) = \min_{y \in C} \phi(y,x).$$

Remark 2.11. If E is a Hilbert space, then Π_C is coincident with the metric projection, denoted by P_C.

Lemma 2.12. [28] *Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let $x \in E$ and $z \in C$. Then the following statements hold:*

(i) $z = \Pi_C(x)$ *if and only if* $\langle y - z, Jz - Jx \rangle \geq 0$, $\forall y \in C$.
(ii) $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x)$, $\forall y \in C$.

Lemma 2.13. [29] *Let E be a 2-uniformly convex Banach space. Then there exists a constant $\delta \geq 1$ such that*

$$\frac{1}{\delta}\|x - y\|^2 \leq \phi(x, y),\ \forall x, y \in E.$$

Lemma 2.14. [10] *Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $A : C \to E^*$ be a pseudo-monotone and hemicontinuous operator. Then*

(i) z *is a solution of VIP (1.1) if and only if* $\langle x - z, Ax \rangle \geq 0$, $\forall x \in C$.
(ii) $VI(C, A)$ *is closed and convex.*

The following fact can be found in [30, Lemma 3.1].

Lemma 2.15. *For any $a, b \in \mathbb{R}$ and $\varepsilon > 0$, the following inequality holds*

$$2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2.$$

Proof. Since $0 \leq \left(\frac{1}{\sqrt{\varepsilon}}a - \sqrt{\varepsilon}b\right)^2 = \frac{a^2}{\varepsilon} - 2ab + \varepsilon b^2$, we have $2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$. $\square$

Lemma 2.16. *Let $\{\mathscr{A}_n\}$ and $\{\mathscr{B}_n\}$ be two nonnegative real sequences such that*

$$\mathscr{A}_{n+1} \leq \mathscr{A}_n - \mathscr{B}_n,\ \forall n \geq 1.$$

Then $\lim_{n\to\infty} \mathscr{A}_n$ *exists and* $\sum_{n=1}^{\infty} \mathscr{B}_n < \infty$.

The following lemma will be needed in the proof of the main result.

Lemma 2.17. *Let C be a nonempty, closed and convex subset of a real 2-uniformly convex Banach space E, which is also uniformly smooth. Let $\{x_n\}$ be a sequence in E. Suppose that the following two conditions hold:*

(i) $\lim_{n\to\infty} \phi(u, x_n)$ *exists for each* $u \in C$;
(ii) every sequential weak limit point of $\{x_n\}$ *belongs to* C.

Suppose, in addition, that J is weakly sequentially continuous on E. Then $\{x_n\}$ converges weakly to some element in C.

Proof. Since $\lim_{n\to\infty} \phi(u, x_n)$ exists, we have that $\{\phi(u, x_n)\}$ is bounded. Applying Lemma 2.13, we obtain that $\{x_n\}$ is bounded. By the reflexivity and the boundedness of $\{x_n\}$, we can suppose that there are two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup u$ and $x_{m_k} \rightharpoonup v$ for some $u, v \in C$. From this, we see that $\lim_{n\to\infty}(\phi(u, x_n) - \phi(v, x_n))$ exists. From the definition of ϕ, we have

$$\begin{aligned} \phi(u, x_n) - \phi(v, x_n) &= \|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2 - (\|v\|^2 - 2\langle v, Jx_n \rangle + \|x_n\|^2) \\ &= \|u\|^2 - \|v\|^2 - 2\langle u - v, Jx_n \rangle \end{aligned}$$

and then

$$\|u\|^2 - \|v\|^2 - 2\lim_{k\to\infty}\langle u-v, Jx_{n_k}\rangle = \|u\|^2 - \|v\|^2 - 2\lim_{k\to\infty}\langle u-v, Jx_{m_k}\rangle.$$

Since J is weakly sequentially continuous, we have $\langle u-v, Ju\rangle = \langle u-v, Jv\rangle$, that is, $\langle u-v, Ju - Jv\rangle = 0$. By the strict convexity of E, we obtain $u = v$. This completes the proof. □

3. Main result

In this section, we propose a modification of Popov's subgradient extragradient method for solving pseudo-monotone variational inequalities in Banach spaces. In order to prove convergence of the proposed method, we need the following assumptions:

(A1) The set C is a nonempty, closed and convex subset of a real 2-uniformly convex Banach space E, which is also uniformly smooth.
(A2) The mapping $A : E \to E^*$ is pseudo-monotone and L-Lipschitz continuous.
(A3) The duality mapping J is weakly sequentially continuous on E.
(A4) The solution set of the VIP is nonempty, that is, $VI(C,A) \neq \emptyset$.

Algorithm 1 Modified Popov's subgradient extragradient algorithm

Step 0: Give $\lambda_0, \lambda_1 > 0$ and $\mu \in \left(0, \frac{\sqrt{2}-1}{\delta}\right)$, where δ is a constant given by Lemma 2.13. Let $x_0, y_0 \in C$ be arbitrary.

Step 1: Compute

$$\begin{cases} x_1 = \Pi_C J^{-1}(Jx_0 - \lambda_0 Ay_0), \\ y_1 = \Pi_C J^{-1}(Jx_1 - \lambda_1 Ay_0). \end{cases} \tag{3.1}$$

Step 2: Given the current iterate x_n, y_n and y_{n-1}, calculate x_{n+1} as follows:

$$x_{n+1} = \Pi_{T_n} J^{-1}(Jx_n - \lambda_n Ay_n), \tag{3.2}$$

where

$$T_n = \{x \in E : \langle x - y_n, Jx_n - \lambda_n Ay_{n-1} - Jy_n\rangle \leq 0\}. \tag{3.3}$$

Step 3: Compute

$$y_{n+1} = \Pi_C J^{-1}(Jx_{n+1} - \lambda_{n+1} Ay_n), \tag{3.4}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\sqrt{2}\|x_{n+1}-y_n\|^2 + \frac{1}{\sqrt{2}}\|y_n - y_{n-1}\|^2}{2\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle}, \lambda_n\right\} & \text{if } \langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.5}$$

If $x_{n+1} = x_n$ and $y_{n-1} = y_n$ (or $x_{n+1} = y_n = y_{n+1}$), then stop and y_n is a solution of the VIP. Otherwise, go to **Step 1**.

Lemma 3.1. *Let $\{\lambda_n\}$ be a sequence generated by (3.5). Then $\{\lambda_n\}$ is nonincreasing and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}.$$

Moreover,

$$2\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle \le \frac{\mu}{\lambda_{n+1}}\Big(\sqrt{2}\|x_{n+1}-y_n\|^2+\frac{1}{\sqrt{2}}\|y_n-y_{n-1}\|^2\Big),\ \forall n\ge 1.$$

Proof. It is obvious from (3.5) that $\lambda_{n+1}\le\lambda_n$ for all $n\ge 1$. For the case $\langle x_{n+1}-y_n, Ay_n-Ay_{n-1}\rangle>0$, Lemma 2.15 and the L-Lipschitz continuity of A, we have

$$\mu\frac{\sqrt{2}\|x_{n+1}-y_n\|^2+\frac{1}{\sqrt{2}}\|y_n-y_{n-1}\|^2}{2\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle}\ge\frac{2\mu\|x_{n+1}-y_n\|\|y_n-y_{n-1}\|}{2L\|x_{n+1}-y_n\|\|y_n-y_{n-1}\|}=\frac{\mu}{L}.$$

Clearly,

$$\lambda_{n+1}\ge\min\Big\{\frac{\mu}{L},\lambda_n\Big\}.$$

By induction, we immediately obtain that $\{\lambda_n\}$ is bounded from below by $\min\{\frac{\mu}{L},\lambda_1\}$. Thus there exists $\lambda:=\lim_{n\to\infty}\lambda_n\ge\min\{\frac{\mu}{L},\lambda_1\}$.

On the other hand, from the definition of λ_n, we have

$$\begin{aligned}\lambda_{n+1}&=\min\Big\{\mu\frac{\sqrt{2}\|x_{n+1}-y_n\|^2+\frac{1}{\sqrt{2}}\|y_n-y_{n-1}\|^2}{2\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle},\lambda_n\Big\}\\&\le\mu\frac{\sqrt{2}\|x_{n+1}-y_n\|^2+\frac{1}{\sqrt{2}}\|y_n-y_{n-1}\|^2}{2\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle}.\end{aligned}$$

This implies that

$$2\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle \le \frac{\mu}{\lambda_{n+1}}\Big(\sqrt{2}\|x_{n+1}-y_n\|^2+\frac{1}{\sqrt{2}}\|y_n-y_{n-1}\|^2\Big),\ \forall n\ge 1.$$

□

Lemma 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 1. If $x_{n+1}=x_n$ and $y_{n-1}=y_n$, then y_n is a solution of the VIP.*

Proof. If $x_{n+1}=x_n$, then $x_n=\Pi_{T_n}J^{-1}(Jx_n-\lambda_nAy_n)$. By Lemma 2.12 (i), we have

$$\begin{aligned}\langle x-x_n, Jx_n-J\circ J^{-1}(Jx_n-\lambda_nAy_n)\rangle &= \langle x-x_n, Jx_n-Jx_n+\lambda_nAy_n\rangle\\&=\lambda_n\langle x-x_n, Ay_n\rangle\ge 0,\ \forall x\in T_n.\end{aligned}$$

This implies that $\langle x-y_n, Ay_n\rangle\ge\langle x_n-y_n, Ay_n\rangle$, $\forall x\in T_n$. It is easy to see that $C\subset T_n$, and

$$\langle x-y_n, Ay_n\rangle\ge\langle x_n-y_n, Ay_n\rangle,\ \forall x\in C. \tag{3.6}$$

On the other hand, by the definition of T_n, we have

$$\langle x-y_n, Jx_n-\lambda_nAy_{n-1}-Jy_n\rangle\le 0,\ \forall x\in T_n.$$

Since $x_n=x_{n+1}$, we have $x_n\in T_n$. If $y_{n-1}=y_n$, then $\langle x_n-y_n, Jx_n-\lambda_nAy_n-Jy_n\rangle\le 0$. From (2.3), we see that

$$\langle x_n-y_n, Ay_n\rangle\ge\langle x_n-y_n, Jx_n-Jy_n\rangle\ge\frac{1}{2}\phi(x_n,y_n)\ge 0. \tag{3.7}$$

Combining (3.6) and (3.7), we thus get $\langle x-y_n, Ay_n\rangle\ge 0$, $\forall x\in C$. Hence, y_n is a solution of the VIP. □

Lemma 3.3. *Assume that Assumptions $(A1)-(A4)$ hold. Let $\{x_n\}$ be a sequence generated by Algorithm 1. For each $n \geq 1$, we have*

$$\mathscr{A}_{n+1} \leq \mathscr{A}_n - (1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n) - (1-\sqrt{2}\theta_n-\theta_{n+1})\phi(x_{n+1},y_n),\ \forall p \in VI(C,A),$$

where $\mathscr{A}_n = \phi(p,x_n)+\theta_n\phi(x_n,y_{n-1})$ and $\theta_n = \frac{\mu\delta\lambda_n}{\lambda_{n+1}}$.

Proof. Let $p \in VI(C,A)$. By Lemma 2.12 (ii), we have

$$\begin{aligned}
&\phi(p,x_{n+1})\\
&=\phi(p,\Pi_{T_n}J^{-1}(Jx_n-\lambda_nAy_n))\\
&\leq \phi(p,J^{-1}(Jx_n-\lambda_nAy_n))-\phi(x_{n+1},J^{-1}(Jx_n-\lambda_nAy_n))\\
&=V(p,Jx_n-\lambda_nAy_n)-V(x_{n+1},Jx_n-\lambda_nAy_n)\\
&=\|p\|^2-2\langle p,Jx_n\rangle+2\lambda_n\langle p,Ay_n\rangle+\|Jx_n-\lambda_nAy_n\|^2-\|x_{n+1}\|^2\\
&\quad+2\langle x_{n+1},Jx_n\rangle-2\lambda_n\langle x_{n+1},Ay_n\rangle-\|Jx_n-\lambda_nAy_n\|^2\\
&=\|p\|^2-2\langle p,Jx_n\rangle+\|x_n\|^2-\Big(\|x_{n+1}\|^2-2\langle x_{n+1},Jx_n\rangle+\|x_n\|^2\Big)-2\lambda_n\langle x_{n+1}-p,Ay_n\rangle\\
&=\phi(p,x_n)-\phi(x_{n+1},x_n)-2\lambda_n\langle x_{n+1}-p,Ay_n\rangle\\
&=\phi(p,x_n)-\phi(x_{n+1},x_n)-2\lambda_n\langle x_{n+1}-y_n,Ay_n\rangle-2\lambda_n\langle y_n-p,Ay_n\rangle.
\end{aligned} \tag{3.8}$$

Since $y_n \in C$, we have $\langle y_n-p,Ap\rangle \geq 0$. By the pseudo-monotonicity of A, we have $\langle y_n-p,Ay_n\rangle \geq 0$. It follows from (3.8) that

$$\phi(p,x_{n+1}) \leq \phi(p,x_n)-\phi(x_{n+1},x_n)-2\lambda_n\langle x_{n+1}-y_n,Ay_n\rangle. \tag{3.9}$$

From (2.3), we see that

$$\phi(x_{n+1},x_n)=\phi(x_{n+1},y_n)+\phi(y_n,x_n)+2\langle x_{n+1}-y_n,Jy_n-Jx_n\rangle. \tag{3.10}$$

Substituting (3.10) and (3.9), we have

$$\begin{aligned}
\phi(p,x_{n+1}) &\leq \phi(p,x_n)-\phi(x_{n+1},y_n)-\phi(y_n,x_n)-2\langle x_{n+1}-y_n,Jy_n-Jx_n\rangle-2\lambda_n\langle x_{n+1}-y_n,Ay_n\rangle\\
&=\phi(p,x_n)-\phi(x_{n+1},y_n)-\phi(y_n,x_n)+2\lambda_n\langle x_{n+1}-y_n,Ay_{n-1}-Ay_n\rangle\\
&\quad+2\langle x_{n+1}-y_n,Jx_n-\lambda_nAy_{n-1}-Jy_n\rangle.
\end{aligned} \tag{3.11}$$

Since $x_{n+1} \in T_n$, we have $\langle x_{n+1}-y_n,Jx_n-\lambda_nAy_{n-1}-Jy_n\rangle \leq 0$. This implies that

$$\phi(p,x_{n+1}) \leq \phi(p,x_n)-\phi(x_{n+1},y_n)-\phi(y_n,x_n)+2\lambda_n\langle x_{n+1}-y_n,Ay_{n-1}-Ay_n\rangle. \tag{3.12}$$

From Lemma 3.1, we see that

$$2\lambda_n\langle x_{n+1}-y_n,Ay_{n-1}-Ay_n\rangle \leq \mu\frac{\lambda_n}{\lambda_{n+1}}\Big(\sqrt{2}\|x_{n+1}-y_n\|^2+\frac{1}{\sqrt{2}}\|y_{n-1}-y_n\|^2\Big). \tag{3.13}$$

Now, we estimate $\|y_{n-1}-y_n\|^2$. Observe that

$$\begin{aligned}\|y_{n-1}-y_n\|^2 &\le \Big(\|y_n-x_n\|+\|x_n-y_{n-1}\|\Big)^2\\ &\le \|y_n-x_n\|^2+2\|y_n-x_n\|\|x_n-y_{n-1}\|+\|x_n-y_{n-1}\|^2\\ &\le \|y_n-x_n\|^2+(\sqrt{2}+1)\|y_n-x_n\|^2+\frac{1}{\sqrt{2}+1}\|x_n-y_{n-1}\|^2+\|x_n-y_{n-1}\|^2\\ &= (\sqrt{2}+2)\|y_n-x_n\|^2+\sqrt{2}\|x_n-y_{n-1}\|^2. \end{aligned}\tag{3.14}$$

Substituting (3.14) into (3.13) and using Lemma 2.13, we have

$$\begin{aligned}&2\lambda_n\langle x_{n+1}-y_n, Ay_{n-1}-Ay_n\rangle\\ &\le \mu\frac{\lambda_n}{\lambda_{n+1}}\Big(\sqrt{2}\|x_{n+1}-y_n\|^2+(\sqrt{2}+1)\|y_n-x_n\|^2+\|x_n-y_{n-1}\|^2\Big)\\ &\le \mu\frac{\lambda_n}{\lambda_{n+1}}\Big(\delta\sqrt{2}\phi(x_{n+1},y_n)+\delta(\sqrt{2}+1)\phi(y_n,x_n)+\delta\phi(x_n,y_{n-1})\Big)\\ &= \theta_n\Big(\sqrt{2}\phi(x_{n+1},y_n)+(\sqrt{2}+1)\phi(y_n,x_n)+\phi(x_n,y_{n-1})\Big),\end{aligned}\tag{3.15}$$

where $\theta_n=\frac{\mu\delta\lambda_n}{\lambda_{n+1}}$. Substituting (3.15) into (3.12), we have

$$\begin{aligned}\phi(p,x_{n+1}) &\le \phi(p,x_n)-\phi(x_{n+1},y_n)-\phi(y_n,x_n)\\ &\quad+\theta_n\Big(\sqrt{2}\phi(x_{n+1},y_n)+(\sqrt{2}+1)\phi(y_n,x_n)+\phi(x_n,y_{n-1})\Big)\\ &= \phi(p,x_n)+\theta_n\phi(x_n,y_{n-1})-(1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n)-(1-\sqrt{2}\theta_n)\phi(x_{n+1},y_n).\end{aligned}\tag{3.16}$$

Adding the term $\theta_{n+1}\phi(x_{n+1},y_n)$ to both sides of (3.16), we get

$$\begin{aligned}\phi(p,x_{n+1})+\theta_{n+1}\phi(x_{n+1},y_n) &\le \phi(p,x_n)+\theta_n\phi(x_n,y_{n-1})-(1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n)\\ &\quad -(1-\sqrt{2}\theta_n-\theta_{n+1})\phi(x_{n+1},y_n).\end{aligned}\tag{3.17}$$

Then (3.17) reduces to the following inequality:

$$\mathscr{A}_{n+1}\le\mathscr{A}_n-(1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n)-(1-\sqrt{2}\theta_n-\theta_{n+1})\phi(x_{n+1},y_n),\tag{3.18}$$

where $\mathscr{A}_n=\phi(p,x_n)+\theta_n\phi(x_n,y_{n-1})$. □

Lemma 3.4. *Assume that Assumptions $(A1)-(A4)$ hold. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z\in E$. If $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$, then $z\in VI(C,A)$.*

Proof. Let $\{x_{n_k}\}$ be the subsequence of $\{x_n\}$ such that $x_{n_k}\rightharpoonup z\in E$. Since $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $\{y_n\}\subset C$, we have $y_{n_k}\rightharpoonup z$ and $z\in C$. From the definition of y_{n_k}, we see that

$$y_{n_k+1}=\Pi_C J^{-1}(Jx_{n_k+1}-\lambda_{n_k+1}Ay_{n_k}).$$

By Lemma 2.12 (i), we have

$$\langle x-y_{n_k+1}, Jy_{n_k+1}-Jx_{n_k+1}+\lambda_{n_k+1}Ay_{n_k}\rangle\ge 0,\ \forall x\in C,\tag{3.19}$$

which implies that

$$\lambda_{n_k+1}\langle x-y_{n_k+1}, Ay_{n_k}\rangle\ge\langle x-y_{n_k+1}, Jx_{n_k+1}-Jy_{n_k+1}\rangle,\ \forall x\in C.$$

Moreover, we have

$$\langle x-y_{n_k},Ay_{n_k}\rangle \geq \frac{1}{\lambda_{n_k}}\langle x-y_{n_k+1},Jx_{n_k+1}-Jy_{n_k+1}\rangle+\langle y_{n_k+1}-y_{n_k},Ay_{n_k}\rangle.$$

Since $\lim_{k\to\infty}\lambda_{n_k}=\lambda>0$, $\{Ay_{n_k}\}$ is bounded and J is norm-to-norm uniform continuous, we have

$$\liminf_{k\to\infty}\langle x-y_{n_k},Ay_{n_k}\rangle \geq 0. \tag{3.20}$$

Let $\{\varepsilon_k\}$ be a decreasing sequence of positive real numbers such that $\varepsilon_k\to 0$ as $k\to\infty$. For each ε_k, we denote by N the smallest positive integer such that

$$\langle x-y_{n_k},Ay_{n_k}\rangle+\varepsilon_k \geq 0,\ \forall k\geq N. \tag{3.21}$$

It is clear that (3.21) can be written as

$$\langle x+\varepsilon_k v_{n_k}-y_{n_k},Ay_{n_k}\rangle \geq 0,\ \forall k\geq N \tag{3.22}$$

for some $v_{n_k}:=w\in E$ satisfying $\langle v_{n_k},Ay_{n_k}\rangle=1$ (since $Ay_{n_k}\neq 0$). By the pseudo-monotonicity of A, we have

$$\langle x+\varepsilon_k v_{n_k}-y_{n_k},A(x+\varepsilon_k v_{n_k})\rangle \geq 0. \tag{3.23}$$

Since $y_{n_k}\rightharpoonup z$, $\varepsilon_k\to 0$ and A is Lipschitz continuous (hence it is continuous), it follows from (3.23) that

$$\langle x-z,Ax\rangle \geq 0,\ \forall x\in C. \tag{3.24}$$

By Lemma 2.14 (ii), we obtain $z\in VI(C,A)$. □

Theorem 3.5. *Assume that Assumptions $(A1)-(A4)$ hold. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ converges weakly to an element in $VI(C,A)$.*

Proof. Since $\lim_{n\to\infty}\lambda_n$ exists and $\mu\in\left(0,\frac{\sqrt{2}-1}{\delta}\right)$, we have

$$\lim_{n\to\infty}(1-(\sqrt{2}+1)\theta_n)=\lim_{n\to\infty}(1-\sqrt{2}\theta_n-\theta_{n+1})=1-(\sqrt{2}+1)\mu\delta=1-\frac{\mu\delta}{\sqrt{2}-1}>0.$$

Thus there exists $n_0\in\mathbb{N}$ such that

$$1-(\sqrt{2}+1)\theta_n>0 \text{ and } 1-\sqrt{2}\theta_n-\theta_{n+1}>0,\ \forall n\geq n_0,$$

which implies that

$$(1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n)+(1-\sqrt{2}\theta_n-\theta_{n+1})\phi(x_{n+1},y_n)\geq 0,\ \forall n\geq n_0.$$

Now, we can write (3.18) in the following form:

$$\mathscr{A}_{n+1}\leq \mathscr{A}_n-\mathscr{B}_n,\ \forall n\geq n_0,$$

where $\mathscr{B}_n=(1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n)+(1-\sqrt{2}\theta_n-\theta_{n+1})\phi(x_{n+1},y_n)$. By Lemma 2.16, we obtain $\lim_{n\to\infty}\mathscr{A}_n$ exists and

$$\sum_{n=n_0}^{\infty}\mathscr{B}_n=\sum_{n=n_0}^{\infty}\Big[(1-(\sqrt{2}+1)\theta_n)\phi(y_n,x_n)+(1-\sqrt{2}\theta_n-\theta_{n+1})\phi(x_{n+1},y_n)\Big]<\infty.$$

Thus we have $\lim_{n\to\infty}\phi(y_n,x_n)=\lim_{n\to\infty}\phi(x_{n+1},y_n)=0$. Using Lemma 2.13, we get

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.25}$$

By the uniform continuity of J, we have

$$\lim_{n\to\infty}\|Jy_n - Jx_n\| = \lim_{n\to\infty}\|Jx_{n+1} - Jy_n\| = 0,$$

which implies that

$$\|Jy_n - Jy_{n-1}\| \leq \|Jy_n - Jx_n\| + \|Jx_n - Jy_{n-1}\| \to 0.$$

Also, by the norm to norm uniform continuity of J^{-1}, we have

$$\lim_{n\to\infty}\|y_n - y_{n-1}\| = 0. \tag{3.26}$$

Since $\lim_{n\to\infty}\mathscr{A}_n$ exists and from (3.25), we have that $\{\phi(p,x_n)\}$ is bounded. It follows from Lemma 2.13 that $\{x_n\}$ is bounded. By the reflexivity of E, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z \in E$. As proved in Lemma 3.4, we have $z \in VI(C,A)$. In summary, we have shown that

(i) $\lim_{n\to\infty}\phi(p,x_n)$ exists for every $p \in VI(C,A)$;
(ii) every sequential weak limit point of $\{x_n\}$ is in $VI(C,A)$.

Therefore, by Lemma 2.17, we conclude that $\{x_n\}$ converges weakly to an element in $VI(C,A)$. This completes the proof. □

4. NUMERICAL EXPERIMENTS

In this section, we provide several numerical experiments to illustrate the convergence and the efficiency of our Algorithm 1 in solving the variational inequality problem. Moreover, we also compare it with subgradient extragradient method (SEM) (1.5), Popov's subgradient extragradient method (PSEM) (1.6), Halpern's subgradient extragradient method (HSEM) proposed in ([11, Algorithm 3.5]) and modified subgradient extragradient method with an Armijo line search rule (L-MSEM) proposed in ([31, Algorithm 3.1]). We have used the following values of the control parameters for the whole numerical part:

(i) subgradient extragradient method (SEM):

$$\lambda = \frac{0.7}{L},\ D_n = \|x_{n+1} - x_n\|;$$

(ii) Popov's subgradient extragradient method (PSEM):

$$\lambda = \frac{0.7}{3L},\ x_0 = y_0,\ D_n = \|x_{n+1} - x_n\|;$$

(iii) Halpern's subgradient extragradient method (HSEM):

$$\lambda = \frac{0.7}{L},\ \alpha_n = \frac{1}{n+2},\ D_n = \|x_{n+1} - x_n\|;$$

(iv) modified subgradient extragradient method with an Armijo line search rule (L-MSEM):

$$\mu = 0.80,\ \gamma = 1,\ l = 0.20,\ D_n = \|x_{n+1} - x_n\|;$$

(v) Algorithm 1 (modified Popov's subgradient extragradient method (M-PSEM)):

$$\mu = 0.33,\ \lambda_0 = \lambda_1 = 0.20,\ x_0 = y_0,\ D_n = \|x_{n+1} - x_n\|.$$

Example 4.1. We consider the HpHard problem which is taken from [32]. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be an operator defined by $Ax = Mx + q$ with $q \in \mathbb{R}^m$ and

$$M = NN^T + B + D,$$

where N is an $m \times m$ matrix, B is an $m \times m$ skew-symmetric matrix and D is an $m \times m$ positive definite diagonal matrix. The feasible set is

$$C = \{x \in \mathbb{R}^m : Qx \leq b\},$$

where Q is an $100 \times m$ matrix and b is a nonnegative vector in $\mathbb{R}^m$. It is clear that A is monotone and Lipschitz continuous with $L = \|M\|$ (hence the variational inequality has a unique solution). For $q = 0$, the solution set of the corresponding variational inequality is $VI(C, A) = \{0\}$. We perform numerical experiments with the starting point $x_0 = (1, 1, \ldots, 1)^T$ and use $\|x_{n+1} - x_n\| < \varepsilon = 10^{-3}$ to stop the iterative process. The numerical results of all methods have been reported in the Figures 1-3.

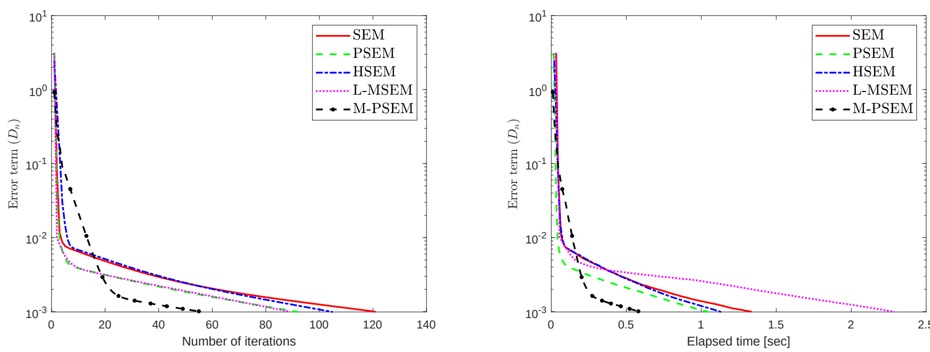

FIGURE 1. Numerical comparison of M-PSEM with existing algorithms when $m = 10$ for Example 4.1.

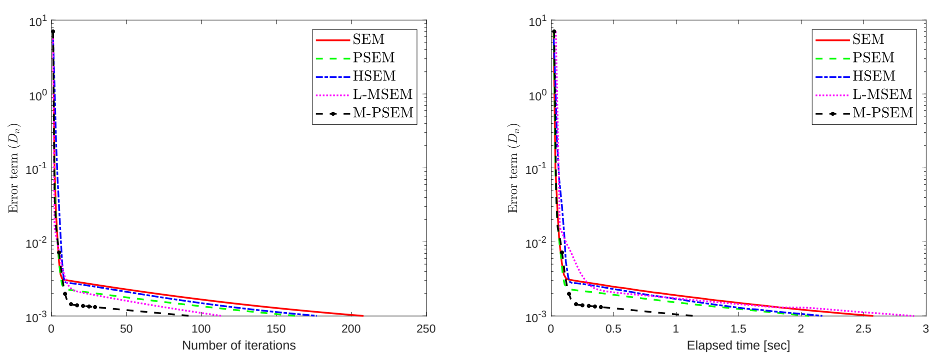

FIGURE 2. Numerical comparison of M-PSEM with existing algorithms when $m = 50$ for Example 4.1.

Example 4.2. In this example, we take $E = L_2([0, 1])$ with the norm

$$\|x\|_2 = \left(\int_0^1 |x(t)|^2 dt \right)^{1/2}$$

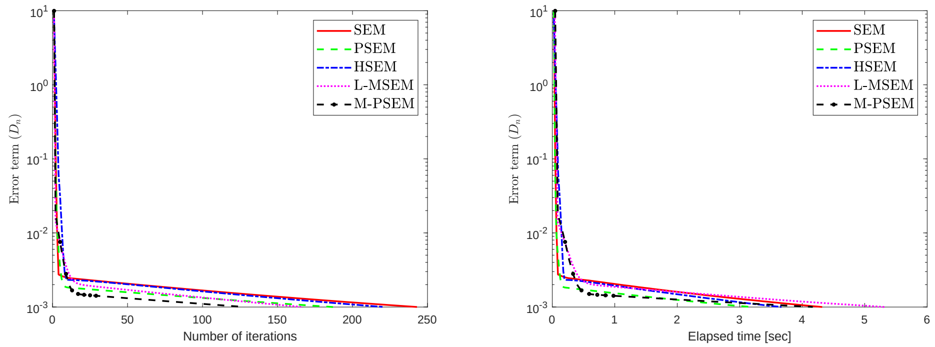

FIGURE 3. Numerical comparison of M-PSEM with existing algorithms when $m = 100$ for Example 4.1.

and the inner product

$$\langle x, y \rangle = \int_0^1 x(t)y(t)dt$$

for all $x, y \in L_2([0,1])$. The feasible set is $C = \{x \in E : \|x\| \leq 1\}$. Define an integral operator $A : C \to E$ by

$$Ax(t) = \int_0^1 \Big(x(t) - f(t,s)g(x(s))\Big)ds - h(t), \ x \in C \text{ and } t \in [0,1],$$

where

$$f(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \ g(x) = \cos x \text{ and } h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

It was shown that A is monotone (hence it is pseudo-monotone) and Lipschitz continuous with $L = 2$ (see [33]). The solution set of the corresponding variational inequality problem is $VI(C,A) = \{0\}$. We perform numerical experiments with three different starting points and use $\|x_{n+1} - x_n\| < \varepsilon = 10^{-4}$ to stop the iterative process. The numerical results of all methods have been reported in the Figures 4-6.

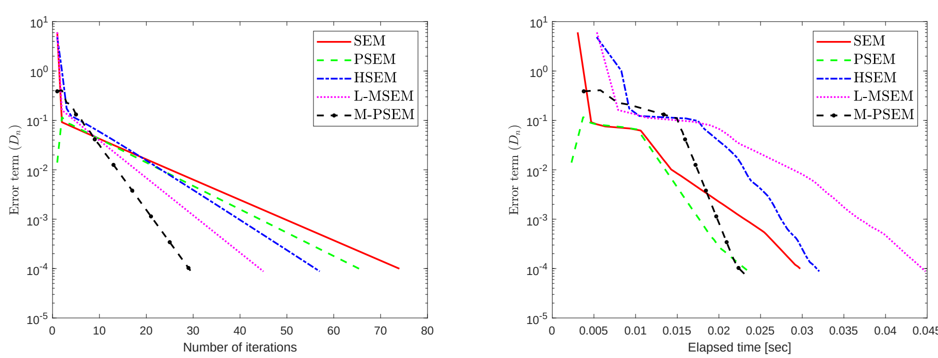

FIGURE 4. Numerical comparison of M-PSEM with existing algorithms when $x_0 = 1$ for Example 4.2.

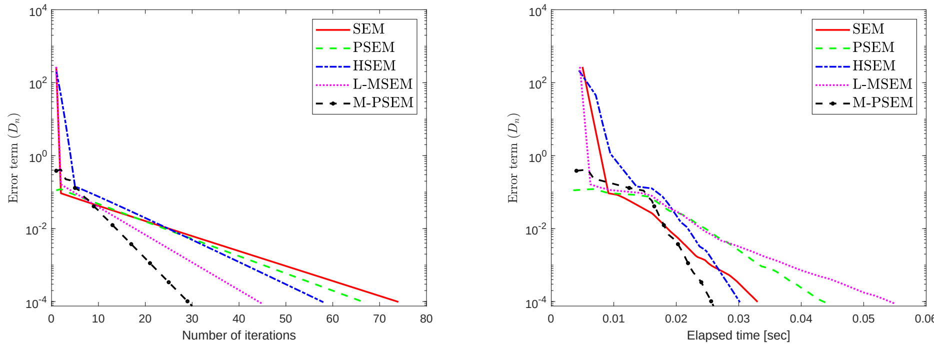

FIGURE 5. Numerical comparison of M-PSEM with existing algorithms when $x_0 = t$ for Example 4.2.

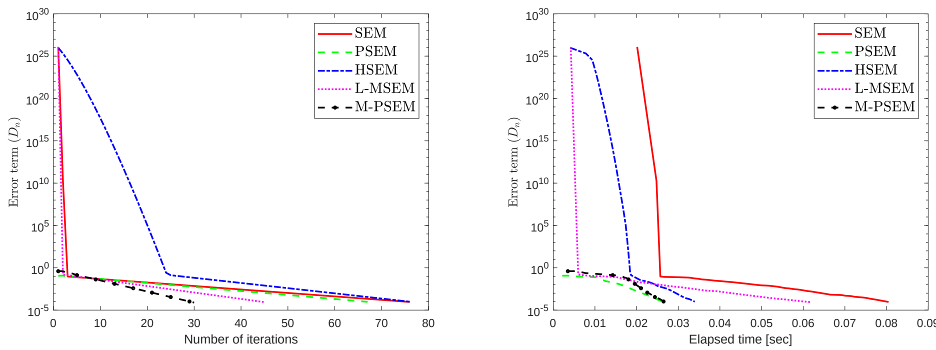

FIGURE 6. Numerical comparison of M-PSEM with existing algorithms when $x_0 = e^t$ for Example 4.2.

Example 4.3. Consider the following pseudo-monotone variational inequalities with

$$Ax = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 - 4x_2x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_3x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_4 \\ x_1 + x_2 + x_3 + x_4 - 4x_1x_2x_3 \end{pmatrix}.$$

The feasible set is $C = \{x \in \mathbb{R}^4 : 1 \leq x_1 \leq 5,\ i = 1,2,3,4\}$. It is easy to see that Ax is not monotone on C. Using the Monte Carlo approach [34], it can be shown that A is pseudo-monotone on C. This problem has unique solution $z = (5,5,5,5)^T$. Since the Lipschitz constant is unknown, thus SEM, PSEM and HSEM do not applicable in this problem. We only perform the numerical experiments of L-PSEM and M-MSEM with three different starting points x_0 and use $\|x_{n+1} - x_n\| < \varepsilon$ to stop the iterative process. The numerical results of mentioned methods have been reported in the Tables 1 and 2.

In the last example, we use our proposed algorithm to solve the pseudo-convex minimization problem in a finite dimension space $E = \mathbb{R}^m$. Let C be a nonempty, closed and convex subset of E. Recall that a differentiable function $f : E \to \mathbb{R}$ is called *pseudo-convex* if

$$\langle \nabla f(x), y - x \rangle \geq 0 \implies f(y) \geq f(x),\ \forall x, y \in C,$$

TABLE 1. Numerical tests of L-PSEM with different starting points for Example 4.3.

ε	0.01	0.001	0.0001	0.00001	0.01	0.001	0.0001	0.00001
x_0	No. of Iter.	No. of Iter.	No. of Iter.	No. of Iter.	Time(s)	Time(s)	Time(s)	Time(s)
$(-2,2,8,10)^T$	3	3	3	3	0.0633	0.0701	0.0865	0.0723
$(-1,2,2,5)^T$	3	3	3	3	0.0723	0.0721	0.0789	0.0768
$(3,1,-5,-2)^T$	3	3	3	3	0.0770	0.0821	0.0871	0.0799

TABLE 2. Numerical tests of M-PSEM with different starting points for Example 4.3.

ε	0.01	0.001	0.0001	0.00001	0.01	0.001	0.0001	0.00001
x_0	No. of Iter.	No. of Iter.	No. of Iter.	No. of Iter.	Time(s)	Time(s)	Time(s)	Time(s)
$(-2,2,8,10)^T$	5	5	5	5	0.0298	0.0234	0.0395	0.0415
$(-1,2,2,5)^T$	5	5	5	5	0.0394	0.0440	0.0387	0.0382
$(3,1,-5,-2)^T$	4	4	4	4	0.0806	0.0264	0.0271	0.0269

where ∇f is the gradient of f. The *pseudo-convex minimization problem* is to find an element $z \in C$ such that

$$f(z) = \min_{x \in C} f(x), \tag{4.1}$$

where f is differentiable and pseudo-convex. This problem (4.1) is equivalent to the following variational inequality problem [35, 36]:

$$\langle \nabla f(z), x - z \rangle \geq 0, \ \forall x \in C. \tag{4.2}$$

It is known that a differentiable function is pseudo-convex if and only if its gradient is a pseudo-monotone mapping (see [37]).

Example 4.4. Consider the quadratic fractional programming problem in the following form [34]:

$$\begin{cases} \min f(x) = \dfrac{x^T Q x + a^T x + a_0}{b^T x + b_0}, \\ \text{subject to } x \in K = \{x \in \mathbb{R}^4 : b^T x + b_0 > 0\}, \end{cases}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2 \text{ and } b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite on $\mathbb{R}^4$ and consequently f is pseudo-convex on K. Hence ∇f is pseudo-monotone. Using the quotient rule, we get

$$\nabla f(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Q + a^T x + a_0)}{(b^T x + b_0)^2}.$$

In this point of view, we can set $A = \nabla f$ in Theorem 3.5. We minimize f over $C = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 10, \ i = 1,2,3,4\} \subset K$. This problem has a unique solution $z = (1,1,1,1)^T \in C$. Since the Lipschitz constant is unknown, thus SEM, PSEM and HSEM do not applicable in this problem.

We only perform the numerical experiments of L-PSEM and M-MSEM with three different starting points x_0 and use $\|x_{n+1}-x_n\|<\varepsilon$ to stop the iterative process. The numerical results of mentioned methods have been reported in the Tables 3 and 4.

TABLE 3. Numerical tests of L-PSEM with different starting points for Example 4.4.

ε	0.01	0.001	0.0001	0.00001	0.01	0.001	0.0001	0.00001
x_0	No. of Iter.	No. of Iter.	No. of Iter.	No. of Iter.	Time(s)	Time(s)	Time(s)	Time(s)
$(10,10,10,10)^T$	58	101	141	187	0.6008	1.4598	1.9081	3.7862
$(10,20,30,40)^T$	73	113	156	233	0.9732	1.7781	2.2009	4.5341
$(20,-20,20,-20)^T$	81	141	199	243	1.1023	1.8970	4.0032	4.9032

TABLE 4. Numerical tests of M-PSEM with different starting points for Example 4.4.

ε	0.01	0.001	0.0001	0.00001	0.01	0.001	0.0001	0.00001
x_0	No. of Iter.	No. of Iter.	No. of Iter.	No. of Iter.	Time(s)	Time(s)	Time(s)	Time(s)
$(10,10,10,10)^T$	49	91	133	175	0.2203	0.4282	0.7989	0.8634
$(10,20,30,40)^T$	69	102	148	191	0.2197	0.6281	1.2019	1.3451
$(20,-20,20,-20)^T$	87	158	201	295	0.5214	1.3122	1.9514	2.4151

5. CONCLUSION

In recent years, several variants of the Popov's subgradient extragradient method have been studied intensively by many authors. Note that most of them were studied in Hilbert spaces. In this paper, we extend the Popov's subgradient extragradient method to Banach spaces. The weak convergence theorem of the proposed algorithm was proved without the knowledge of the Lipschitz constant of the mapping. Several numerical experiments are performed to illustrate the performance of our algorithm.

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Strong convergence of a generalized forward–backward splitting method in reflexive Banach spaces

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ABSTRACT

In this paper, we study the so-called *generalized monotone quasi-inclusion problem* which is a generalization and extension of well-known monotone quasi-inclusion problem. We propose a forward–backward splitting method for solving this problem in the framework of reflexive Banach spaces. Based on Bregman distance function, we prove a strong convergence result of the proposed algorithm to a common zero of the problem. As an application, we apply the main result to the variational inequality problem. Finally, we provide some numerical examples to demonstrate our algorithm performance. The results presented in this paper improve and extend many known results in the literature.

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1. Introduction

Let E be a real Banach space. Let $A : E \to E$ and $B : E \multimap E$ be single and set-valued operators, respectively. Consider the following so-called *monotone quasi-inclusion problem*:

$$\text{find } z \in E \text{ such that } 0 \in (A + B)z, \tag{1}$$

where 0 is the zero vector in E. The solutions set of the problem (1) is denoted by $(A + B)^{-1}0 = \{x \in E : 0 \in (A + B)x\}$. Many practical nonlinear problems arising in applied sciences such as in image recovery, signal processing and machine learning can be formulated as this problem (see [1–3]). Moreover, this problem includes the core of many mathematical problems, as special cases, such as: variational inequalities, split feasibility problem, minimization problem, Nash equilibrium problem in noncooperative games and so on (see [4–6]).

A well-known method for approximating a solution of the problem (1) is the *forward–backward splitting algorithm* which was introduced in [7,8]. This

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method is defined in a Hilbert space H by $x_1 \in H$ and

$$x_{n+1} = (I + \lambda B)^{-1}(I - \lambda A)x_n, \quad \forall n \geq 1, \tag{2}$$

where $\lambda > 0$ and I denotes the identity operator of H. Note that the operators $(I + \lambda B)^{-1}$ and $I - \lambda A$ are usually called the *backward operator* and the *forward operator*, respectively. It is also known that, when $A = 0$, this method becomes the *proximal point algorithm* which was considered and studied in [9]. In fact, under appropriate conditions, the sequence generated by (2) converges weakly to a solution of the problem (1).

Based on Halpern-type iteration, Takahashi et al. [10] proposed the following modified forward–backward splitting algorithm in a Hilbert space H: for given $x_1, u \in H$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J^B_{\lambda_n}(x_n - \lambda_n A x_n)), \quad \forall n \geq 1, \tag{3}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$ is the resolvent of a maximal monotone operator B, A is an α-inverse strongly monotone mapping, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$. They proved that the sequence $\{x_n\}$ generated by (3) converges strongly to a solution of the problem (1) under appropriate conditions.

Later, López et al. [6] extended the Halpern-type forward–backward splitting method (3) to a q-uniformly smooth and uniformly convex Banach spaces E. They developed an iterative scheme with errors a_n and b_n in the following way: for given $x_1, u \in E$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J^B_{\lambda_n}(x_n - \lambda_n(Ax_n + a_n)) + b_n), \quad \forall n \geq 1, \tag{4}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$ is the resolvent of an m-accretive operator B, A is an α-inverse strongly accretive mapping, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$. They also proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a solution of the problem (1) under some appropriate conditions.

In 2016, Cholamjiak [11] introduced the following generalized forward–backward splitting method for solving the problem (1) in a q-uniformly smooth and uniformly convex Banach spaces E, with A is an α-inverse strongly accretive operator and B is an m-accretive operator: for any $u, x_1 \in E$ and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \delta_n J^B_{\lambda_n}(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{5}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1$. It was shown that the sequence $\{x_n\}$ generated by (5) converges strongly to an element in $(A + B)^{-1}0$.

In recent years, the forward–backward splitting algorithm for solving the monotone quasi-inclusion problem has been studied and extended by numerous authors in different styles (see [12–15]).

On the other hand, we study the following *generalized monotone quasi-inclusion problem*:

$$\text{find } z \in E \text{ such that } 0 \in \bigcap_{i=1}^{N} (A_i + B_i)z, \tag{6}$$

where $A_i : E \to E$ and $B_i : E \multimap E$, $i = 1, 2, \ldots, N$ are finite family of single and set-valued operators, respectively. It is well-known that if $N = 1$, then the problem (6) becomes the monotone quasi-inclusion problem (1).

Very recently, Chang et al. [12] proposed the following strong convergence theorem of the splitting algorithm for finding a common zero of the problem (6) in a q-uniformly smooth and uniformly convex Banach space.

Theorem 1.1: *Let E be a q-uniformly smooth and uniformly convex Banach space. For each $i = 1, 2, \ldots, N$, let $A_i : E \to E$ be an α-inverse strongly accretive of order q and $B_i : E \multimap E$ be an m-accretive operator such that $\Omega := \bigcap_{i=1}^{N} (A_i + B_i)^{-1}0 \neq \emptyset$. Let $g : E \to E$ be a ρ-contractive mapping with $\rho \in (0, 1/q)$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E$ and*

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) S_\lambda x_n, \quad \forall n \geq 1, \tag{7}$$

where

$$S_\lambda := \beta_0 I + \beta_1 J_\lambda^{B_1}(I - \lambda A_1) + \beta_2 J_\lambda^{B_2}(I - \lambda A_2) + \cdots + \beta_N J_\lambda^{B_N}(I - \lambda A_N)$$

in which $\{\beta_i\}_{i=0}^{N} \subset (0, 1)$ with $\sum_{i=0}^{N} \beta_i = 1$ and $J_\lambda^{B_i} := (I + \lambda B_i)^{-1}$ for $i = 1, 2, \ldots, N$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $0 < \lambda \leq (\alpha q/\kappa_q)^{1/q-1}$, *where κ_q is the q-uniform smoothness coefficient of E,*

then $\{x_n\}$ defined by (7) converges strongly to a point $z = Q_\Omega g(z)$ which is a solution of the problem (6), where Q_Ω is a sunny nonexpansive retraction of E onto Ω.

Motivated by the previous works, it is noteworthy to mention the following questions:

(1) Can we extend Theorem 1.1 originally proposed by Chang et al. [12] to a general reflexive Banach space?
(2) Can we remove the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and relax the condition $0 < \lambda \leq (\alpha q/\kappa_q)^{1/q-1}$ in Theorem 1.1?
(3) Most of all the methods for solving the monotone quasi-inclusion problem used the norm distance function. Can we extend those works by use the Bregman distance function?

The purpose of this paper is to provide affirmative answers to the above questions. Thus we propose a new forward–backward splitting method for solving the generalized monotone quasi-inclusion problem (6) with A_i, $i = 1, 2, \ldots, N$ is a finite family of Bregman inverse strongly monotone mappings and B_i, $i = 1, 2, \ldots, N$ is a finite family of maximal monotone mappings. Based on Bregman distance function, we prove a strong convergence result of the proposed algorithm in reflexive Banach spaces.

This paper is organized as follows: In Section 2, we collect some definitions and preliminaries that will be needed throughout the paper. In Section 3, we propose a strong convergence of the proposed algorithm. In Section 4, we provide an application of our main result to the variational inequality problem in reflexive Banach spaces. Finally, in Section 5, some numerical examples to demonstrate our algorithm performance are provided.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers, respectively. Let E be a real reflexive Banach space with its dual space E^*. We write $\langle x^*, x\rangle$ for the value of a functional x^* in E^* at x in E, that is, $\langle x^*, x\rangle = x^*(x)$. We denote by $x_n \rightharpoonup x$ the weak convergence of $\{x_n\}$ to x and $x_n \to x$ the strong convergence of $\{x_n\}$ to x. Let $f : E \to (-\infty, \infty]$ be a function. We denote by dom$f$ the domain of f, that is, $\mathrm{dom}f = \{x \in E : f(x) < \infty\}$. A function $f : E \to (-\infty, \infty]$ is said to be *proper* if $\mathrm{dom}f \neq \emptyset$. It is also said to be lower semicontinuous if the set $\{x \in E : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$. The function f is also said to be *convex* if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in E$ and $\alpha \in [0, 1]$. For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, the *subdifferential* of f defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \quad \forall y \in E\},\ x \in E.$$

The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, \infty]$ defined by

$$f^*(x^*) = \sup_{x\in E}\{\langle x^*, x\rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x^*, x\rangle$.

For any $x \in \mathrm{int}(\mathrm{dom}f)$ and $y \in E$, the *directional derivative* of a convex function f at x in the direction $y \in E$ given by

$$f'_+(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{8}$$

The function f is said to be *Gâteaux differentiable* at x if the limit (8) exists for each y. In this case, $f'_+(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \mathrm{int}(\mathrm{dom}f)$. The function f is said to be *Fréchet differentiable* at x if the limit (8) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (8) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (Fréchet differentiable, respectively), then f is continuous and ∇f is norm-to-weak* continuous (continuous, respectively) (see [16,17]).

A proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$ is said to be:

(1) *essentially smooth* if ∂f is both locally bounded and single-valued on its domain;
(2) *essentially strictly convex* if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom$f$;
(3) *Legendre* if it is both essentially smooth and essentially strictly convex.

For a Legendre function f on a reflexive Banach space E, we have the following properties:

(i) f is Legendre if and only if f^* is Legendre (see [[17], Corollary 5.5]);
(ii) $(\partial f)^{-1} = \partial f^*$ (see [[17], p. 83]);
(iii) ∇f is a bijection, and it satisfies

$$\nabla f = (\nabla f^*)^{-1}, \mathrm{ran}\nabla f = \mathrm{dom}\nabla f^* = \mathrm{int}(\mathrm{dom}\, f^*) \quad \text{and}$$
$$\nabla f^* = \mathrm{dom}\nabla f = \mathrm{int}(\mathrm{dom} f)$$

(see [[17], Theorem 5.10]).

Several interesting examples of Legendre functions are presented in [18]. One important and interesting example of a Legendre function is $f(x) = 1/p\|x\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f is coincident with the *generalized duality mapping* J_p $(1 < p < \infty)$ which is given by

$$J_p(x) = \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}, \quad \forall x \in E.$$

In particular, $J_2 = J$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable function. The function $D_f : \mathrm{dom} f \times \mathrm{int}(\mathrm{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y\rangle \tag{9}$$

is called *Bregman distance respect to* f [19]. We remark that the Bregman distance is like to a metric but does not satisfy the triangle inequality nor symmetry. Clearly, $D_f(x, x) = 0$, but $D_f(x, y) = 0$ does not imply $x = y$. In this case, when f is Legendre, this indeed holds (see [[18], Lemma 7.3 (vi)]).

Remark 2.1: If E is a uniformly smooth and uniformly convex Banach space and $f = \frac{1}{2}\|\cdot\|^2$, then $\nabla f = J$ and we have

$$\begin{aligned} D_{\frac{1}{2}\|\cdot\|^2}(x,y) &= \frac{1}{2}(\|x\|^2 - \|y\|^2 - 2\langle x-y, J(y)\rangle) \\ &= \frac{1}{2}(\|x\|^2 + \|y\|^2 - 2\langle x, J(y)\rangle) \\ &= \frac{1}{2}\phi(x,y) \end{aligned}$$

for all $x, y \in E$. Such a ϕ is called the *Lyapunov function* which was studied in [20,21]. Also, if E is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$ for all $x, y \in E$.

The Bregman distance has the following two important properties called the *three-point identity* and the *four-point identity*, respectively: for any $x \in \mathrm{dom}f$ and any $y, z \in \mathrm{int}(\mathrm{dom}f)$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle, \tag{10}$$

and for any $y, \omega \in \mathrm{dom}f$ and any $x, z \in \mathrm{int}(\mathrm{dom}f)$,

$$D_f(y,x) - D_f(y,z) - D_f(\omega,x) + D_f(\omega,z) = \langle \nabla f(z) - \nabla f(x), y - \omega\rangle. \tag{11}$$

Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable convex function. The *modulus of total convexity* of f at $x \in \mathrm{dom}f$ is the function $\nu_f(x,\cdot) : [0,\infty) \to [0,\infty]$ defined by

$$\nu_f(x,t) = \inf\{D_f(y,x) : y \in \mathrm{dom}f, \|y-x\| = t\}.$$

The function f is called *totally convex at* x if $\nu_f(x,t) > 0$, whenever $t > 0$. The function f is called *totally convex* if it is totally convex at any point $x \in \mathrm{int}(\mathrm{dom}f)$ and is said to be *totally convex on bounded sets* if $\nu_f(K,t) > 0$ for any nonempty bounded subset K of E and $t > 0$, where the modulus of total convexity of the function f on the set K is the function $\nu_f : \mathrm{int}(\mathrm{dom}f) \times [0,\infty) \to [0,\infty]$ defined by

$$\nu_f(K,t) = \inf\{\nu_f(x,t) : x \in K \cap \mathrm{dom}f\}.$$

The function f is called *uniformly convex on bounded subsets of* E if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0,\infty) \to [0,\infty)$ defined by

$$\rho_r(t) = \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

where $B_r = \{x \in E : \|x\| \leq r\}$ for all $r > 0$. It is well-known that f is uniformly convex on bounded sets if and only if f is totally convex on bounded sets (see [22], Theorem 2.10). For a deeper information on totally convex and uniformly convex functions, we refer the reader to [23,24].

A *Bregman projection with respect to* f of $x \in \text{int}(\text{dom}f)$ onto the nonempty, closed and convex subset of dom$f$ is the unique vector $P_C^f(x) \subset C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \quad (12)$$

It can be characterized by the following variational inequality [25,26]:

$$\langle \nabla f(x) - \nabla f(P_C^f(x)), y - P_C^f(x) \rangle \leq 0, \quad \forall y \in C. \quad (13)$$

Moreover, we have

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall y \in C. \quad (14)$$

Remark 2.2: If E is a uniformly convex and uniformly smooth Banach space and $f = \frac{1}{2}\|\cdot\|^2$, then P_C^f coincides with the generalized projection Π_C [27], and if E is a Hilbert space, then P_C^f coincides the metric projection P_C.

Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function. Following [20,28], we make use of the function $V_f : E \times E^* \to [0, \infty)$ which is given by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*. \quad (15)$$

Clearly, V_f has the following properties:

$$D_f(x, \nabla f^*(x^*)) = V_f(x, x^*), \quad \forall x \in E,\ x^* \in E^* \quad (16)$$

and

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad \forall x \in E,\ x^*, y^* \in E^*. \quad (17)$$

We also known that if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, \infty]$ is a proper weak* lower semicontinuous convex function (see [29]). Hence V_f is convex in the second variable. Thus for all $v \in E$,

$$D_f\left(v, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(v, x_i), \quad (18)$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N$ with $\sum_{i=1}^N t_i = 1$.

Let C be a nonempty subset of a reflexive Banach space E. A point $x \in C$ is a fixed point of T if $x = Tx$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. A point z in C is said to be an *asymptotic fixed point* [21] of T if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup z$ as $n \to \infty$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the asymptotic fixed point set of T by $\widehat{F}(T)$.

Let C be a nonempty subset of int(dom$f$). Recall that a mapping $T : C \to$ int(dom$f$) is said to be:

(1) *Bregman quasi-nonexpansive* (*BQNE*) if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C,\ p \in F(T);$$

(2) *Bregman relatively nonexpansive* (*BRNE*) if $F(T) \neq \emptyset, \widehat{F}(T) = F(T)$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C,\ p \in F(T);$$

(3) *Bregman strongly nonexpansive* (*BSNE*) with $\widehat{F}(T) \neq \emptyset$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C,\ p \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that $\lim_{n\to\infty} D_f(x_n, Tx_n) = 0$.

Remark 2.3: From the definitions, it is obvious that if $\widehat{F}(T) = F(T) \neq \emptyset$, then BSNE $\Rightarrow$ BRNE $\Rightarrow$ BQNE.

Lemma 2.4 ([30]): *Let $f : E \to \mathbb{R}$ be a Legendre function and let C be a nonempty, closed and convex subset of E. If $T : C \to E$ is a BQNE operator, then $F(T)$ is closed and convex.*

Lemma 2.5: *Let $f : E \to \mathbb{R}$ be a Legendre function. Let $\{T_i\}_{i=1}^N : E \to E$ be a BQNE mapping such $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\beta_i\}_{i=0}^N \subset (0, 1)$ satisfy $\sum_{i=0}^N \beta_i = 1$. Define a mapping $S : E \to E$ by $Sx := \nabla f^*(\beta_0 \nabla f(x) + \sum_{i=1}^N \beta_i \nabla f(T_i x))$ for all $x \in E$. Then S is BQNE such that $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Proof: Let $x \in E$ and $z \in F(T_i)$ for $i = 1, 2, \ldots, N$. From (18), it follows that

$$\begin{aligned}
D_f(z, Sx) &= D_f(z, \beta_0 \nabla f^*(\nabla f(x) + \sum_{i=1}^N \beta_i \nabla f(T_i x))) \\
&\leq \beta_0 D_f(z, x) + \sum_{i=1}^N \beta_i D_f(z, T_i x) \\
&\leq \beta_0 D_f(z, x) + \sum_{i=1}^N \beta_i D_f(z, x) \\
&= D_f(z, x).
\end{aligned}$$

Hence S is a BQNE mapping. Since ∇f is a bijection, we have

$$\nabla f(Sx) - \nabla f(x) = \sum_{i=1}^N \beta_i \nabla f(T_i x) - (1 - \beta_0) \nabla f(x)$$

$$= \sum_{i=1}^{N} \beta_i \nabla f(T_i x) - \sum_{i=1}^{N} \beta_i \nabla f(x)$$

$$= \sum_{i=1}^{N} \beta_i (\nabla f(T_i x) - \nabla f(x)). \tag{19}$$

If $x \in \bigcap_{i=1}^N F(T_i)$, then we obtain from (19) that $\nabla f(Sx) - \nabla f(x) = 0$. This implies $x = Sx$, that is, $x \in F(S)$. So $\bigcap_{i=1}^N F(T_i) \subset F(S)$. To prove the reverse conclusion, let $x \in \bigcap_{i=1}^N F(T_i)$, $y \in E$ and $z = T_i y$ with any $i = 1, 2, \dots, N$. From (10), we have

$$\begin{aligned}\beta_i \langle \nabla f(T_i y) - \nabla f(y), x - y \rangle &= \beta_i \big(D_f(x, y) - D_f(x, T_i y) + D_f(y, T_i y) \big) \\ &\geq \beta_i \big(D_f(x, y) - D_f(x, y) + D_f(y, T_i y) \big) \\ &= \beta_i D_f(y, T_i y). \end{aligned} \tag{20}$$

Taking any $y \in F(S)$. It follows from (19) that $\sum_{i=1}^N \beta_i (\nabla f(T_i y) - \nabla f(y)) = 0$. Thus by (20), we have

$$0 = \sum_{i=1}^{N} \beta_i \langle \nabla f(T_i y) - \nabla f(y), x - y \rangle \geq \sum_{i=1}^{N} \beta_i D_f(y, T_i y).$$

This implies that $D_f(y, T_i y) = 0$ for all $i = 1, 2, \dots, N$. Since f is Legendre, we have $T_i y = y$ for all $i = 1, 2, \dots, N$. So $y \in \bigcap_{i=1}^N F(T_i)$. Therefore $F(S) = \bigcap_{i=1}^N F(T_i)$. ■

For a set-valued operator $A : E \multimap E^*$, we define its domain, range and graph as follows:

$$\text{dom } A = \{x \in E : Ax \neq \emptyset\},$$

$$\text{ran } A = \bigcup \{Ax : x \in \text{dom } A\}$$

and

$$G(A) = \{(x^*, x) \in E^* \times E : x^* \in Ax\}.$$

An operator A is said to be *monotone* if for each $(x^*, x), (y^*, y) \in G(A)$,

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

A monotone operator A is said to be *maximal*, if its graph is not contained in the graph of any other monotone operators on E. It is known that if A is maximal monotone, then the set $A^{-1}0 = \{x \in E : 0 \in Ax\}$ is closed and convex.

Let $f : E \to (-\infty, \infty]$ be a Gâteaux differentiable convex function and $A : E \multimap E^*$ be a maximal monotone operator. Then we can define the resolvent of

A corresponding to f by $Res^f_{\lambda A} = (\nabla f + \lambda A)^{-1} \circ \nabla f$. It is known that $Res^f_{\lambda A}$ is single-valued and $F(Res^f_{\lambda A}) = A^{-1}0$ [31]. The *Yosida approximation* $A_\lambda : E \to E^*$, associated with A for $\lambda > 0$, is the mapping defined by

$$A_\lambda(x) = \frac{1}{\lambda}\big(\nabla f(x) - \nabla f\big(Res^f_{\lambda A}(x)\big)\big), \quad \forall x \in E. \tag{21}$$

From [[25], Proposition 2.7], we know that $(Res^f_{\lambda A}(x), A_\lambda(x)) \in G(A)$ and $0 \in Ax$ if and only if $0 \in A_\lambda(x)$ for all $x \in E$ and $\lambda > 0$. A mapping $A : E \multimap E^*$ satisfying $\mathrm{ran}(\nabla f - \lambda A) \subset \mathrm{ran}(\nabla f)$ is called *Bregman inverse strongly monotone* if dom A $\cap$ int(dom$f$)$\neq \emptyset$ and for any $x, y \in$ int(dom$f$) and each $u \in Ax, v \in Ay$,

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0. \tag{22}$$

Remark 2.6: The class of BISM is more general than the class of inverse strongly monotone in Hilbert spaces. Indeed, if E is a Hilbert space and $f = \frac{1}{2}\|\cdot\|^2$, then the class of BISM becomes the class of inverse strongly monotone [25].

For any operator $A^f : E \multimap E$ associated with A for $\lambda > 0$ is defined by

$$A^f = \nabla f^* \circ (\nabla f - \lambda A). \tag{23}$$

Note that dom $A^f \subset$ dom$A$ $\cap$ int(dom$f$) and ran $A^f \subset$ int(dom$f$). It is known that the operator A is Bregman inverse strongly monotone if and only if A^f is a single-valued mapping (see [[32], Lemma 3.5(c) and (d), p. 2109]).

Lemma 2.7: *Let $A : E \to E^*$ be a BISM mapping and $B : E \multimap E^*$ be a maximal monotone operator. Define a mapping $T_\lambda x := Res^f_{\lambda B} \circ A^f(x)$ for $x \in E$ and $\lambda > 0$. Hence $F(T_\lambda) = (A+B)^{-1}0$.*

Proof: Let $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned}
x = T_\lambda x &\Leftrightarrow x = Res^f_{\lambda B} \circ A^f(x) \\
&\Leftrightarrow x = (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda A)x) \\
&\Leftrightarrow x = (\nabla f + \lambda B)^{-1} \circ (\nabla f - \lambda A)x) \\
&\Leftrightarrow \nabla f(x) - \lambda Ax \in \nabla f(x) + \lambda Bx \\
&\Leftrightarrow 0 \in \lambda(A+B)x \\
&\Leftrightarrow 0 \in (A+B)x \\
&\Leftrightarrow x \in (A+B)^{-1}0.
\end{aligned}$$

Hence $F(T_\lambda) = (A+B)^{-1}0$. ∎

Lemma 2.8 (**[33]**)**:** *Let $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A : E \to E^*$ be a BISM mapping and $B : E \multimap E^*$ be a maximal monotone operator. Then the following hold:*

(i) $D_f(z, Res^f_{\lambda B} \circ A^f(x)) + D_f(Res^f_{\lambda B} \circ A^f(x), x) \leq D_f(z, x)$ for all $z \in (A + B)^{-1}0$, $x \in E$ and $\lambda > 0$;
(ii) $Res^f_{\lambda B} \circ A^f$ *is a BRNE mapping.*

Lemma 2.9 (**[25]**)**:** *Let $f : E \to \mathbb{R}$ be a totally convex function. Suppose that $x \in E$, if $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.10 (**[34]**)**:** *Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then, $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.11 (**[35]**)**:** *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1,$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 (**[36]**)**:** *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_\ell}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_\ell} < \Gamma_{n_\ell+1}$ for all $\ell \in \mathbb{N}$. Define the sequence $\{\sigma(n)\}_{n\geq n_0}$ of integers as follows:*

$$\sigma(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\sigma(n_0) \leq \sigma(n_0 + 1) \leq \ldots$ *and* $\sigma(n) \to \infty$;
(ii) $\Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\sigma(n)+1}$, $\forall n \geq n_0$.

3. Main result

First, we have the following main theorem.

Theorem 3.1: *Let E be a real reflexive Banach space. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A_i : E \to E^*$, $i = 1, 2, \ldots, N$ be a*

BISM mapping and $B_i : E \multimap E^*$, $i = 1, 2, \ldots, N$ be a maximal monotone operator. Suppose that $\Omega := \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) S_\lambda x_n), \quad \forall n \geq 1, \tag{24}$$

where

$$\begin{aligned} S_\lambda := \beta_0 \nabla f + \beta_1 \nabla f(Res^f_{\lambda B_1} \circ A^f_1) \\ + \beta_2 \nabla f(Res^f_{\lambda B_2} \circ A^f_2) + \cdots + \beta_N \nabla f(Res^f_{\lambda B_N} \circ A^f_N) \end{aligned} \tag{25}$$

in which $\{\beta_i\}_{i=0}^{N} \subset (0, 1)$ with $\sum_{i=0}^{N} \beta_i = 1$, $\{\alpha_n\} \subset (0, 1)$ and $\lambda > 0$. Suppose that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = \Pi^f_\Omega(u)$.

Proof: Put $T^i_\lambda := Res^f_{\lambda B_i} \circ A^f_i$ for $i = 1, 2, \ldots, N$. Clearly, $F(T^i_\lambda) = (A_i + B_i)^{-1}0$ for each $i = 1, 2, \ldots, N$. From definition of S_λ, we can write $\{x_n\}$ as

$$\begin{aligned} y_n &= \nabla f^*(\beta_0 \nabla f(x_n) + \textstyle\sum_{i=1}^{N} \beta_i \nabla f(T^i_\lambda x_n)), \\ x_{n+1} &= \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(y_n)). \end{aligned} \tag{26}$$

Since T^i_λ is a BQNE operator for each $i = 1, 2, \ldots, N$. Thus by Lemma 2.5, we have

$$F(S_\lambda) = \bigcap_{i=1}^{N} F(T^i_\lambda) = \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0. \tag{27}$$

Let $v \in \Omega := \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 = \bigcap_{i=1}^{N} F(T^i_\lambda)$. Thus by (18), we see that

$$\begin{aligned} D_f(v, y_n) &= D_f(v, \nabla f^*(\beta_0 \nabla f(x_n) + \sum_{i=1}^{N} \beta_i \nabla (T^i_\lambda x_n))) \\ &\leq \beta_0 D_f(v, x_n) + \sum_{i=1}^{N} \beta_i D_f(v, T^i_\lambda x_n) \\ &\leq \beta_0 D_f(v, x_n) + \sum_{i=1}^{N} \beta_i D_f(v, x_n) \\ &= D_f(v, x_n). \end{aligned} \tag{28}$$

It follows that

$$\begin{aligned} D_f(v, x_{n+1}) &= D_f(v, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(y_n))) \\ &\leq \alpha_n D_f(v, u) + (1 - \alpha_n) D_f(v, y_n) \end{aligned}$$

$$\leq \alpha_n D_f(v,u) + (1-\alpha_n) D_f(v,x_n)$$
$$\leq \max\{D_f(v,u), D_f(v,x_n)\}. \tag{29}$$

By induction, we obtain

$$D_f(v,x_n) \leq \max\{D_f(v,u), D_f(v,x_1)\}, \quad \forall n \geq 1. \tag{30}$$

This shows that $\{D_f(v,x_n)\}$ is bounded. Since f is totally convex, it follows from Lemma 2.9 that $\{x_n\}$ is bounded. Let $z = \Pi^f_\Omega(u)$. By Lemma 2.8 (i), we have

$$\begin{aligned} D_f(z,y_n) &\leq \beta_0 D_f(z,x_n) + \sum_{i=1}^N \beta_i D_f(z, T^i_\lambda x_n) \\ &\leq \beta_0 D_f(z,x_n) + \sum_{i=1}^N \beta_i (D_f(z,x_n) - D_f(T^i_\lambda x_n, x_n)) \\ &= D_f(z,x_n) - \sum_{i=1}^N \beta_i D_f(T^i_\lambda x_n, x_n). \end{aligned} \tag{31}$$

It follows that

$$\begin{aligned} D_f(z,x_{n+1}) &\leq \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z,y_n) \\ &\leq \alpha_n D_f(z,u) + (1-\alpha_n)[D_f(z,x_n) - \sum_{i=1}^N \beta_i D_f(T^i_\lambda x_n, x_n)] \\ &= \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z,x_n) - (1-\alpha_n) \sum_{i=1}^N \beta_i D_f(T^i_\lambda x_n, x_n), \end{aligned} \tag{32}$$

which implies that

$$(1-\alpha_n) \sum_{i=1}^N \beta_i D_f(T^i_\lambda x_n, x_n) \leq D_f(z,x_n) - D_f(z,x_{n+1}) + \alpha_n M, \tag{33}$$

where $M = \sup_{n\in\mathbb{N}}\{|D_f(z,u) - D_f(z,x_n)|\}$.

The rest of the our proof will be divided into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(z,x_n)\}_{n\geq n_0}$ is non-increasing. So, we have $\lim_{n\to\infty} D_f(z,x_n)$ exists. This implies that

$$D_f(z,x_n) - D_f(z,x_{n+1}) \to 0. \tag{34}$$

By our assumptions, it follows from (33) that

$$D_f(T^i_\lambda x_n, x_n) \to 0 \tag{35}$$

for $i = 1, 2, \ldots, N$. Moreover, we have

$$D_f(y_n, x_n) \le \beta_0 D_f(x_n, x_n) + \sum_{i=1}^{N} \beta_i D_f(T_\lambda^i x_n, x_n) \to 0. \tag{36}$$

By Lemma 2.10, we also have

$$\|T_\lambda^i x_n - x_n\| \to 0 \tag{37}$$

for $i = 1, 2, \ldots, N$ and

$$\|y_n - x_n\| \to 0. \tag{38}$$

Since f is uniformly Fréchet differentiable, it follows that ∇f is uniformly continuous on bounded subset of E (see [[37], Theorem 1.8, p. 13]). Thus we have

$$\|\nabla f(y_n) - \nabla f(x_n)\| \to 0 \tag{39}$$

and hence

$$\begin{aligned} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| &\le \|\nabla f(x_{n+1}) - \nabla f(y_n)\| + \|\nabla f(y_n) - \nabla f(x_n)\| \\ &\le \alpha_n \|\nabla f(u) - \nabla f(y_n)\| + \|\nabla f(y_n) - \nabla f(x_n)\| \\ &\to 0. \end{aligned} \tag{40}$$

Since f is strongly coercive and uniformly convex on bounded sets, it follows that ∇f^* is uniformly continuous on bounded subset of E^* (see [[38], Proposition 3.6.4]). Thus we have

$$\|x_{n+1} - x_n\| \to 0. \tag{41}$$

By the reflexivity of a Banach space E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ as $k \to \infty$ and

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_n - z \rangle = \lim_{k\to\infty} \langle \nabla f(u) - \nabla f(z), x_{n_k} - z \rangle. \tag{42}$$

From (37), we note that $\|T_\lambda^i x_{n_k} - x_{n_k}\| \to 0$ for each $i = 1, 2, \ldots, N$. Hence $\hat{x} \in \widehat{F}(T_\lambda^i) = F(T_\lambda^i)$ for each $i = 1, 2, \ldots, N$. This implies that $\hat{x} \in \bigcap_{i=1}^N F(T_\lambda^i) = \bigcap_{i=1}^N (A_i + B_i)^{-1}0$. So

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_n - z \rangle \le 0. \tag{43}$$

From (41), we also have

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_{n+1} - z \rangle \le 0. \tag{44}$$

Finally, we show that $x_n \to z$. From (17) and (28), we have

$$\begin{aligned}
&D_f(z, x_{n+1}) \\
&\quad = D_f(z, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\
&\quad = V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\
&\quad \le V_f\big(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n) - \alpha_n(\nabla f(u) - \nabla f(z))\big) \\
&\qquad + \langle \alpha_n(\nabla f(u) - \nabla f(z)), x_{n+1} - z\rangle \\
&\quad = V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(y_n) + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\
&\quad \le \alpha_n V_f(z, \nabla f(z)) + (1-\alpha_n) V_f(z, \nabla f(y_n)) + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\
&\quad = \alpha_n D_f(z, z) + (1-\alpha_n) D_f(z, y_n) + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\
&\quad \le (1-\alpha_n) D_f(z, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle.
\end{aligned} \tag{45}$$

Thus by Lemma 2.11, we can conclude that $\lim_{n\to\infty} D_f(z, x_n) = 0$. Therefore, $x_n \to z$.

Case 2. Suppose that there exists a subsequence $\{n_\ell\}$ of $\{n\}$ such that $D_f(z, x_{n_\ell}) < D_f(z, x_{n_\ell+1})$ for all $\ell \in \mathbb{N}$. Define a positive integer sequence $\{\sigma(n)\}$ by

$$\sigma(n) = \max\{k \le n : D_f(z, x_k) < D_f(z, x_{k+1})\} \tag{46}$$

for all $n \ge n_0$ (for some n_0 large enough). By Lemma 2.12, we have $\{\sigma(n)\}$ is a non-decreasing sequence such that $\sigma_n \to \infty$ as $n \to \infty$ and

$$D_f(z, x_{\sigma(n)}) - D_f(z, x_{\sigma(n)+1}) \le 0. \tag{47}$$

From (33), we have

$$D_f(T_\lambda^i x_{\rho(n)}, x_{\rho(n)}) \to 0 \tag{48}$$

for $i = 1, 2, \dots, N$. By the similar way proposed in **Case 1**, we can conclude that

$$\|T_\lambda^i x_{\rho(n)} - x_{\rho(n)}\| \to 0 \tag{49}$$

for $i = 1, 2, \dots, N$ and

$$\|x_{\rho(n)+1} - x_{\rho(n)}\| \to 0. \tag{50}$$

Furthermore, we can also show that

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(z), x_{\sigma(n)+1} - z\rangle \le 0 \tag{51}$$

and

$$D_f(z, x_{\sigma(n)+1}) \le (1-\alpha_{\sigma(n)}) D_f(z, x_{\sigma(n)}) + \alpha_{\sigma(n)} \langle \nabla f(u) - \nabla f(z), x_{\sigma(n)+1} - z\rangle. \tag{52}$$

Using Lemma 2.11 again, we obtain $\lim_{n\to\infty} D_f(z, x_{\sigma(n)}) = 0$. We also have

$$\lim_{n\to\infty} D_f(z, x_{\sigma(n)+1}) = \lim_{n\to\infty} D_f(z, x_{\sigma(n)}) = 0. \tag{53}$$

By Lemma 2.12, we see that

$$D_f(z, x_n) \le D_f(z, x_{\sigma(n)+1}). \tag{54}$$

This implies that $\lim_{n\to\infty} D_f(z, x_n) = 0$. Therefore, $x_n \to z$. This completes the proof. ∎

Remark 3.2: We remark that our work generalizes and improves many works in the following ways:

(1) Our result generalizes the results of [6,11,13] from the problem of finding a zero of the sum of two monotone operators in a q-uniformly smooth and uniformly convex Banach space to the problem of finding a common zero of the sum of a finite family of monotone operators in reflexive Banach spaces.
(2) The method of proof of our result is very different from the method of proof of [6,11–13,15] in the sense that we use the Bregman distance function while mentioned works used the norm distance function.
(3) The results of Chang et al. [[12], Theorem 3.1] and Chang et al. [[13], Theorem 3.1] always assume that λ satisfies the condition $0 < \lambda \le (\alpha q/\kappa_q)^{1/q-1}$, where κ_q is the q-uniform smoothness coefficient of E (see [39] for more detail). However, our result is proved without the strict assumption $\lambda \le (\alpha q/\kappa_q)^{1/q-1}$. Furthermore, we can remove the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ in mentioned works.

If we put $A_i = 0$ for all $i = 1, 2, \dots, N$, then we have the following generalized proximal point algorithm in reflexive Banach spaces.

Corollary 3.3: *Let E be a real reflexive Banach space. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $B_i : E \multimap E^*$, $i = 1, 2, \dots, N$ be a maximal monotone operator. Suppose that $\Omega := \bigcap_{i=1}^{N} B_i^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) S_\lambda x_n), \quad \forall n \ge 1, \tag{55}$$

where

$$S_\lambda := \beta_0 \nabla f + \beta_1 \nabla f(Res^f_{\lambda B_1}) + \beta_2 \nabla f(Res^f_{\lambda B_2}) + \cdots + \beta_N \nabla f(Res^f_{\lambda B_N})$$

in which $\{\beta_i\}_{i=0}^{N} \subset (0, 1)$ with $\sum_{i=0}^{N} \beta_i = 1$, $\{\alpha_n\} \subset (0, 1)$ and $\lambda > 0$. Suppose that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = \Pi^f_\Omega(u)$.

If we put $N = 1$, then we have the following result for the classical forward–backward splitting method in reflexive Banach spaces.

Corollary 3.4: *Let E be a real reflexive Banach space. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A : E \to E^*$ be a BISM mapping and $B : E \multimap E^*$ be a maximal monotone operator. Suppose that $\Omega := (A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) S_\lambda x_n), \quad \forall n \geq 1, \tag{56}$$

where

$$S_\lambda := \beta_0 \nabla f + (1-\beta_0) \nabla f(Res^f_{\lambda B} \circ A^f)$$

in which $\beta_0 \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\lambda > 0$. Suppose that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = \Pi^f_\Omega(u)$.

If E is a uniformly convex Banach space which also uniformly smooth and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then we have the following result.

Corollary 3.5: *Let E be a real uniformly convex and uniformly smooth Banach space. Let $A_i : E \to E^*$, $i = 1, 2, \ldots, N$ be a BISM mapping with respect to the functional $f = \frac{1}{2}\|\cdot\|^2$ and $B_i : E \multimap E^*$, $i = 1, 2, \ldots, N$ be a maximal monotone operator. Suppose that $\Omega := \bigcap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n J(u) + (1-\alpha_n) S_\lambda x_n), \quad \forall n \geq 1, \tag{57}$$

where

$$\begin{aligned} S_\lambda := {} & \beta_0 J + \beta_1 J(Q_\lambda^{B_1} J^{-1}(J - \lambda A_1)) \\ & + \beta_2 J(Q_\lambda^{B_2} J^{-1}(J - \lambda A_2)) + \cdots + \beta_N J(Q_\lambda^{B_N} J^{-1}(J - \lambda A_N)) \end{aligned}$$

in which $\{\beta_i\}_{i=0}^N \subset (0,1)$ with $\sum_{i=0}^N \beta_i = 1$, $\{\alpha_n\} \subset (0,1)$ and $\lambda > 0$. Suppose that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = \Pi_\Omega(u)$.

We consequently obtain the following result in a real Hilbert space.

Corollary 3.6: *Let H be a real Hilbert space. Let $A_i : H \to H$, $i = 1, 2, \ldots, N$ be an inverse strongly monotone mapping and $B_i : H \multimap H$, $i = 1, 2, \ldots, N$ be a maximal monotone operator. Suppose that $\Omega := \bigcap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in H$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) S_\lambda x_n, \quad \forall n \geq 1, \tag{58}$$

where

$$S_\lambda := \beta_0 I + \beta_1 (I + \lambda B_1)^{-1}(I - \lambda A_1) \\ + \beta_2 (I + \lambda B_2)^{-1}(I - \lambda A_2) + \cdots + \beta_N (I + \lambda B_N)^{-1}(I - \lambda A_N)$$

in which $\{\beta_i\}_{i=0}^N \subset (0,1)$ *with* $\sum_{i=0}^N \beta_i = 1$, $\{\alpha_n\} \subset (0,1)$ *and* $\lambda > 0$. *Suppose that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* $z = P_\Omega(u)$.

4. Application to the variational inequality problem

Let E be a real reflexive Banach space. Let $f : E \to (-\infty, \infty]$ be a Legendre and totally convex function, $A : E \to E^*$ be a BISM mapping and C be a nonempty, closed and convex subset of dom$A$. The *variational inequality problem* (VIP) is to find $z \in C$ such that

$$\langle Az, x - z\rangle \geq 0, \quad \forall x \in C. \tag{59}$$

The set of solutions of VIP is denoted by $VI(C, A)$. Recall that an *indicator function* of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases} \tag{60}$$

It is known that i_C is proper convex, lower semicontinuous and convex function and its subdifferential ∂i_C is maximal monotone (see [40]). Moreover, we know that

$$\partial i_C(x) = \begin{cases} N_C(x), & \text{if } x \in C; \\ \emptyset, & \text{if } x \notin C, \end{cases} \tag{61}$$

where N_C is the normal cone of C given by

$$N_C(x) = \{x^* \in E^* : \langle x^*, y - x\rangle \leq 0, \ \forall y \in C\}. \tag{62}$$

Thus we can define the resolvent associated with ∂i_C for $\lambda > 0$ by

$$Res^f_{\lambda \partial i_C}(x) = (\nabla f + \lambda \partial i_C)^{-1} \circ \nabla f(x), \quad \forall x \in E.$$

So we have for any $x \in E$ and $y \in C$,

$$\begin{aligned} z = Res^f_{\lambda \partial i_C}(x) &\Leftrightarrow \nabla f(x) \in \nabla f(z) + \partial i_C(z) \\ &\Leftrightarrow \nabla f(x) \in \nabla f(z) + \lambda N_C(z) \\ &\Leftrightarrow \nabla f(x) - \nabla f(z) \in \lambda N_C(z) \\ &\Leftrightarrow \frac{1}{\lambda}\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \quad \forall y \in C \end{aligned}$$

$$\Leftrightarrow \langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \quad \forall y \in C$$
$$\Leftrightarrow z = P_C^f(x),$$

where P_C^f is the Bregman projection from E onto C.

Proposition 4.1 ([26]): *Let $f : E \to (-\infty, \infty]$ be a Legendre and totally convex function. Let $A : E \to E^*$ be a BISM mapping. If C is a nonempty, closed and convex subset of* dom $A \cap \text{int}(\text{dom} f)$, *then $VI(C, A) = F(P_C^f \circ A^f)$.*

In fact, if we set $B_i = \partial i_{C_i}$ for all $i = 1, 2, \ldots, N$ in Theorem 3.1, then by Proposition 4.1 we obtain the following result.

Theorem 4.2: *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A_i : E \to E^*$, $i = 1, 2, \ldots, N$ be a BISM mapping. Let C_i,$i = 1, 2, \ldots, N$, be nonempty, closed and convex subsets of E such that $C_i \subseteq$ dom $A_i \cap$ int(dom$f$). Suppose that $\Omega := \bigcap_{i=1}^N VI(C_i, A_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) S_\lambda x_n), \quad \forall n \geq 1, \tag{63}$$

where

$$S_\lambda := \beta_0 \nabla f + \beta_1 \nabla f(P_{C_1}^f \circ A_1^f) + \beta_2 \nabla f(P_{C_2}^f \circ A_2^f) + \cdots + \beta_N \nabla f(P_{C_N}^f \circ A_N^f) \tag{64}$$

in which $\{\beta_i\}_{i=0}^N \subset (0, 1)$ with $\sum_{i=0}^N \beta_i = 1$, $\{\alpha_n\} \subset (0, 1)$ and $\lambda > 0$. Suppose that $\lim_{n\to\infty} \alpha_n = 0$ *and $\sum_{n=1}^\infty \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = \Pi_\Omega^f(u)$.*

5. Numerical experiments

In this section, we provide some numerical experiments which show the efficiency and implementation of the proposed algorithm. The numerical tests are performed in MATLAB R2014a running on an HP Compaq 510, Core(TM) 2 Duo CPU T5870 with 2.0 GHz and 2 GB RAM.

Example 5.1: Consider the minimization problem:

$$\min_{x \in \mathbb{R}^3} h(x),$$

where $h(x) = \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (2, -3, 4)^T x + 1$ for all $x = (w_1, w_2, w_3) \in \mathbb{R}^3$. Let $F(x) = \frac{1}{2}\|x\|_2^2 + (2, -3, 4)^T x + 1$ and $G(x) = \|x\|_1$. This problem is equivalent to the following problem: find an element $x^* \in \mathbb{R}^3$ such that

$$\nabla F(x^*) + \partial G(x^*) \ni 0, \tag{65}$$

where $\nabla F(x) = x + (2, -3, 4)$ and

$$\partial G(x) = \left\{ (\xi_1, \xi_2, \xi_3) : \ \xi_i = \begin{cases} 1 & \text{if } w_i > 0, \\ -1 & \text{if } w_i < 0, \\ [-1, 1] & \text{if } w_i = 0, \end{cases} \quad \text{for all } i = 1, 2, 3 \right\}. \quad (66)$$

Then the element $x^* = (w_1^*, w_2^*, w_3^*)$ is a solution of the problem (65) if and only if

$$\begin{aligned} w_1^* + \xi_1 + 2 &\ni 0, \\ w_2^* + \xi_2 - 3 &\ni 0, \\ w_3^* + \xi_3 + 4 &\ni 0, \end{aligned}$$

where (ξ_1, ξ_2, ξ_3) are defined by (66). This implies that $x^* = (-1, 2, -3)$ is a unique solution of the problem (65) and $\min_{x \in \mathbb{R}^3} h(x) = F(x^*) + G(x^*) = -7$.

It is easy to see that $A = \nabla F$ is BISM and $B = \partial G$ is maximal monotone. From [41], we know that

$$\begin{aligned} J_\lambda^{\partial G}(x) &= (I + \lambda \partial G)^{-1}(x) \\ &= \big(\max\{|w_1| - \lambda, 0\} \operatorname{sgn}(w_1), \max\{|w_2| \\ &\quad - \lambda, 0\} \operatorname{sgn}(w_2), \max\{|w_3| - \lambda, 0\} \operatorname{sgn}(w_3) \big) \end{aligned}$$

for $\lambda > 0$.

In this numerical experiment, we perform numerical tests of our algorithm and compare it with algorithms of Takahashi et al. [Algorithm (4)] and Cholamjiak [Algorithm (5)]. The parameters in the algorithms are chosen as follows:

- Our algorithm: $\lambda = 1.45, \alpha_n = 1/n, \beta_0 = 0.001, \beta_1 = 0.999$
- Algorithm (3): $\lambda_n = 0.45, \alpha_n = 1/n, \beta_n = 0.8$
- Algorithm (5): $\lambda_n = 0.45, \alpha_n = 1/2n, \beta_n = 0.75 - 1/4n, \delta_n = 0.25 - 1/4n$

We use the stopping rule $\text{TOL}_n = \|x_{n+1} - x_n\| < \varepsilon$ with $\varepsilon = 10^{-6}$ to stop the iterative process. When $x_1 = (-10, 5, 10)$ and $u = (1, -1, -2)$, we obtain the numerical results in Table 1.

The behaviours of TOL_n in Table 1 are illustrated in Figure 1.

Table 1. Table of numerical results for Example 5.1.

Algorithms	TOL_n	n	x_n	$h(x_n)$	Time (s)
Our algorithm	9.99605×10^{-7}	1608	$(-0.99914, 1.99871, -2.99957)$	-6.99999	0.016
Algorithm (3)	9.99985×10^{-7}	2893	$(-0.99845, 1.99768, -2.99922)$	-6.99999	0.031
Algorithm (5)	9.99950×10^{-7}	4083	$(-0.99782, 1.99673, -2.99891)$	-6.99999	0.047

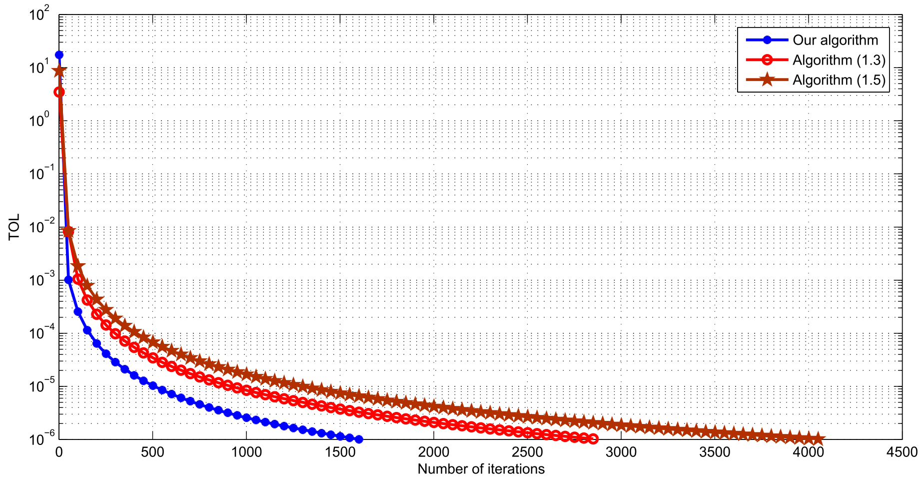

Figure 1. The behaviour of TOL_n in Table 1.

In the last example, we present numerical results of our proposed algorithm to solve the variational inequality problem.

Example 5.2: Let C_i and Q_i, $i = 1, 2, 3$ be nonempty, closed and convex subsets of $\mathbb{R}^4$ such that $\bigcap_{i=1}^3 C_i \neq \emptyset$. We consider the problem of finding an element $x^* \in \mathbb{R}^4$ such that

$$x^* \in \bigcap_{i=1}^{3} \arg\min_{x \in C_i} f_i(x) \neq \emptyset, \tag{67}$$

where $f_i(x) = \|x - P_{Q_i}x\|$ for all $x \in \mathbb{R}^4$ and $i = 1, 2, 3$. It is easy to see that above problem is equivalent to the following problem: find an element $x^* \in \mathbb{R}^4$ such that

$$x^* \in \bigcap_{i=1}^{3} \arg\min_{x \in C_i} g_i(x), \tag{68}$$

where $g_i(x) = \frac{1}{2}\|x - P_{Q_i}x\|^2$ for all $x \in \mathbb{R}^4$ and $i = 1, 2, 3$. We know that an element $x^* \in \mathbb{R}^4$ is a solution of the problem (68) if and only if

$$x^* \in \bigcap_{i=1}^{3} VI(C_i, \nabla g_i), \tag{69}$$

with $\nabla g_i = I - P_{Q_i}$ for all $i = 1, 2, 3$. We now consider the problem (67) with C_i and Q_i, $i = 1, 2, 3$ defined as follows:

$$\begin{aligned}
C_1 &= \{(x_1, x_2, x_3, x_4) : \; x_1 + x_2 - x_3 - x_4 = 1\}, \\
C_2 &= \{(x_1, x_2, x_3, x_4) : \; x_1 - x_2 - x_3 + x_4 = -1\}, \\
C_3 &= \{(x_1, x_2, x_3, x_4) : \; x_1 + 2x_2 - x_3 + 2x_4 = 6\},
\end{aligned}$$

Table 2. Table of numerical results for Example 5.2.

ε	n	TOL_n	x_n	Time (s)
$\lambda = 0.25$				
10^{-5}	2337	9.993713×10^{-6}	(0.017550, 1.999930, 0.010653, 1.013655)	0.125
10^{-6}	7629	9.999805×10^{-7}	(0.005429, 2.000000, 0.003332, 1.004194)	0.421
10^{-7}	24124	9.999752×10^{-8}	(0.001717, 2.000000, 0.001053, 1.001326)	1.359
$\lambda = 2.5$				
10^{-5}	1296	9.998373×10^{-6}	(0.009583, 2.000000, 0.008349, 1.002469)	0.078
10^{-6}	4097	9.999497×10^{-7}	(0.003031, 2.000000, 0.002641, 1.000781)	0.218
10^{-7}	12955	9.999143×10^{-8}	(0.000958, 2.000000, 0.000835, 1.000247)	0.719
$\lambda = 3.75$				
10^{-5}	1246	9.990594×10^{-6}	(0.009128, 2.000000, 0.008272, 1.001712)	0.077
10^{-6}	3938	9.996261×10^{-7}	(0.002888, 2.000000, 0.002617, 1.000541)	0.217
10^{-7}	12450	9.999422×10^{-8}	(0.000913, 2.000000, 0.000827, 1.000171)	0.702

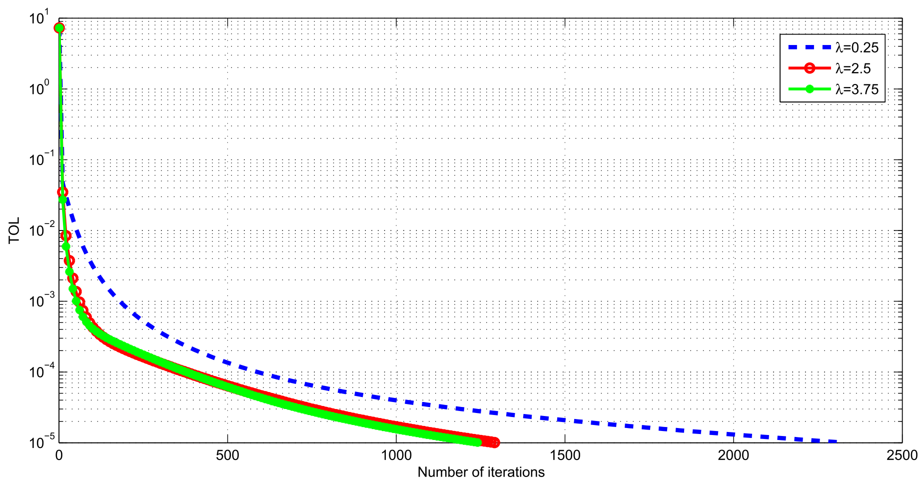

Figure 2. The behaviour of TOL_n with the stopping rule $\text{TOL}_n < 10^{-5}$.

and

$$Q_1 = \{(x_1, x_2, x_3, x_4) : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 9\},$$
$$Q_2 = \{(x_1, x_2, x_3, x_4) : \; (x_1 - 1)^2 + (x_2 + 1)^2 + x_3^2 + x_4^2 \leq 11\},$$
$$Q_3 = \{(x_1, x_2, x_3, x_4) : \; (x_1 + 1)^2 + (x_2 - 1)^2 + (x_3 + 1)^2 + (x_4 - 2)^2 \leq 10\}.$$

Denote by Ω the set of solutions of the problem (67). Putting $C := C_1 \cap C_2 \cap C_3$, it is easy to see that

$$C = \{(t, 2, t, 1) : \; t \in \mathbb{R}\}$$

and $C \cap Q_i \neq \emptyset$ for all $i = 1, 2, 3$. Thus we have

$$\Omega = \{x^* = (t, 2, t, 1) : \; x^* \in Q_i, i = 1, 2, 3\} = \{(t, 2, t, 1) : \; t \in [0, 1]\}.$$

In order to find a solution of the problem (67), we apply Theorem 4.2 for solving the problem (69) with $A_i = \nabla g_i$ for all $i = 1, 2, 3$, $x_1 = (3, 4, 5, 6)$,

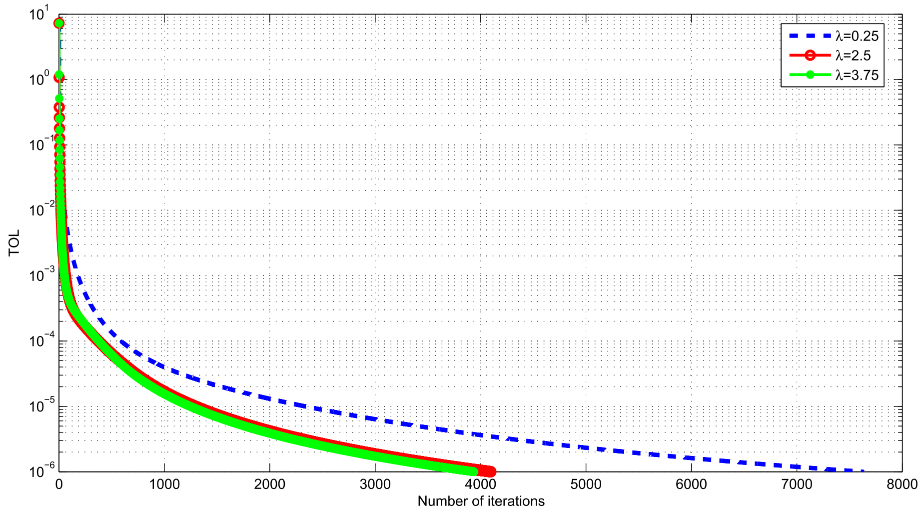

Figure 3. The behaviour of TOL_n with the stopping rule $\mathrm{TOL}_n < 10^{-6}$.

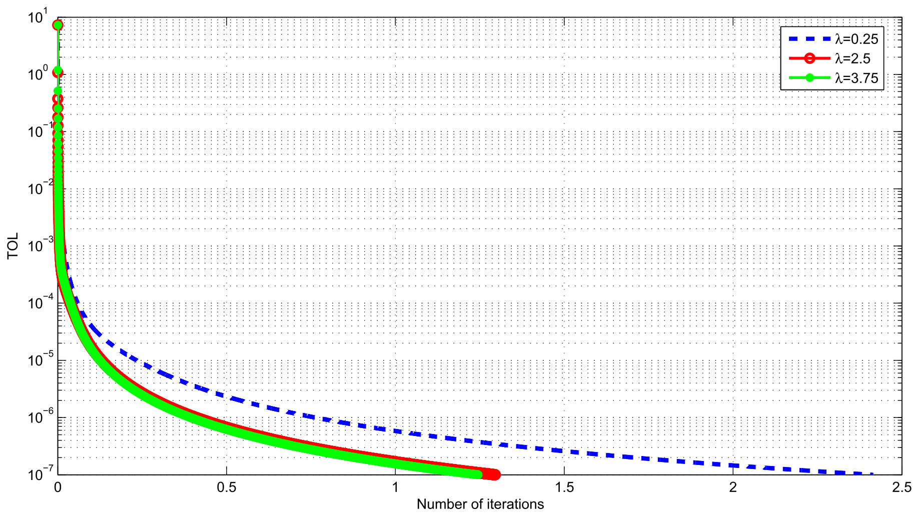

Figure 4. The behaviour of TOL_n with the stopping rule $\mathrm{TOL}_n < 10^{-7}$.

$u = (1, -2, 3, 4)$, $\beta_i = 1/4$ for all $i = 0, 1, 2, 3$ and $\alpha_n = 1/n$ for all $n \geq 1$. We consider the different three cases of step-sizes $\lambda = 0.25$, $\lambda = 2.5$ and $\lambda = 3.75$, and we use the stopping rule $\mathrm{TOL}_n = \|x_{n+1} - x_n\| < \varepsilon$ to stop the iterative process. So we obtain the numerical results in Table 2.

Remark 5.3: In the case that $u = (1, -2, 3, 4)$, we obtain $x^* = P_\Omega(u) = (0, 2, 0, 1)$.

The behaviours of TOL_n in Table 2 are illustrated in Figures 2–4 as follows:

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The generalized viscosity explicit rules for solving variational inclusion problems in Banach spaces

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ABSTRACT

In this paper, we propose a generalized viscosity explicit method for finding zeros of the sum of two accretive operators in the framework of Banach spaces. The strong convergence theorem of such method is proved under some suitable assumption on the parameters. As applications, we apply our main result to the variational inequality problem, the convex minimization problem and the split feasibility problem. The numerical experiments to illustrate the behaviour of the proposed method including compare it with other methods are also presented.

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1. Introduction

The starting point of this paper, we consider initial value problem (IVP) in the following form:

$$x'(t) = f\big(x(t)\big), \quad x(t_0) = x_0. \tag{1}$$

In real life, many mathematical model have been formulated as this problem. It is well known that most of ordinary differential equations (ODEs) are not analytically solvable. Numerical methods have become a powerful method for numerically solving time-dependent ordinary and partial differential equations, as is required in computer simulations of physical processes such as groundwater flow and the wave equation. One of famous method is known as *implicit midpoint method* (or modified Euler's method) (see [1–3] for more detail). Given a time interval $[t_0, T]$, the method firstly computes the step size $h = (T - t_0)/N$, where N is the number of steps of h and select the mesh $\{t_n\}_{n=0}^N$ of time steps $t_n \in [t_0, T]$, through the formula $t_n = t_0 + nh$ for $n = 0, 1, \dots, N - 1$. It provides to generate a sequence $\{y_n\}_{n=0}^N$ of approximation of solution at each time

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step t_n, i.e. $y_n \approx x(t_n)$. The implicit midpoint method (IMM) is given by the following procedure:

$$y_0 = x_0,$$
$$y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right), \quad n = 0, 1, \ldots, N-1. \tag{2}$$

It is known that if $f : \mathbb{R}^M \to \mathbb{R}^M$ is a Lipschitz continuous and sufficiently smooth function, then the sequence $\{y_n\}$ converges to the exact solution of (1) as $h \to 0$ uniformly on $t \in [t_0, T]$. If the function f is written as $f(x) = x - g(x)$, then (2) becomes

$$y_0 = x_0,$$
$$y_{n+1} = y_n + h\left[\frac{y_n + y_{n+1}}{2} - g\left(\frac{y_n + y_{n+1}}{2}\right)\right], \quad n = 0, 1, \ldots, N-1 \tag{3}$$

and the critical points of (1) is the fixed point problems $x = g(x)$.

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. We denote by I the identity operator on H. Let $T : C \to C$ be a non-linear mapping. The fixed points set of T is denoted by $F(T) := \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is called *non-expansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall\, x, y \in C.$$

A mapping $f : C \to C$ is called a *contraction*, if there exists constant $\theta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \theta \|x - y\| \quad \forall\, x, y \in C.$$

In recent years, several types of iterative method have been constructed for fixed point problems in various settings. One classical method, due to Mann' iteration [4] which is defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \quad \forall\, n \ge 0, \tag{4}$$

where T is a self-mapping on C and $\{\alpha_n\}$ is a sequence in $[0, 1]$. It is know that Mann's iteration process has only weak convergence.

Motivated by IMM (2) and Mann's iteration (4), Alghamdi et al. [5] introduced the following two algorithms for a non-expansive mapping T: for given $x_0 \in H$ and

$$x_{n+1} = x_n - t_n\left[\frac{x_n + x_{n+1}}{2} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right] \quad \forall\, n \ge 0, \tag{5}$$

$$x_{n+1} = (1 - t_n) x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall\, n \ge 0, \tag{6}$$

where $\{t_n\} \subset (0, 1)$. They proved that the above two algorithms converge weakly to a point in $F(T)$.

In 2015, Xu et al. [6] applied the viscosity approximation method introduced by Moudafi [7] to the IMM for a non-expansive mapping T. They proposed the following *viscosity implicit midpoint method*: for given $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall\, n \geq 0, \tag{7}$$

where f is a contractive mapping on C and $\{\alpha_n\}$ is a sequence in $(0,1)$. It was proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a fixed point of T.

Later, Ke and Ma [8] improved the viscosity implicit midpoint method (7) by replacing the midpoint by any point of interval $[x_n, x_{n+1}]$. They introduced the following *generalized viscosity implicit method* to approximating the fixed point of a non-expansive mapping T: for given $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T(t_n x_n + (1-t_n) x_{n+1}) \quad \forall\, n \geq 0. \tag{8}$$

They also proved that the sequence $\{x_n\}$ generated by (8) converges strongly to a point in $F(T)$.

However, it is noted that the computation by implicit method is not a simple task in general because this method need to compute at every time steps and it can be much harder to implement. To overcome this difficulty, we consider the method so-called an *explicit midpoint method* (EMM) which given by the following finite difference scheme [9, 10]:

$$\begin{aligned} y_0 &= x_0, \\ \bar{y}_{n+1} &= y_n + hf(y_n), \\ y_{n+1} &= y_n + hf\left(\frac{y_n + \bar{y}_{n+1}}{2}\right) \quad \forall\, n \geq 0. \end{aligned} \tag{9}$$

It is generally remarked that the EMM (9) calculates the system status at a future time from the currently known system status while IMM (2) calculates the system status involving both the current state of the system and the later one (see [9, 11]).

In 2017, Marino et al. [12] combined the generalized viscosity implicit midpoint method (8) with the EMM (9) for solving the fixed point problem of a quasi-non-expansive mapping T. They introduced the following *generalized viscosity explicit midpoint method*: for any $x_1 \in C$ and

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) T x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) T(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1. \end{aligned} \tag{10}$$

They also showed that the sequence $\{x_n\}$ generated by (10) converges strongly to a fixed point of T under certain assumptions imposed on the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$.

On the other hand, let us consider the following *variational inclusion problems*: find $x^* \in H$ such that

$$0 \in (A + B)x^*, \tag{11}$$

where $A : H \to H$ is an operator, $B : H \to 2^H$ is a set-valued operator and 0 is a zero vector in H. The solutions set of (11) is denoted by $(A + B)^{-1}0 := \{x \in H : 0 \in (A + B)x\}$. This problem includes, as special cases, convex programming, variational inequalities, equilibrium problem, split feasibility problem and minimization problem. To be more precise, some concrete problems in signal processing, image recovery, statistical regression and machine learning can be modelled mathematically as this form (see [13–16]).

One of the most successful methods for solving problem (11) is *forward–backward algorithm* (FBA) ([17–20]) which is given by $x_1 \in H$ and

$$x_{n+1} = (I + \lambda B)^{-1}(x_n - \lambda A x_n) \quad \forall\, n \geq 1, \tag{12}$$

where $\lambda > 0$. In the context of this method, the operators $(I + \lambda B)^{-1}$ and $I - \lambda A$ are often referred to as the backward and forward operators, respectively. However, this method has only weak convergence.

In order to obtain strong convergence result, Takahashi et al. [21] (see also [22]) proposed the following modified FBA based on Halpern's iteration: for any $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n) \quad \forall\, n \geq 1, \tag{13}$$

where $A : H \to H$ is a monotone operator, $B : H \to 2^H$ is a maximal monotone operator and $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$ is a resolvent operator of B. They proved that the sequence $\{x_n\}$ generated by (13) converges strongly to a point in $(A + B)^{-1}0$.

López et al. [23] proposed the following modified FBA with error sequences $\{a_n\}, \{b_n\}$ in q-uniformly smooth and uniformly convex Banach spaces E: for given $u, x_1 \in E$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J^B_{\lambda_n}(x_n - \lambda_n(A x_n + a_n)) + b_n) \quad \forall\, n \geq 1. \tag{14}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}, \{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$. They proved that the sequence $\{x_n\}$ generated by (14) converges strongly to a point in $(A + B)^{-1}0$.

In [24], Cholamjiak proposed the following new general type of FBA for accretive operators with error $\{e_n\}$ in Banach spaces E: for given $u, x_1 \in E$ and

$$x_{n+1} = \alpha_n u + \eta_n x_n + \delta_n J^B_{\lambda_n}(x_n - \lambda_n A x_n) + e_n \quad \forall\, n \geq 1, \tag{15}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\eta_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \eta_n + \delta_n = 1$. He proved that the sequence $\{x_n\}$ generated by (15) converges strongly to a point in $(A + B)^{-1}0$ under some appropriate conditions.

Shehu and Cai [25] extended iterative method (13) by combining the viscosity approximation method and FBA in a uniformly smooth and uniformly convex Banach space E: for given $x_1 \in E$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n) \quad \forall\, n \geq 1, \tag{16}$$

where $f : E \to E$ is a contraction with a constant $\theta \in (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset (0,1)$. It was proved that the sequence $\{x_n\}$ generated by (16) converges strongly to a point in $(A+B)^{-1}0$ under some appropriate conditions.

In 2018, Chang et al. [26] proposed the following strong convergence theorem of a generalized viscosity implicit rules for solving the variational inclusion problem (11) in a q-uniformly smooth and uniformly convex Banach space.

Theorem 1.1: *Let E be a q-uniformly smooth and uniformly convex Banach space. Let $A : E \to E$ be an α-isa of order q and $B : E \to 2^E$ be an m-accretive operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $f : E \to E$ be a θ-contractive mapping with $\theta q \in (0,1)$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda}(I - \lambda A)(t_n x_n + (1-t_n)x_{n+1}) \quad \forall\, n \geq 1, \tag{17}$$

where $J^B_\lambda := (I+\lambda B)^{-1}$, κ_q is the q-uniform smoothness coefficient of E, $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ and λ is a positive real number satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C3) $0 < \epsilon \leq t_n \leq t_{n+1} < 1$;
(C4) $0 < \lambda \leq (\alpha q/\kappa_q)^{1/(q-1)}$.

Then $\{x_n\}$ converges strongly to $x^* = Q_{(A+B)^{-1}0} f(x^*)$, where $Q_{(A+B)^{-1}0}$ is a sunny non-expansive retraction of E onto $(A+B)^{-1}0$.

In this paper, motivated and inspired by the works of Chang et al. [26] and Marino et al. [12], we propose a generalized viscosity explicit method for solving the variational inclusion problem (11) in the framework of Banach spaces. We prove its strong convergence of the proposed algorithm under some suitable assumption on the parameters. As applications, we apply our main result to the variational inequality problem, the convex minimization problem and the split feasibility problem. Finally, we provide several numerical experiments to illustrate the behaviour of the proposed method and compare it with other methods. The result obtained in this paper improves and extends many known results in the literature.

2. Basic definitions and preliminaries

In this section, we collect some preliminary results which will be used throughout the paper.

Let E and E^* be a real Banach space and the dual space of E, respectively. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}.$$

The *modulus of smoothness* of E is the function $\theta_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\theta_E(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

Definition 2.1: Suppose that $p, q > 1$. A Banach space E is said to be

(1) *Uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.
(2) *p-Uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$.
(3) *Uniformly smooth* if $\lim_{\tau\to 0} \theta_E(\tau)/\tau = 0$.
(4) *q-Uniformly smooth* if there exists a $c_q > 0$ such that $\theta_E(\tau) \leq c_q\tau^q$ for all $\tau > 0$.

If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth. Further, E is p-uniformly convex (q-uniformly smooth) if and only if E^* is q-uniformly smooth (p-uniformly convex), where $p \geq 2$ and $1 < q \leq 2$ satisfy $1/p + 1/q = 1$. It is well known that a Hilbert space H is 2-uniformly smooth. Typical examples of both uniformly convex and uniformly smooth Banach spaces are ℓ_p and L_p spaces, where $p > 1$. More precisely, ℓ_p and L_p spaces are $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^q, \|\bar{x}\| = \|x\|^{q-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between elements of E and E^*.

In particular, $J_2 := J$ is called the *normalized duality mapping*. If E is smooth, then J_q is single-valued, which is denoted by j_q. If $E{:} = H$ is a real Hilbert space, then $J = I$.

Using the concept of sub-differentials, we know the following inequality:

Lemma 2.2 ([27]): *Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in E$, we have*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle, \tag{18}$$

where $j_q(x+y) \in J_q(x+y)$.

Definition 2.3: Let C a be non-empty, closed and convex subsets of E and Q be a mapping of E onto C. Then Q is said to be:

(1) *Sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$.
(2) *Retraction* if $Qx = x$ for all $x \in C$.
(3) A sunny non-expansive retraction if Q is sunny, non-expansive and a retraction from E onto C.

It is known that if $E := H$ is a real Hilbert space, then a sunny non-expansive retraction Q is coincident with the metric projection from E onto C. Moreover, if E is uniformly smooth and T is a non-expansive mapping of C into itself with $F(T) \neq \emptyset$, then $F(T)$ is a sunny non-expansive retract from E onto C (see [28]). We know that in a uniformly smooth Banach space E, a retraction $Q : E \to C$ is sunny and non-expansive, if and only if $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$ for all $x \in E$ and $y \in C$ (see [29]).

Let $A : E \to 2^E$ be a set-valued operator. We denote the domain of an operator A by $\mathcal{D}(A) = \{x \in E : Ax \neq \emptyset\}$. Let $q > 1$. An operator A is said to be *accretive* of order q if for each $x, y \in \mathcal{D}(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0, \quad u \in Ax \text{ and } v \in Ay.$$

An accretive operator A is said to be α*-inverse strongly accretive* (α-isa) of order q if for each $x, y \in \mathcal{D}(A)$, there exists $\alpha > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha \|Ax - Ay\|^q, \quad u \in Ax \text{ and } v \in Ay.$$

In a real Hilbert space H, $A : C \to H$ is called α-inverse strongly monotone (α-ism).

An accretive operator A is said to be *m-accretive* if and only if A is accretive and $\mathcal{R}(I + \lambda A) = E$ for all $\lambda > 0$, where $\mathcal{R}(I + \lambda A)$ is the range of $I + \lambda A$ (see [30]). For an accretive operator A, we can define a mapping $J_\lambda^A : \mathcal{R}(I + \lambda A) \to \mathcal{D}(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$. Such J_λ^A are called the *resolvents* of A for $\lambda > 0$.

Lemma 2.4 ([31]): *The following statements hold*:

(1) *If J_λ^A is a resolvent of A for $\lambda > 0$, then J_λ^A is a single valued non-expansive mapping with $F(J_\lambda^A) = A^{-1}0$, where $A^{-1}0 = \{x \in \mathcal{D}(A) : 0 \in Ax\}$.*
(2) *In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.*

Let $A : E \to E$ be an α-isa of order q and $B : E \to 2^E$ an m-accretive operator. In what follows, we shall use the following notation:

$$T_\lambda = J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \quad \lambda > 0.$$

Lemma 2.5 ([23]): *The following statements hold:*

(i) *For* $\lambda > 0$, $F(T_\lambda) = (A + B)^{-1}0$.
(ii) *For* $0 < \lambda \leq s$ *and* $x \in E$, $\|x - T_\lambda x\| \leq 2\|x - T_s x\|$.

Lemma 2.6 ([23]): *Let E be a uniformly convex and q-uniformly smooth Banach spaces. Assume that A is a single-valued* α*-isa of order q in E. Let* $r > 0$, *there exists a continuous, strictly increasing and convex function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(0) = 0$ *such that*

$$\|T_\lambda x - T_\lambda y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \lambda^{q-1}\kappa_q)\|Ax - Ay\|^q$$
$$- \phi(\|(I - J_\lambda^B)(I - \lambda A)x - (I - J_\lambda^B)(I - \lambda A)y\|)$$

for all $x, y \in B_r := \{x \in E : \|x\| \leq r\}$, *where* κ_q *is the q-uniform smoothness coefficient of E. In particular, if* $0 < \lambda < (\alpha q/\kappa_q)^{1/(q-1)}$, *then* T_λ *is non-expansive.*

Lemma 2.7 ([32]): *Let C be a non-empty, closed and convex subset of a uniformly smooth Banach space E. Let* $T : C \to C$ *be a non-expansive self-mapping such that* $F(T) \neq \emptyset$ *and* $f : C \to C$ *be a contraction with coefficient* $\theta \in (0, 1)$. *Then a net sequence defined by* $z_t = tf(z_t) + (1 - t)Tz_t$, $\forall\ t \in (0, 1)$ *converges strongly as* $t \to 0$ *to a point* $x^* \in F(T)$.

Lemma 2.8 ([33]): *Assume* $\{s_n\}$ *is a sequence of non-negative real numbers such that*

$$s_{n+1} \leq (1 - \delta_n)s_n + \delta_n\tau_n \quad \forall\, n \geq 1$$

and

$$s_{n+1} \leq s_n - \eta_n + \theta_n \quad \forall\, n \geq 1,$$

where $\{\delta_n\}$ *is a sequence in* $(0, 1)$, $\{\eta_n\}$ *is a sequence of non-negative real numbers and* $\{\tau_n\}$, *and* $\{\theta_n\}$ *are real sequences such that*

(i) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(ii) $\lim_{n\to\infty} \theta_n = 0$;
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence of real numbers* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main result

In this section, we propose a generalized viscosity implicit rule for solving the variational inclusion problem (11) and prove its strong convergence theorem of the generated sequence by the proposed method.

Theorem 3.1: *Let E be a real uniformly convex and q-uniformly smooth Banach space E. Let $A : E \to E$ be an α-isa of order q and let $B : E \to 2^E$ is an m-accretive operator. Let $f : E \to E$ be a contraction with a constant $\theta \in (0,1)$. Assume that $(A+B)^{-1}0 \neq \emptyset$. For any $x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda_n}(I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \end{aligned} \tag{19}$$

where $\{\lambda_n\} \subset (0,\infty)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0,1)$ which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} (1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < (\alpha q/\kappa_q)^{1/(q-1)}$.

Then $\{x_n\}$ converges strongly to an element $x^* = Q_{(A+B)^{-1}} f(x^*)$, where $Q_{(A+B)^{-1}0}$ is a sunny non-expansive retraction of E onto $(A+B)^{-1}0$.

Proof: For each $n \geq 1$, put $T_n := J^B_{\lambda_n}(I - \lambda_n A)$. Let $z \in (A+B)^{-1}0$ and by the non-expansivity of T_n, we have

$$\begin{aligned} \|\bar{x}_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(T_n x_n - T_n z)\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|T_n x_n - T_n z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z)\| \\ &\leq \alpha_n \|f(x_n) - z\| + (1-\alpha_n)\|T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z\| \\ &\leq \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| \\ &\quad + (1-\alpha_n)\|t_n(z_n - z) + (1-t_n)(\bar{x}_{n+1} - z)\| \\ &\leq \alpha_n \theta \|x_n - z\| + (1-\alpha_n)(t_n\|x_n - z\| + (1-t_n)\|\bar{x}_{n+1} - z\|) \\ &\quad + \alpha_n \|f(z) - z\| \\ &= \alpha_n \theta \|x_n - z\| + (1-\alpha_n) t_n \|x_n - z\| + (1-\alpha_n)(1-t_n)\|x_n - z\| \\ &\quad + \alpha_n \|f(z) - z\| \\ &= (1-(1-\theta)\alpha_n)\|x_n - z\| + (1-\theta)\alpha_n \frac{\|f(z) - z\|}{1-\theta} \\ &\leq \max\left\{ \|x_n - z\|, \frac{\|f(z) - z\|}{1-\theta} \right\}. \end{aligned}$$

By induction, we obtain

$$\|x_n - z\| \leq \max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1-\theta}\right\} \quad \forall\, n \geq 1.$$

Hence $\{x_n\}$ is bounded. For each $n \geq 1$, put $z_n := t_n x_n + (1-t_n)\bar{x}_{n+1}$. Let $x^* = Q_{(A+B)^{-1}0} f(x^*)$. By Lemma 2.6, we have

$$\begin{aligned}\|T_n z_n - x^*\|^q &= \|J^B_{\lambda_n}(I - \lambda_n A) z_n - J^B_{\lambda_n}(I - \lambda_n A)x^*\|^q \\ &\leq \|z_n - x^*\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\ &\quad - \phi(\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n A x^*\|). \end{aligned} \tag{20}$$

In a similar way, we also have

$$\begin{aligned}\|T_n x_n - x^*\|^q \leq \|x_n - x^*\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\ - \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|). \end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - x^*\|^q &\leq t_n\|x_n - x^*\|^q + (1-t_n)\|\bar{x}_{n+1} - x^*\|^q \\ &\leq t_n\|x_n - x^*\|^q + (1-t_n) \\ &\quad \times \left[\beta_n\|x_n - x^*\|^q + (1-\beta_n)\|T_n x_n - x^*\|^q\right] \\ &\leq t_n\|x_n - x^*\|^q + (1-t_n) \\ &\quad \times \Big[\beta_n\|x_n - x^*\|^q + (1-\beta_n)\Big(\|x_n - x^*\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q) \\ &\quad \times \|Ax_n - Ax^*\|^q - \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|)\Big)\Big] \\ &\leq \|x_n - x^*\|^q - (1-t_n)(1-\beta_n)\Big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\ &\quad + \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|)\Big). \end{aligned} \tag{21}$$

Substituting (21) into (20), we get

$$\begin{aligned}\|T_n z_n - x^*\|^q \leq \|x_n - x^*\|^q - (1-t_n)(1-\beta_n)\Big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q) \\ \times \|Ax_n - Ax^*\|^q + \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|)\Big) \\ - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\ - \phi(\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n A x^*\|). \end{aligned} \tag{22}$$

From Lemma 2.2 and (22), we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^q\\
&\quad =\|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(T_nz_n-x^*)\|^q\\
&\quad =\|\alpha_n(f(x_n)-f(x^*))+\alpha_n(f(x^*)-x^*)+(1-\alpha_n)(T_nz_n-x^*)\|^q\\
&\quad \le \|\alpha_n(f(x_n)-f(x^*))+(1-\alpha_n)(T_nz_n-x^*)\|^q\\
&\qquad +q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle\\
&\quad \le \alpha_n\|f(x_n)-f(x^*)\|^q+(1-\alpha_n)\|T_nz_n-x^*\|^q\\
&\qquad +q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle\\
&\quad \le \alpha_n\|f(x_n)-f(x^*)\|^q+(1-\alpha_n)\big[\|x_n-x^*\|^q-(1-t_n)(1-\beta_n)\\
&\qquad \times\big(\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)\|Ax_n-Ax^*\|^q\\
&\qquad +\phi(\|x_n-\lambda_nAx_n-T_nx_n+\lambda_nAx^*\|)\big)-\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)\|Az_n-Ax^*\|^q\\
&\qquad -\phi(\|z_n-\lambda_nAz_n-T_nz_n+\lambda_nAx^*\|)\big]+q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle\\
&\quad \le \big(1-(1-\theta)\alpha_n\big)\|x_n-x^*\|^q-K_n\big(\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)\|Ax_n-Ax^*\|^q\\
&\qquad +\phi(\|x_n-\lambda_nAx_n-T_nx_n+\lambda_nAx^*\|)\big)\\
&\qquad -(1-\alpha_n)\big(\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)\|Az_n-Ax^*\|^q\\
&\qquad +\phi(\|z_n-\lambda_nAz_n-T_nz_n+\lambda_nAx^q\|)\big)\\
&\qquad +q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle,
\end{aligned}\tag{23}
$$

where $K_n:=(1-\alpha_n)(1-t_n)(1-\beta_n)$. We note that $\liminf_{n\to\infty}K_n>0$ and $\liminf_{n\to\infty}\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)>0$. For each $n\ge 1$, we set

$$
\begin{aligned}
s_n&:=\|x_n-x^*\|^q,\\
\delta_n&:=(1-\theta)\alpha_n,\\
\tau_n&:=\frac{q}{1-\theta}\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle,\\
\eta_n&:=K_n(\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)\|Ax_n-Ax^*\|^q\\
&\quad +\phi(\|x_n-\lambda_nAx_n-T_nx_n+\lambda_nAx^*\|))\\
&\quad +(1-\alpha_n)(\lambda_n(\alpha q-\lambda_n^{q-1}\kappa_q)\|Az_n-Ax^*\|^q\\
&\quad +\phi(\|z_n-\lambda_nAz_n-T_nz_n+\lambda_nAx^*\|)),\\
\theta_n&:=q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

Then (23) reduces to the following formulae:

$$
s_{n+1}\le(1-\delta_n)s_n+\delta_n\tau_n\quad\forall\, n\ge 1 \tag{24}
$$

and

$$s_{n+1} \leq s_n - \eta_n + \theta_n \quad \forall\, n \geq 1. \tag{25}$$

By $(C1)$, we see that $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\lim_{n\to\infty} \theta_n = 0$. In order to complete the proof, using Lemma 2.8, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies that $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ for any subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. So by our assumptions and the properties of ϕ, we obtain

$$\lim_{k\to\infty} \|Az_{n_k} - Ax^*\| = \lim_{k\to\infty} \|z_{n_k} - \lambda_{n_k} Az_{n_k} - T_{n_k} z_{n_k} + \lambda_{n_k} Ax^*\| = 0$$

and

$$\lim_{k\to\infty} \|Ax_{n_k} - Ax^*\| = \lim_{k\to\infty} \|x_{n_k} - \lambda_{n_k} Ax_{n_k} - T_{n_k} x_{n_k} + \lambda_{n_k} Ax^*\| = 0.$$

Consequently,

$$\lim_{k\to\infty} \|T_{n_k} z_{n_k} - z_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T_{n_k} x_{n_k} - x_{n_k}\| = 0. \tag{26}$$

Since $\liminf_{n\to\infty} \lambda_n > 0$, there is $\lambda > 0$ such that $\lambda_n \geq \lambda$ for all $n \geq 1$. In particular, $\lambda_{n_k} \geq \lambda$ for all $k \geq 1$. Then, by Lemma 2.5 (ii), we have

$$\|T_\lambda x_{n_k} - x_{n_k}\| \leq 2\|T_{n_k} x_{n_k} - x_{n_k}\|.$$

From (26), we obtain

$$\lim_{k\to\infty} \|T_\lambda x_{n_k} - x_{n_k}\| = 0. \tag{27}$$

Let $z_t = tf(z_t) + (1-t)T_\lambda z_t$, $\forall t \in (0,1)$. Then it follows from Lemma 2.7 that $\{z_t\}$ converges strongly to a fixed point $x^* \in F(T_\lambda) = (A+B)^{-1}0$. From Lemma 2.2, we have

$$\begin{aligned}
\|z_t - x_{n_k}\|^q &= \|t(f(z_t) - x_{n_k}) + (1-t)(T_\lambda z_t - x_{n_k})\|^q \\
&\leq (1-t)^q \|T_\lambda z_t - x_{n_k}\|^q + qt\langle f(z_t) - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\
&= (1-t)^q \|T_\lambda z_t - x_{n_k}\|^q + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle \\
&\quad + qt\langle z_t - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\
&\leq (1-t)^q \big(\|T_\lambda z_t - T_\lambda x_{n_k}\| + \|T_\lambda x_{n_k} - x_{n_k}\|\big)^q \\
&\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt\|z_t - x_{n_k}\|^q \\
&\leq (1-t)^q \big(\|z_t - x_{n_k}\| + \|T_\lambda x_{n_k} - x_{n_k}\|\big)^q \\
&\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt\|z_t - x_{n_k}\|^q,
\end{aligned}$$

which implies that

$$\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle \le \frac{(1-t)^q}{qt}\left(\|z_t - x_{n_k}\| + \|T_\lambda x_{n_k} - x_{n_k}\|\right)^q + \frac{qt-1}{qt}\|z_t - x_{n_k}\|^q.$$

From (27), we obtain

$$\limsup_{k\to\infty}\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle \le \frac{(1-t)^q}{qt}M + \frac{qt-1}{qt}M = \left(\frac{(1-t)^q + qt - 1}{qt}\right)M, \qquad (28)$$

where $M = \limsup_{k\to\infty}\|z_t - x_{n_k}\|^q$, $t \in (0,1)$. We see that $((1-t)^q + qt - 1)/qt \to 0$ as $t \to 0$. Since j_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_t \to x^*$, we have

$$\|j_q(x_{n_k} - z_t) - j_q(x_{n_k} - x^*)\| \to 0 \quad \text{as } t \to 0.$$

So we have

$$\begin{aligned}
&|\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle - \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle| \\
&\quad = |\langle f(z_t) - f(x^*) + f(x^*) - x^* + x^* - z_t, j_q(x_{n_k} - z_t)\rangle \\
&\qquad - \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle| \\
&\quad = |\langle f(z_t) - f(x^*), j_q(x_{n_k} - z_t)\rangle + \langle f(x^*) - x^*, j_q(x_{n_k} - z_t)\rangle \\
&\qquad + \langle x^* - z_t, j_q(x_{n_k} - z_t)\rangle - \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle| \\
&\quad \le |\langle f(x^*) - x^*, j_q(x_{n_k} - z_t) - j_q(x_{n_k} - x^*)\rangle| + |\langle f(z_t) - f(x^*), j_q(x_{n_k} - z_t)\rangle| \\
&\qquad + |\langle x^* - z_t, j_q(x_{n_k} - z_t)\rangle| \\
&\quad \le \|f(x^*) - x^*\|\|j_q(x_{n_k} - z_t) - j_q(x_{n_k} - x^*)\| \\
&\qquad + (1+\theta)\|z_t - x^*\|\|x_{n_k} - z_t\|^{q-1}.
\end{aligned}$$

Hence as $t \to 0$, we have

$$\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle \to \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle.$$

From (28), as $t \to 0$, it follows that

$$\limsup_{k\to\infty}\langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle \le 0. \qquad (29)$$

On the other hand, we have

$$\begin{aligned}
\|T_{n_k} z_n - x_{n_k}\| &\le \|T_{n_k} z_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \\
&\le \|T_{n_k} z_{n_k} - z_{n_k}\| + (1 - t_{n_k})(1 - \beta_{n_k})\|T_{n_k} x_{n_k} - x_{n_k}\| \\
&\le \|T_{n_k} z_{n_k} - z_{n_k}\| + \|T_{n_k} x_{n_k} - x_{n_k}\|.
\end{aligned}$$

This together with (26) implies that

$$\lim_{k\to\infty} \|T_{n_k}z_{n_k} - x_{n_k}\| = 0. \tag{30}$$

Further, we have

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\le \|x_{n_k+1} - T_{n_k}z_{n_k}\| + \|T_{n_k}z_{n_k} - x_{n_k}\| \\ &\le \alpha_{n_k}\|f(x_{n_k}) - T_{n_k}z_{n_k}\| + \|T_{n_k}z_{n_k} - x_{n_k}\|.\end{aligned}$$

This together with (30) implies

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{31}$$

Combining (29) and (31), we get

$$\limsup_{k\to\infty} \langle f(x^*) - x^*, j_q(x_{n_k+1} - x^*)\rangle \le 0. \tag{32}$$

This implies that $\lim_{k\to\infty} \tau_{n_k} \le 0$. Then, by Lemma 2.8, we conclude that $\lim_{n\to\infty} s_n = 0$. Hence $x_n \to x^*$ as $n \to \infty$. This completes the proof. ■

Remark 3.2: We point out main issue on our work as follows:

(1) The method of proof of our result is very different from ones of [12, 21, 26, 34–37]. In particular, we remove the assumptions '$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$' and '$0 < \epsilon \le t_n \le t_{n+1} < 1$' in Theorem 3.1 of [8, 26, 34]. Moreover, we remove the assumption $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ in Theorem 3.1 of [37].
(2) The method of proof of our result is more simple with respect to the result of Chang et al. [26].

From [38], we obtain the following results.

Corollary 3.3: *Let $E := \ell_q$ (or L_q) with $1 < q \le 2$. Let $A : E \to E$ be an α-isa of order q and let $B : E \to 2^E$ is an m-accretive operator. Let $f : E \to E$ be a contraction with a constant $\theta \in (0,1)$. Assume that $(A+B)^{-1}0 \ne \emptyset$. For any $x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}\bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n)J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)J^B_{\lambda_n}(I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \ \forall n \ge 1,\end{aligned} \tag{33}$$

where $\{\lambda_n\} \subset (0,\infty)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0,1)$ which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < (\alpha q/\kappa_q)^{1/(q-1)}$, *where* $\kappa_q = (1 + t_q^{q-1})/((1 + t_q)^{q-1})$ *and* t_q *is the unique solution of the equation* $(q-2)t^{q-1} + (q-1)t^{q-2} - 1 = 0,\ 0 < t < 1$.

Then $\{x_n\}$ *converges strongly to an element* $x^* = Q_{(A+B)^{-1}}f(x^*)$, *where* $Q_{(A+B)^{-1}0}$ *is a sunny non-expansive retraction of E onto* $(A+B)^{-1}0$.

Corollary 3.4: *Let* $E := \ell_p$ *(or* L_p*) with* $2 \le p < \infty$. *Let* $A : E \to E$ *be an* α*-isa of order 2 and let* $B : E \to 2^E$ *is an m-accretive operator. Let* $f : E \to E$ *be a contraction with a constant* $\theta \in (0,1)$. *Assume that* $(A+B)^{-1}0 \ne \emptyset$. *For any* $x_1 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) J_{\lambda_n}^B (I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \ge 1, \end{aligned} \tag{34}$$

where $\{\lambda_n\} \subset (0,\infty)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} (1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha/(p-1)$.

Then $\{x_n\}$ *converges strongly to an element* $x^* = Q_{(A+B)^{-1}}f(x^*)$, *where* $Q_{(A+B)^{-1}0}$ *is a sunny non-expansive retraction of E onto* $(A+B)^{-1}0$.

Corollary 3.5: *Let H be a Hilbert space H. Let* $A : H \to H$ *be an* α*-ism and let* $B : H \to 2^H$ *be a maximal monotone operator. Let* $f : H \to H$ *be a contraction mapping with a constant* $\theta \in (0,1)$. *Suppose that* $(A+B)^{-1}0 \ne \emptyset$. *For any* $x_1 \in H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) J_{\lambda_n}^B (I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \ge 1, \end{aligned} \tag{35}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty} (1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha$.

Then $\{x_n\}$ *converges strongly to an element* $x^* = P_{(A+B)^{-1}0}f(x^*)$, *where* $P_{(A+B)^{-1}0}$ *is a metric projection of H onto* $(A+B)^{-1}0$.

4. Some applications

4.1. Application to variational inequality problem

Let C be a non-empty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear monotone operator. The *variational inequality problem* (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, z - x^* \rangle \geq 0 \quad \forall\, z \in C. \tag{36}$$

The set of solutions of VIP is denoted by $VI(C, A)$. Let i_C be an indicator function of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases} \tag{37}$$

Denote N_C the normal cone of C, i.e.

$$N_C(u) = \{z \in H : \langle z, v - u \rangle \leq 0,\ \forall\, v \in C\}.$$

It is also known that i_C is proper convex and lower semi continuous function and sub-differential ∂i_C is maximal monotone operator (see [39]). We define the resolvent operator $J_\lambda^{\partial i_C}$ of i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) := (I + \lambda \partial i_C)^{-1}(x) \quad \forall\, x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{z \in H : i_C(v) + \langle z,\ v - u \rangle \leq i_C(u),\ \forall\, u \in H\} \\ &= \{z \in H : \langle z, v - u \rangle \leq 0,\ \forall v \in C\} = N_C(u), \quad u \in C. \end{aligned}$$

So we have

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) \Leftrightarrow x - u \ &\in\ \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, v - u \rangle \leq 0 \quad \forall\, v \in C \\ &\Leftrightarrow u = P_C(x), \end{aligned}$$

where P_C is the metric projection from H onto C. Further, we also have $(A + \partial i_C)^{-1}0 = VI(C, A)$ (see [37]).

If we set $B = \partial i_C$ in Theorem 3.1, then we obtain the following result.

Theorem 4.1: *Let $A : C \to H$ be an α-ism such that $VI(C, A) \neq \emptyset$. Let $f : C \to C$ be a contraction with a constant $\theta \in (0, 1)$. For any $x_1 \in C$, let $\{x_n\}$ be a sequence* *generated by*

$$\bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)P_C(x_n - \lambda_n A x_n),$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_C(I-\lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \tag{38}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$.

Then $\{x_n\}$ *converges strongly to a point in* $VI(C, A)$.

4.2. Application to convex minimization problem

Let $g : H \to \mathbb{R}$ be a convex smooth function and $h : H \to \mathbb{R}$ be a proper convex and lower semicontinuous function. The *convex minimization problem* is to find $x^* \in H$ such that

$$g(x^*) + h(x^*) = \min_{x\in H}\{g(x) + h(x)\}. \tag{39}$$

By Fermat's rule, it is known that the problem (39) is equivalent to the problem of finding $x^* \in H$ such that

$$0 \in \nabla g(x^*) + \partial h(x^*),$$

where ∇g is a gradient of g and ∂h is a subdifferential of h. It is also known if ∇g is $(1/\alpha)$-Lipschitz continuous, then it is also α-ism (see [40]). In fact, we can set $A = \nabla g$ and $B = \partial h$ in Theorem 3.1. So we obtain the following result.

Theorem 4.2: *Let* $g : H \to \mathbb{R}$ *be a convex and differentiable function with* $(1/\alpha)$-*Lipschitz continuous gradient* ∇g *and let* $h : H \to \mathbb{R}$ *be a convex and lower semicontinuous function such that* $g + h$ *attains a minimizer. Let* $f : H \to H$ *be a contraction with a constant* $\theta \in (0, 1)$. *For any* $x_1 \in H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)J_{\lambda_n}^{\partial h}(x_n - \lambda_n \nabla g(x_n)),$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{\lambda_n}^{\partial h}(I-\lambda_n \nabla g)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \tag{40}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha$.

Then $\{x_n\}$ *converges strongly to minimizer of* $g+h$.

4.3. Application to split feasibility problem

Let C and Q be non-empty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a linear bounded operator with its adjoint T^*. The *split feasibility problem* (SFP) is to find

$$x^* \in C \quad \text{such that } Tx^* \in Q. \tag{41}$$

The SFP can be used to model the intensity-modulated radiation therapy (see [41–43]). To solve the SFP (41), we can rewrite it as the following convexly

Table 1. Comparison of Algorithm (19), Algorithm (17), Algorithm (15) and Algorithm (16) for Example 5.1.

		Algorithm (19)	Algorithm (17)	Algorithm (16)	Algorithm (15)
Case 1	No. of Iter.	34	43	44	1806
Case 2	No. of Iter.	34	43	44	803

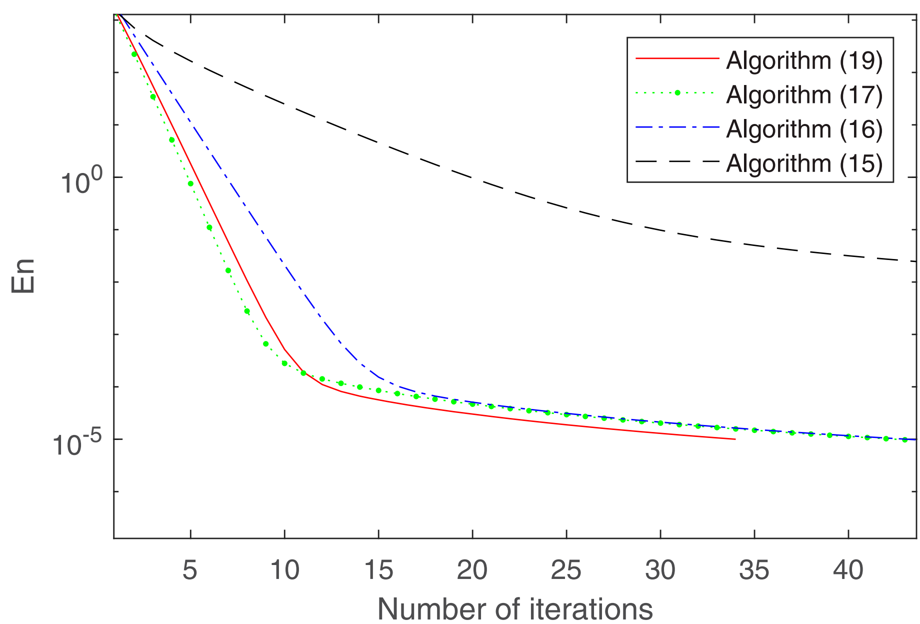

Figure 1. The error plotting of iterations in Case 1.

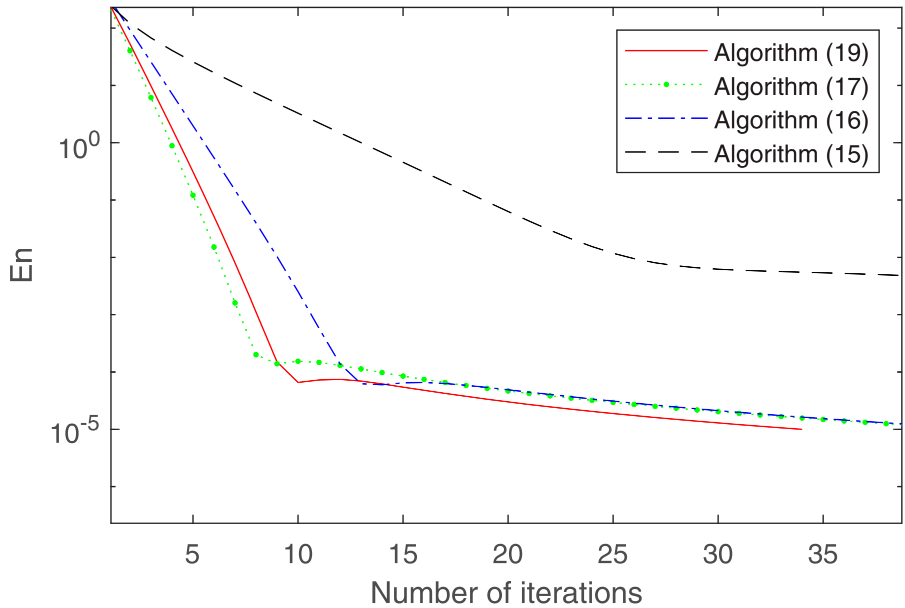

Figure 2. The error plotting of iterations in Case 2.

Table 2. Comparison of Algorithm (19), Algorithm (16) and Algorithm (15) for Example 5.3.

		Algorithm (19)	Algorithm (16)	Algorithm (15)
Case 1	No. of Iter.	2028	4052	12,204
	CPU	0.4311	0.4707	1.4134
Case 2	No. of Iter.	3776	7547	22,718
	CPU	13.8184	13.9072	41.6642
Case 3	No. of Iter.	7347	14,688	44,201
	CPU	100.2134	100.3403	302.2716

constrained minimization problem:

$$\min_{x\in C} g(x),$$

where $g(x) := \frac{1}{2}\|(I - P_Q)Tx\|^2$. Note that the function g is differentiable convex and has a Lipschitz gradient given by $\nabla g = T^*(I - P_Q)T$. Further, ∇g is $1/\|T\|^2$-ism, where $\|T\|^2$ is the spectral radius of T^*T (see [13]). Thus, we have the SFP equivalent to the variational inclusion problem (11) with $A = \nabla g$ and $B = \partial i_C$. It follows that

$$\begin{aligned} 0 \in \nabla g(x^*) + \partial i_C(x^*) &\Leftrightarrow 0 \in x^* + \lambda \partial i_C(x^*) - (x^* - \lambda \nabla g(x^*)) \\ &\Leftrightarrow x^* - \lambda \nabla g(x^*) \in x^* + \lambda \partial i_C(x^*) \\ &\Leftrightarrow x^* = (I + \lambda \partial i_C)^{-1}(x^* - \lambda \nabla g(x^*)) \\ &\Leftrightarrow x^* = P_C(x^* - \lambda \nabla g(x^*)). \end{aligned}$$

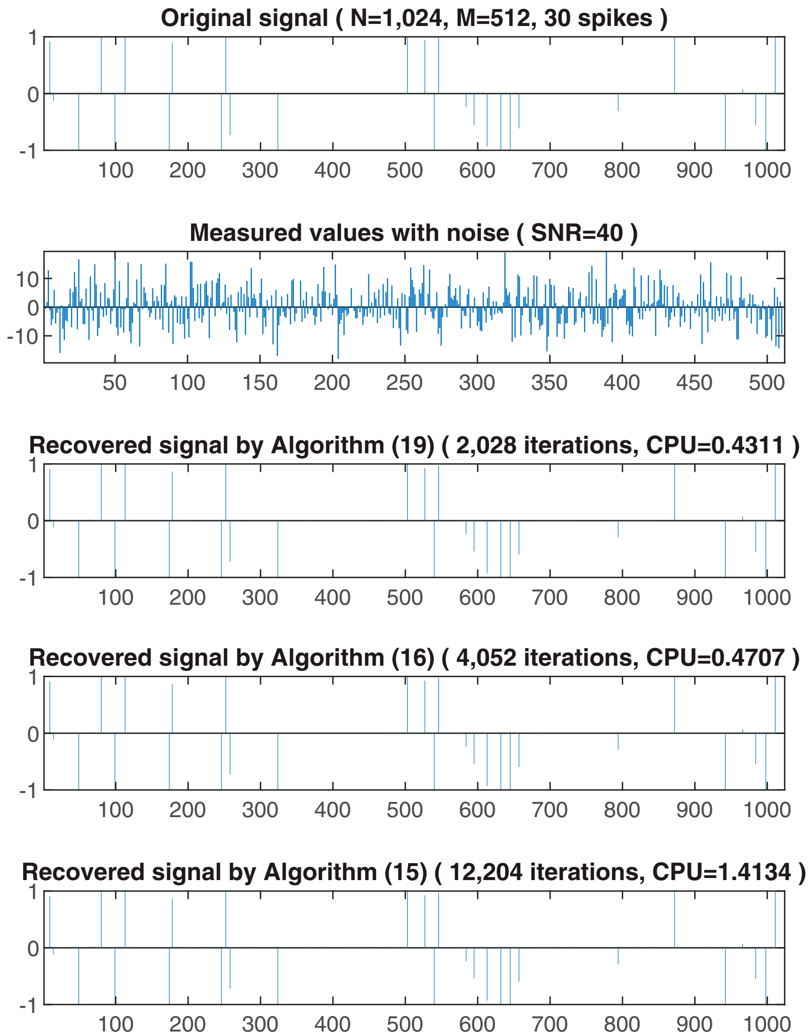

Figure 3. Comparison of recovered signal by using different algorithms in Case 1.

Theorem 4.3: *Let C and Q be non-empty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with its adjoint T^* and $T \neq 0$. Let $f : C \to C$ be a contraction with a constant $\theta \in (0, 1)$. Suppose that the solution sets of SFP is non-empty. For any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) P_C(x_n - \lambda_n T^*(I - P_Q)Tx_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) P_C(I - \lambda_n T^*(I - P_Q)T) \\ &\quad \times (t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \end{aligned} \tag{42}$$

where $\{\lambda_n\} \subset (0, 2/\|T\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$ which satisfy the following conditions:

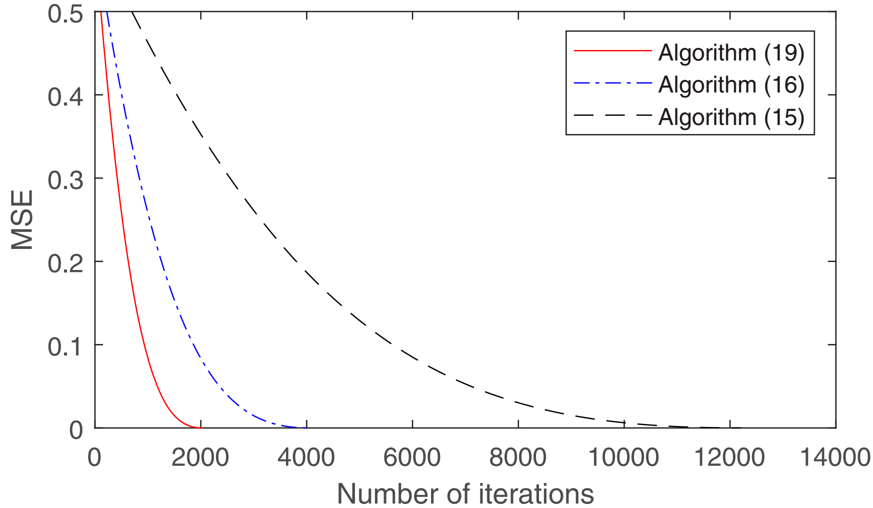

Figure 4. MSE versus number of iterations in Case 1.

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2/\|T\|^2$.

Then $\{x_n\}$ converges strongly to solution of SFP.

5. Numerical experiments

In this section, we provide numerical experiments to illustrate the behaviour of the our Algorithm (19) and also compare it with Algorithm (17) in [26], Algorithm (15) in [24] and Algorithm (16) in [25].

Example 5.1: We consider the example in an infinite dimensional Banach spaces outside Hilbert spaces which is taken from [24] (see also [44]). Let $E = \ell_3$ and $x = (x_1, x_2, x_3, \dots) \in \ell_3$. Let $A, B : \ell_3 \to \ell_3$ be defined by

$$Ax = 2x + (1,1,1,0,0,0,0,\dots) \quad \text{and} \quad Bx = 5x \quad \text{for } x \in \ell_3.$$

It is to see that A is 1/2-isa of order 2 and B is an m-accretive operator with $\mathcal{R}(I + \lambda B) = \ell_3$ for all $\lambda > 0$. Moreover,

$$J_\lambda^B(x - \lambda Ax) = \frac{1-2\lambda}{1+5\lambda}x - \frac{\lambda}{1+5\lambda}(1,1,1,0,0,0,0,\dots),$$

for all $x \in \ell_3$. It is not difficult to check that $(A+B)^{-1}0 = \{(-\frac{1}{7}, -\frac{1}{7}, -\frac{1}{7}, 0, 0, 0, 0, \dots)\}$.

Since, in ℓ_3, we have $q = 2$ and $\kappa_2 = 2$. Due to $\alpha = \frac{1}{2}$, then we can choose $\lambda_n = \frac{1}{10}$ for all $n \in \mathbb{N}$. We take $\alpha_n = (1/2n)$, $\beta_n = 1/(3(n+1))$, $\delta_n = n/(3(n+3))$,

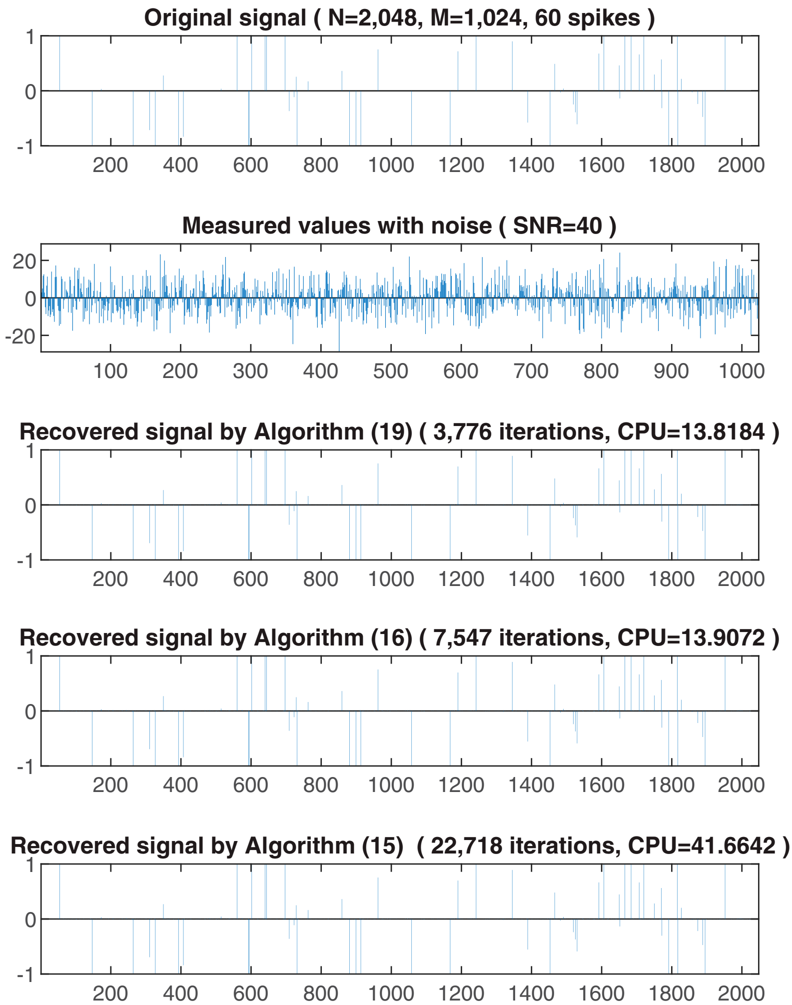

Figure 5. Comparison of recovered signal by using different algorithms in Case 2.

$\eta_n = 1 - (1/2n) - (n/(3(n+3)))$, $t_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ and $f(x) = x/2$ in those algorithms. In our numerical experiments, we consider the following two cases of starting point x_1:

Case 1 : $x_1 = (71.23, -42.51, -1.42, 0, 0, 0, \ldots)$;

Case 2 : $x_1 = (-27.53, -22.47, 4.64, 0, 0, 0, \ldots)$.

Let u be randomly generated in ℓ_3. We choose the stopping criterion is $E_n = \|x_{n+1} - x_n\| < 10^{-5}$. The numerical results are reported in Table 1 and Figures 1 and 2.

Remark 5.2: From Table 1 and Figures 1 and 2, we see that our Algorithm (19) has a number of iterations less than Algorithm (17) of Chang et al. [26], Algorithm (16) of Shehu and Cai [25] and Algorithm (15) of Cholamjiak [24].

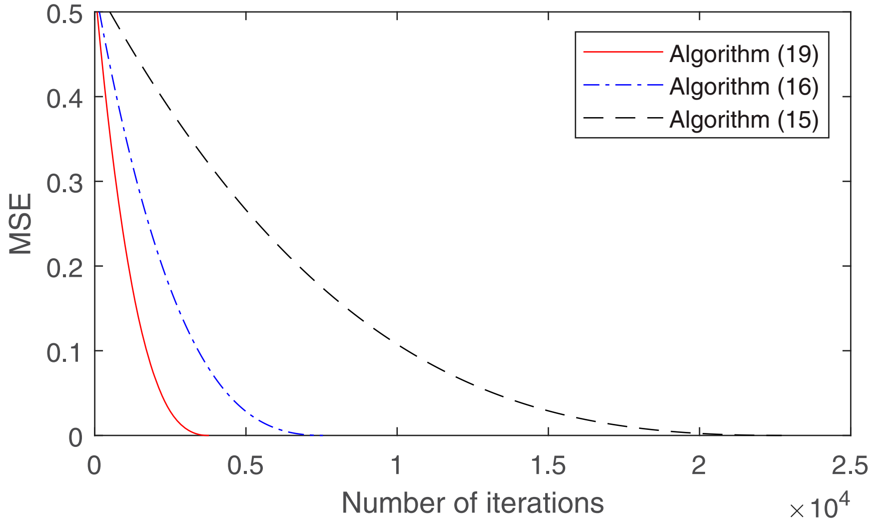

Figure 6. MSE versus number of iterations in Case 2.

It is shown that our proposed algorithm has good convergence behaviour.

Example 5.3: In this example, we consider the signal recovery by compressed sensing which refers to a signal acquisition and reconstruction technique. In signal processing, compressed sensing can be modelled as the following under determinated linear equation system:

$$y = Tx + \varepsilon, \tag{43}$$

where $x \in \mathbb{R}^N$ is a vector with m non-zero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $T : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear observation operator. It is know that problem (43) can be seen as solving the following LASSO problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y - Tx\|_2^2 \quad \text{subject to } \|x\|_1 \leq t, \tag{44}$$

where $t > 0$ is a given constant. In particular, if $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$, then the LASSO problem can be considered as the SFP (41).

The sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m non-zero elements. The matrix $T \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio SNR $= 40$. The process is started with $t = m$ and starting point x_1 is randomly generated in $\mathbb{R}^N$. The

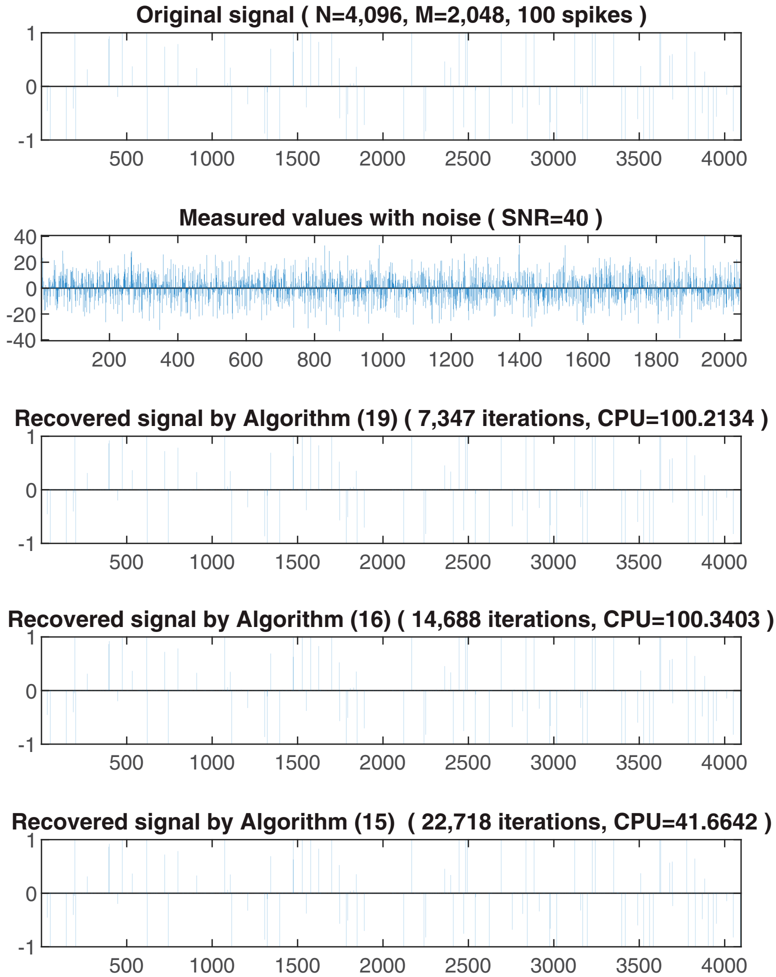

Figure 7. Comparison of recovered signal by using different algorithms in Case 3.

restoration accuracy is measured by the mean squared error as follows:

$$E_n = \frac{1}{N}\|x_n - x^*\|_2^2 < 10^{-5}, \tag{45}$$

where x^* is an estimated signal of x.

We perform numerical computations for Algorithm (19) and also compare with Algorithm (15) and Algorithm (16). Take $\alpha_n = (1/(1500(n+5)))$, $\beta_n = (1/(3(n+10)))$, $t_n = (n/(5700(n+1)))$, $\delta_n = (n/(3(n+3)))$, $\lambda_n = 1/\|T\|^2$ for all $n \in \mathbb{N}$, $f(x) = x/2$ and $u = (1, 1, \ldots, 1) \in \mathbb{R}^N$.

In our numerical experiments, we consider the following three cases of N, M and m:

Case 1 : $N = 1024$, $M = 512$ and $m = 30$;

Case 2 : $N = 2048$, $M = 1024$ and $m = 60$;

Case 3 : $N = 4096$, $M = 2048$ and $m = 100$.

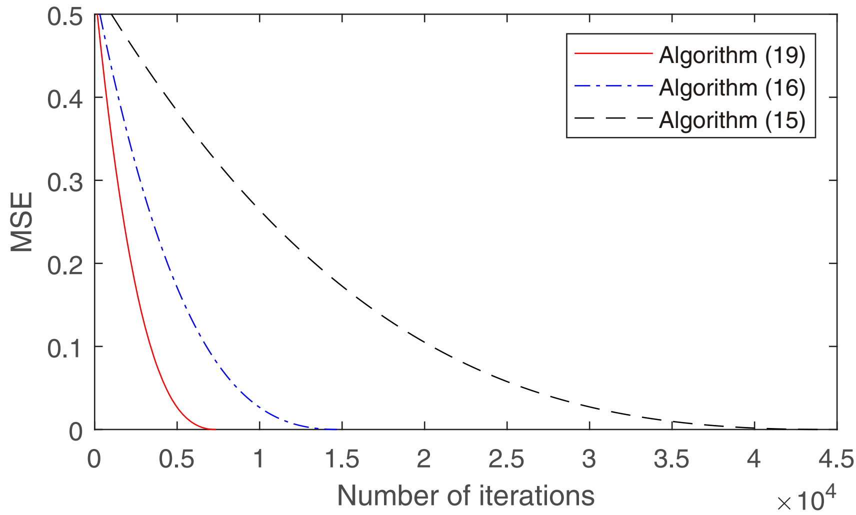

Figure 8. MSE versus number of iterations in Case 3.

Then the numerical results are reported in Table 2 and Figures 3–8.

Remark 5.4: From Table 2 and Figures 3–8, we see that our Algorithm (19) has a number of iterations and cpu time less than Algorithm (16) of Shehu and Cai [25] and Algorithm (15) of Cholamjiak [24]. It is shown that our algorithm highly improves those algorithms. This is the primary advantage of our algorithm in comparison with other algorithms.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Iterative Methods for Solving the Monotone Inclusion Problem and the Fixed Point Problem in Banach Spaces

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Abstract In this work, we propose two iterative algorithms for solving the monotone inclusion problem and the fixed point problem of a relatively nonexpansive mapping in the framework of Banach spaces. We prove the strong convergence theorems of the proposed algorithms under some suitable assumptions. Furthermore, some numerical experiments of proposed algorithms to compressed sensing in signal recovery are presented. Our results improve and generalize many recent and important results in the literature.

MSC: 47H09; 47H10; 47J25; 47J05

Keywords: maximal monotone operator; Banach space; strong convergence; extragradient algorithm

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1. Introduction

Let E be a real Banach space. Consider the following so-called *monotone inclusion problem*: find $x^* \in E$ such that

$$0 \in (A + B)x^*, \tag{1.1}$$

where $A : E \to E$ and $B : E \to 2^E$ are single and set-valued mappings, respectively and 0 is a zero vector in E. In particular case, when $A = 0$, then the problem (1.1) becomes the inclusion problem introduced by Rockafellar [1] and when $E = \mathbb{R}^n$, then the problem

*Corresponding author.

(1.1) becomes the generalized equation introduced by Robinson [2]. The set of solutions of the problem (1.1) is denoted by $(A+B)^{-1}0$. Many practical nonlinear problems arising in applied sciences such as in machine learning, image processing, statistical regression and linear inverse problem can be formulated as this problem (see [3–5]).

A well-known method for solving the problem (1.1) in Hilbert spaces H, is the *forward-backward algorithm* [6] which is defined by the following manner:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = J_\lambda^B(x_n - \lambda A x_n), \ \forall n \geq 1, \end{cases} \tag{1.2}$$

where $J_\lambda^B := (I + \lambda B)^{-1}$ is a resolvent of B for $\lambda > 0$. Here, I denotes the identity operator of H. It was proved that the sequence generated by (1.2) converge weakly to a point in $(A+B)^{-1}0$ under the assumption that A is α-cocoercivity, that is,

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in H$$

and λ is chosen in $(0, 2\alpha)$.

In order to get strong convergence, Takashashi et al. [7] introduced the following modified forward–backward algorithm in Hilbert spaces H:

$$\begin{cases} x_1, u \in H, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n) J_{\lambda_n}^B(x_n - \lambda_n A x_n)), \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where A is an α-cocoercive mapping on H and $\{\lambda_n\} \subset (0, \infty)$. They also proved the strong convergence of the generated by (1.3) converges strongly to a point in $(A+B)^{-1}0$ under appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

López et al. [8] established a strong convergence theorem of the forward-backward algorithm (1.2) in a q-uniformly smooth and uniformly convex Banach spaces E. They introduced a modified forward-backward algorithm with errors a_n and b_n in the following way:

$$\begin{cases} x_1, u \in E, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(J_{\lambda_n}^B(x_n - \lambda_n(A x_n + a_n)) + b_n), \ \forall n \geq 1, \end{cases} \tag{1.4}$$

where $J_{\lambda_n}^B := (I + \lambda_n B)^{-1}$ is the resolvent of an m-accretive operator B, A is an α-cocoercive mapping, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$. They also proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point in $(A+B)^{-1}0$.

In recent years, various modifications of forward-backward algorithm have been constructed and modified by many authors in several settings (see, *e.g.*, [9–16]). It can be seen that, the cocoercivity of A of most of methods is strong assumption. To avoid this strong assumption, Tseng [17] introduced the following algorithm in Hilbert spaces H, later it is known as *Tseng's splitting algorithm*:

$$\begin{cases} x_1 \in H, \\ y_n = J_{\lambda_n}^B(I - \lambda_n A) x_n, \\ x_{n+1} = y_n - \lambda_n(A y_n - A x_n), \ \forall n \geq 1, \end{cases} \tag{1.5}$$

where A is Lipschitz continuous with a constant $L > 0$. It was shown that the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of (1.1) provided the step-size λ_n is chosen in $\left(0, \frac{1}{L}\right)$.

On the other hand, the fixed point problem is problem of finding a point $x^* \in E$ such that

$$x^* = Tx^*, \tag{1.6}$$

where $T : E \to E$ is a nonlinear mapping. The set of solutions of problem (1.6) is denoted by $F(T) = \{x \in E : x = Tx\}$. In real life, many mathematical models have been formulated as this problem.

In this paper, we study the following problem: find $x^* \in E$ such that

$$x^* \in F(T) \cap (A + B)^{-1}0. \tag{1.7}$$

Currently, there have been many authors who interested in finding a common solution of the fixed point problem (1.6) and the monotone inclusion problem (1.1) (see, *e.g.*, [16, 18–23]).

Motivated by the works in the literature, we introduce two Halpern-Tseng type for solving the monotone inclusion problem and the fixed point problem of a relatively nonexpansive mapping in the framework of Banach spaces. We prove the strong convergence results of the proposed methods under some appropriate conditions. Finally, we provide numerical experiments to compressed sensing in signal recovery. The results presented in this paper are improve and generalize many known results in this direction.

2. Preliminaries

Let E be a real Banach space with its dual space E^*. We denote $\langle x, f\rangle$ by the value of a functional f in E^* at x in E, that is, $\langle x, f\rangle = f(x)$. For a sequence $\{x_n\}$ in E, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The set of all real numbers is denoted by $\mathbb{R}$, while $\mathbb{N}$ stands for the set of nonnegative integers. Let S_E denote the unit sphere of E. The space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. The space E is said to be *uniformly smooth* if the limit (2.1) converges uniformly in $x, y \in S_E$. It is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. The space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S_E, \|x - y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. Let $p \geq 2$. The space E is said to be *p-uniformly convex* if there is a $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Let $1 < q \leq 2$. The space E is said to be *q-uniformly smooth* if there exists a $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$, where ρ_E is the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in S_E\right\}$$

for all $t \geq 0$. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is observe that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [24]). If E is uniformly convex then E is reflexive and strictly convex and if E is uniformly smooth then E is reflexive and smooth (see [25]). Moreover, we know that for every $p > 1$, L_p and ℓ_p spaces are $\min\{p, 2\}$-uniformly smooth and $\max\{p, 2\}$-uniformly convex, while Hilbert space is 2-uniformly smooth and 2-uniformly convex (see [26] for more details).

Definition 2.1. Let C be a nonempty subset of E. Recall that a mapping $A : C \to E^*$ is said to be:

(i) *cocoercive* if there exists a constant $\gamma > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2$ for all $x, y \in C$;
(ii) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$;
(iii) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$;
(iv) *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Remark 2.2. It is easy to see that if A is cocoercive, then A is monotone and Lipschitz continuous but converse is not true in general.

Definition 2.3. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*.

If E is a Hilbert space, then $J = I$ is the identity mapping on E. It is known that E is smooth if and only if J is single-valued from E into E^* and if E is a reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E. Moreover, if E is uniformly smooth then J is norm-to-norm uniformly continuous on bounded subsets of E (see [25] for more details).

Lemma 2.4. [27, 28] *(i) Let E be a 2-uniformly smooth Banach space. Then there exists a constant $\kappa > 0$ such that*

$$\|x - y\|^2 \leq \|x\|^2 - 2\langle y, Jx\rangle + \kappa\|y\|^2,\ \forall x, y \in E.$$

(ii) Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that

$$\|x - y\|^2 \geq \|x\|^2 - 2\langle y, Jx\rangle + c\|y\|^2,\ \forall x, y \in E.$$

Remark 2.5. It is well-known that $\kappa = c = 1$ whenever E is a Hilbert space. Moreover, we refer to [28] for the exact values of the constants κ and c.

Next, we recall the following Lyapunov function which introduced in [29]:

Definition 2.6. Let E be a smooth Banach space. The Lyapunov functional $\phi : E\times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall x, y \in E.$$

In the particular case in which E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2,\ \forall x, y \in E$$

and

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z),\ \forall x, y, z \in E,\ \alpha \in [0, 1]. \quad (2.2)$$

In addition, the function ϕ satisfies the following *three point identity*:

$$\phi(x, y) = \phi(x, z) - \phi(y, z) + 2\langle y - x, Jy - Jz\rangle, \ \forall x, y, z \in E.$$

Lemma 2.7. [30] *Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that*

$$c\|x - y\|^2 \leq \phi(x, y), \ \forall x, y \in E,$$

where c is the constant in Lemma 2.4 (ii).

Lemma 2.8. [31] *Let E be a uniformly convex Banach space. Then there exists a continuous strictly increasing convex function $g : [0, 2r) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z) - \alpha(1 - \alpha)g(\|Jy - Jz\|)$$

for all $\alpha \in [0, 1]$, $x \in E$ and $y, z \in B_r := \{\omega : \|\omega\| \leq r\}$ for some $r > 0$.

The following important fact can be found in [32]. For two sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E. Then

$$\|x_n - y_n\| \to 0 \ \Leftrightarrow \ \|Jx_n - Jy_n\| \to 0 \ \Leftrightarrow \ \phi(x_n, y_n) \to 0. \tag{2.3}$$

Let C be a nonempty subset of a smooth Banach space E. A point $p \in C$ is a fixed point of T if $p = Tp$ and we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *relatively nonexpansive* if it satisfies the following conditions:

(i) $F(T) \neq \emptyset$;

(ii) $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$;

(iii) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, it follows that $p \in F(T)$.

Remark 2.9. If T satisfies (i) and (ii), then T is called *relatively quasi-nonexpansive*. In a Hilbert space H, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $T : C \to C$ is relatively quasi-nonexpansive, then it is quasi-nonexpansive, that is, $\|Tx - p\| \leq \|x - p\|$ for all $p \in F(T)$ and $x \in C$.

Lemma 2.10. [33] *Let E be a strictly convex and smooth Banach space. Let C be a closed and convex subset of E. If $T : C \to C$ be a relatively nonexpansive mapping, then $F(T)$ is closed and convex.*

We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in [29]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \ \forall x \in E, \ x^* \in E^*.$$

Obviously, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.11. [29] *Let E be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \ \forall x \in E, \ x^*, y^* \in E^*.$$

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed convex subset of E. Then we know that for any $x \in E$, there exists a unique point $z \in C$ such that

$$\phi(z, x) = \min_{y\in C} \phi(y, x).$$

Such a mapping $\Pi_C : E \to C$ defined by $z = \Pi_C(x)$ is called the *generalized projection*. If E is a Hilbert space, then Π_C is coincident with the metric projection denoted by P_C.

Lemma 2.12. [29] *Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed, and convex subset of E. For each $x \in E$ and $z \in C$. Then the following statements hold:*

(i) $z = \Pi_C(x)$ *if and only if* $\langle y - z, Jx - Jz \rangle \leq 0$, $\forall y \in C$.

(ii) $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x)$, $\forall y \in C$.

Let $B : E \to 2^{E^*}$ be a multi-valued mapping. The effective domain of B is denoted by $D(B) = \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $R(B) = \bigcup\{Bx : x \in D(B)\}$. The set of zeros of B is denoted by $B^{-1}0 = \{x \in D(B) : 0 \in Bx\}$. A multi-valued mapping B from E into E^* is said to be *monotone* if

$$\langle x - y, u - v \rangle \geq 0,\ \forall x, y \in D(B),\ u \in Bx \text{ and } v \in By.$$

A monotone operator B on E is said to be *maximal* if its graph $G(B) = \{(x, y) \in E \times E^* : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone operator on E. In other words, the maximality of B is equivalent to $R(J + \lambda B) = E^*$ for $\lambda > 0$ (see [34, Theorem 1.2]). It is known that if B is maximal monotone, then $B^{-1}0$ is closed and convex (see [35]). For a maximal monotone operator B, we define the resolvent of B by $J_\lambda^B(x) = (J + \lambda B)^{-1}Jx$ for $x \in E$ and $\lambda > 0$. It is also known that $B^{-1}0 = F(J_\lambda^B)$.

Lemma 2.13. [34] *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded mapping. Let $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A + B$ is a maximal monotone mapping.*

Lemma 2.14. *Let E be a reflexive, strictly convex and smooth Banach space. Let $A : E \to E^*$ be a mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then the following statements hold:*

(i) Define a mapping $T_\lambda x := J_\lambda^B \circ J^{-1}(J - \lambda A)x$ *for* $x \in E$ *and* $\lambda > 0$, *then* $F(T_\lambda) = (A + B)^{-1}0$.

(ii) $(A + B)^{-1}0$ *is closed and convex.*

Proof. (i) Let $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned} x = T_\lambda x \quad &\Leftrightarrow \quad x = J_\lambda^B \circ J^{-1}(J - \lambda A)x \\ &\Leftrightarrow \quad x = (J + \lambda B)^{-1} J \circ J^{-1}(J - \lambda A)x \\ &\Leftrightarrow \quad Jx - \lambda Ax \in Jx + \lambda Bx \\ &\Leftrightarrow \quad 0 \in (A + B)x \\ &\Leftrightarrow \quad x \in (A + B)^{-1}0. \end{aligned}$$

Hence $F(T_\lambda) = (A + B)^{-1}0$.

(ii) By Lemma 2.13, we know that $A + B$ is maximal monotone, then we can show that the set $(A + B)^{-1}0 = \{x \in E : 0 \in (A + B)x\}$ is closed and convex. ∎

Lemma 2.15. [36] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence of real numbers such that

(i) $\sum_{n=1}^\infty \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.16. [37] *Let* $\{a_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists an increasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k := \max\{j \leq k : a_j \leq a_{j+1}\}$.

3. Main Results

In this section, we introduce two Halpern-Tseng type for finding a common solution of the monotone inclusion problem and the fixed point problem in Banach spaces. From now on, let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone operator. Let $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap (A+B)^{-1}0 \neq \emptyset$. To prove the strong convergence results, we also need to assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, such that $\{\beta_n\} \subset [a,b] \subset (0,1)$ for some $a, b > 0$ and $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 1 Halpern-Tseng type algorithm

Step 0. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n A x_n). \tag{3.1}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.2}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n J z_n + (1-\beta_n) JTz_n)). \tag{3.3}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.1. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3. Then*

$$\phi(p, z_n) \leq \phi(p, x_n) - \Big(1 - \frac{\kappa \lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n), \ \forall p \in (A+B)^{-1}0,$$

where c *and* κ *are the constants in Lemma 2.4.*

Proof. Let $p \in (A+B)^{-1}0$. By Lemma 2.4 (ii), we have

$$\begin{aligned}
\phi(p,z_n) &= \phi(p,J^{-1}(Jy_n-\lambda_n(Ay_n-Ax_n))) \\
&= V(p,Jy_n-\lambda_n(Ay_n-Ax_n)) \\
&= \|p\|^2-2\langle p,Jy_n-\lambda_n(Ay_n-Ax_n)\rangle+\|Jy_n-\lambda_n(Ay_n-Ax_n)\|^2 \\
&\leq \|p\|^2-2\langle p,Jy_n\rangle+2\lambda_n\langle p,Ay_n-Ax_n\rangle+\|Jy_n\|^2-2\lambda_n\langle y_n,Ay_n-Ax_n\rangle \\
&\quad +\kappa\|\lambda_n(Ay_n-Ax_n)\|^2 \\
&= \|p\|^2-2\langle p,Jy_n\rangle+\|y_n\|^2-2\lambda_n\langle y_n-p,Ay_n-Ax_n\rangle+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
&= \phi(p,y_n)-2\lambda_n\langle y_n-p,Ay_n-Ax_n\rangle+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
&= \phi(p,x_n)+\phi(x_n,y_n)+2\langle x_n-p,Jy_n-Jx_n\rangle-2\lambda_n\langle y_n-p,Ay_n-Ax_n\rangle \\
&\quad +\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
&= \phi(p,x_n)+\phi(x_n,y_n)-2\langle y_n-x_n,Jy_n-Jx_n\rangle+2\langle y_n-p,Jy_n-Jx_n\rangle \\
&\quad -2\lambda_n\langle y_n-p,Ay_n-Ax_n\rangle+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
&= \phi(p,x_n)-\phi(y_n,x_n)+2\langle y_n-p,Jy_n-Jx_n\rangle-2\lambda_n\langle y_n-p,Ay_n-Ax_n\rangle \\
&\quad +\kappa\lambda_n^2\|Ay_n-Ax_n\|^2+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
&= \phi(p,x_n)-\phi(y_n,x_n)+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
&\quad -2\langle y_n-p,Jx_n-Jy_n+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2-\lambda_n(Ax_n-Ay_n)\rangle. \qquad (3.4)
\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}
\phi(p,z_n) \leq\ & \phi(p,x_n)-\Big(1-\frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n,x_n)+\kappa\lambda_n^2\|Ay_n-Ax_n\|^2 \\
& -2\langle y_n-p,Jx_n-Jy_n-\lambda_n(Ax_n-Ay_n)\rangle. \qquad (3.5)
\end{aligned}$$

We now show that

$$\langle y_n-p,Jx_n-Jy_n-\lambda_n(Ax_n-Ay_n)\rangle \geq 0.$$

From the definition of $\{y_n\}$, we note that $Jx_n-\lambda_nAx_n \in Jy_n+\lambda_nBy_n$. Since B is maximal monotone, there exists $v_n \in By_n$ such that $Jx_n-\lambda_nAx_n = Jy_n+\lambda_nv_n$, it follows that

$$v_n = \frac{1}{\lambda_n}\big(Jx_n-Jy_n-\lambda_nAx_n\big). \qquad (3.6)$$

Since $0 \in (A+B)p$ and $Ay_n+v_n \in (A+B)y_n$, it follows from Lemma 2.13 that $A+B$ is maximal monotone. Hence

$$\langle y_n-p,Ay_n+v_n\rangle \geq 0. \qquad (3.7)$$

Substituting (3.6) into (3.7), we have

$$\frac{1}{\lambda_n}\langle y_n-p,Jx_n-Jy_n-\lambda_nAx_n+\lambda_nAy_n\rangle \geq 0,$$

which implies that

$$\langle y_n-p,Jx_n-Jy_n-\lambda_n(Ax_n-Ay_n)\rangle \geq 0. \qquad (3.8)$$

Combining (3.5) and (3.8), we have

$$\phi(p,z_n) \leq \phi(p,x_n)-\Big(1-\frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n,x_n). \qquad (3.9)$$

■

Theorem 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Suppose that $\{\lambda_n\}$ be a sequence in $\big(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\big)$ such that $\{\lambda_n\} \subset [a', b'] \subset \big(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\big)$ for some $a', b' > 0$. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega(u)$.*

Proof. We first show that $\{x_n\}$ is bounded. Let $z \in \Omega$. Since $\lambda_n \in \big(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\big)$, we have $1 - \frac{\kappa\lambda_n^2 L^2}{c} > 0$. This implies by Lemma 3.1 that

$$\phi(z, z_n) \leq \phi(z, x_n). \tag{3.10}$$

Put $w_n = J^{-1}(\beta_n J z_n + (1-\beta_n)JTz_n)$ for all $n \in \mathbb{N}$. Thus by (2.2) and (3.10), we have

$$\begin{aligned}
\phi(z, w_n) &\leq \beta_n\phi(z, z_n) + (1-\beta_n)\phi(z, Tz_n) \\
&\leq \beta_n\phi(z, z_n) + (1-\beta_n)\phi(z, z_n) \\
&\leq \phi(z, x_n).
\end{aligned} \tag{3.11}$$

Using (3.11), we obtain

$$\begin{aligned}
\phi(z, x_{n+1}) &\leq \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, w_n) \\
&\leq \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, x_n) \\
&\leq \max\{\phi(z, u), \phi(z, x_n)\} \\
&\ \ \vdots \\
&\leq \max\{\phi(z, u), \phi(z, x_1)\}.
\end{aligned}$$

This implies that $\{\phi(z, x_n)\}$ is bounded. Applying Lemma 2.7, we have $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Let $x^* = \Pi_\Omega(u)$. From Lemma 2.8 and (3.9), we have

$$\begin{aligned}
\phi(x^*, w_n) &\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, Tz_n) - \beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|) \\
&\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\phi(x^*, z_n) - \beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|) \\
&\leq \beta_n\phi(x^*, z_n) + (1-\beta_n)\Big\{\phi(x^*, x_n) - \Big(1 - \frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n, x_n)\Big\} \\
&\quad -\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|) \\
&\leq \phi(x^*, x_n) - (1-\beta_n)\Big(1 - \frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n, x_n) \\
&\quad -\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|).
\end{aligned} \tag{3.12}$$

Then we have

$$\begin{aligned}
&\phi(x^*, x_{n+1}) \\
\leq\ & \alpha_n\phi(x^*, u) + (1-\alpha_n)\Big\{\phi(x^*, x_n) - (1-\beta_n)\Big(1 - \frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n, x_n) \\
& -\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|)\Big\} \\
=\ & \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n) - (1-\alpha_n)(1-\beta_n)\Big(1 - \frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n, x_n) \\
& -(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jz_n - JTz_n\|).
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\Big(1-\frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n,x_n)+(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\\
\le\ &\phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_nK, \qquad (3.13)
\end{aligned}$$

where $K=\sup_{n\in\mathbb{N}}\{|\phi(x^*,u)-\phi(x^*,x_n)|\}$.

The rest of the proof will be divided into two cases:

Case 1. Suppose that there exists $N\in\mathbb{N}$ such that $\phi(x^*,x_{n+1})\le\phi(x^*,x_n)$ for all $n\ge N$. This implies that $\lim_{n\to\infty}\phi(x^*,x_n)$ exists. By our assumptions, we have from (3.13) that

$$\lim_{n\to\infty}\phi(y_n,x_n)=0 \text{ and } \lim_{n\to\infty}g(\|Jz_n-JTz_n\|)=0. \qquad (3.14)$$

Consequently,

$$\lim_{n\to\infty}\|x_n-y_n\|=0 \text{ and } \lim_{n\to\infty}\|Jz_n-JTz_n\|=0. \qquad (3.15)$$

Moreover, we also have

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0. \qquad (3.16)$$

Since A is Lipschitz continuous, we have

$$\lim_{n\to\infty}\|Ax_n-Ay_n\|=0 \qquad (3.17)$$

and hence

$$\begin{aligned}
\|Jz_n-Jy_n\| &= \lambda_n\|Ax_n-Ay_n\|\\
&\to 0. \qquad (3.18)
\end{aligned}$$

Combining (3.16) and (3.18), we obtain

$$\begin{aligned}
\|Jx_n-Jz_n\| &\le \|Jx_n-Jy_n\|+\|Jy_n-Jz_n\|\\
&\to 0. \qquad (3.19)
\end{aligned}$$

Moreover from (3.15) and (3.19), we obtain

$$\begin{aligned}
\|Jx_{n+1}-Jx_n\| &\le \|Jx_{n+1}-Jw_n\|+\|Jw_n-Jz_n\|+\|Jz_n-Jx_n\|\\
&= \alpha_n\|Ju-Jw_n\|+(1-\beta_n)\|JTz_n-Jz_n\|+\|Jz_n-Jx_n\|\\
&\to 0. \qquad (3.20)
\end{aligned}$$

Then we have from (3.19) and (3.20) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0 \qquad (3.21)$$

and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.22)$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}\in E$ and

$$\limsup_{n\to\infty}\langle x_n-x^*,Ju-Jx^*\rangle=\lim_{k\to\infty}\langle x_{n_k}-x^*,Ju-Jx^*\rangle.$$

From (3.21), we also have $z_{n_k} \rightharpoonup \hat{x}$. Since $\|z_n - Tz_n\| \to 0$ and $I - T$ is demi-closed at zero, we have $\hat{x} \in F(T)$. We next show that $\hat{x} \in (A+B)^{-1}0$. Let $(v,w) \in G(A+B)$, we have $w - Av \in Bv$. Since

$$(J - \lambda_{n_k} A)x_{n_k} \in (J + \lambda_{n_k} B)y_{n_k}.$$

It follows that

$$\frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \in By_{n_k}.$$

Since B is maximal monotone, we have

$$\Big\langle v - y_{n_k}, w - Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k})\Big\rangle \geq 0$$

Using the monotonicity of A, we have

$$\begin{aligned}
\langle v - y_{n_k}, w\rangle &\geq \Big\langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k})\Big\rangle \\
&= \langle v - y_{n_k}, Av - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k}\rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle \\
&\quad + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle.
\end{aligned}$$

Since $y_{n_k} \rightharpoonup \hat{x}$, it follows from (3.16) and (3.17) that

$$\langle v - \hat{x}, w\rangle \geq 0.$$

By the monotonicity of $A + B$, we get $0 \in (A+B)\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}0$. So $\hat{x} \in \Omega := F(T) \cap (A+B)^{-1}0$. Thus we have

$$\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle = \langle \hat{x} - x^*, Ju - Jx^*\rangle \leq 0.$$

From (3.22), we also have

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, Ju - Jx^*\rangle \leq 0. \tag{3.23}$$

Finally, we show that $x_n \to x^*$. By Lemma 2.11, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jw_n)) \\
&= V(x^*, \alpha_n Ju + (1-\alpha_n)Jw_n) \\
&\leq V(x^*, \alpha_n Ju + (1-\alpha_n)Jw_n - \alpha_n(Ju - Jx^*)) \\
&\quad + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\
&= V(x^*, \alpha_n Jx^* + (1-\alpha_n)Jw_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\
&= \phi(x^*, J^{-1}(\alpha_n Jx^* + (1-\alpha_n)Jw_n)) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\
&\leq \alpha_n\phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle.
\end{aligned} \tag{3.24}$$

This together with (3.23) and (3.24), so we can conclude by Lemma 2.15 that $\phi(x^*, x_n) \to 0$. Therefore, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\phi(x^*, x_{n_i})\}$ of $\{\phi(x^*, x_n)\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. By Lemma 2.16, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \tag{3.25}$$

and

$$\phi(x^*, x_k) \leq \phi(x^*, x_{m_k}). \tag{3.26}$$

As proved in the **Case 1**, we obtain

$$\begin{aligned}
& (1-\alpha_{m_k})(1-\beta_{m_k})\Big(1-\frac{\kappa\lambda_{m_k}^2 L^2}{c}\Big)\phi(y_{m_k}, x_{m_k}) \\
& +(1-\alpha_{m_k})\beta_{m_k}(1-\beta_{m_k})g(\|Jz_{m_k} - JTz_{m_k}\|) \\
\leq \;& \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) + \alpha_{m_k} K \\
\leq \;& \alpha_{m_k} K,
\end{aligned}$$

where $K = \sup_{k\in\mathbb{N}}\{|\phi(x^*, u) - \phi(x^*, x_{m_k})|\}$. By our assumptions, we have

$$\lim_{k\to\infty} \phi(y_{m_k}, x_{m_k}) = 0 \text{ and } \lim_{k\to\infty} g(\|Jz_{m_k} - JTz_{m_k}\|) = 0.$$

Consequently,

$$\lim_{k\to\infty} \|x_{m_k} - y_{m_k}\| = 0 \text{ and } \lim_{k\to\infty} \|Jz_{m_k} - JTz_{m_k}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we can show that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty} \langle x_{m_k+1} - x^*, Ju - Jx^*\rangle \leq 0.$$

From (3.24) and (3.25), we have

$$\begin{aligned}
\phi(x^*, x_{m_k+1}) &\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k}) + \alpha_{m_k}\langle x_{m_k+1} - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k+1}) + \alpha_{m_k}\langle x_{m_k+1} - x^*, Ju - Jx^*\rangle.
\end{aligned}$$

This implies that

$$\phi(x^*, x_{m_k+1}) \leq \langle x_{m_k+1} - x^*, Ju - Jx^*\rangle.$$

Then we have

$$\limsup_{k\to\infty} \phi(x^*, x_{m_k+1}) \leq 0. \tag{3.27}$$

Combining (3.26) and (3.27) we obtain

$$\limsup_{k\to\infty} \phi(x^*, x_k) \leq 0.$$

Hence $\limsup_{k\to\infty} \phi(x^*, x_k) = 0$ and so $x_k \to x^*$. This completes the proof. ∎

If we take $T = I$ in Theorem 3.2, then we obtain the following result regarding the monotone quasi-inclusion problem (1.1).

Corollary 3.3. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Assume that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, u \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(Jy_n - \lambda_n(Ay_n - Ax_n))), \ \forall n \geq 1, \end{cases} \tag{3.28}$$

where $\{\lambda_n\}$ is a sequence in $\left(0, \frac{1}{L}\right)$ such that $\{\lambda_n\} \subset [a', b'] \subset \left(0, \frac{1}{L}\right)$ for some $a', b' > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.28) converges strongly to $x^* \in (A+B)^{-1}0$, where $x^* = \Pi_{(A+B)^{-1}0}(u)$.

We next propose a strong convergence theorem of another modification of Tseng's splitting algorithm with line search for solving the monotone inclusion problem and the fixed point problem in Banach spaces. It is noted that this proposed algorithm does not required to know the Lipschitz constant of the Lipschitz continuous mapping.

Algorithm 2 Halpern-Tseng type algorithm with Armijo-type line search

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Step 0. Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in \left(0, \sqrt{\frac{c}{\kappa}}\right)$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \tag{3.29}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{3.30}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.31}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n Jz_n + (1-\beta_n)JTz_n)). \tag{3.32}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.4. *The Armijo line search rule defined by (3.30) is well defined and*

$$\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma.$$

Proof. Since A is L-Lipschitz continuous on E, we have

$$\|Ax_n - A(J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n))\| \leq L\|x_n - J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n)\|.$$

Using the fact that $L > 0$ and $\mu > 0$, we get

$$\frac{\mu}{L}\|Ax_n - A(J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n))\| \leq \mu\|x_n - J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n)\|.$$

This implies that (3.30) holds for all $\gamma l^{m_n} \leq \frac{\mu}{L}$ and so λ_n is well defined. Obviously, $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$, then the lemma is proved. Otherwise, if $\lambda_n < \gamma$, then we have from

(3.30) that

$$\|Ax_n - A(J^B_{\frac{\lambda_n}{l}} J^{-1}(Jx_n - \frac{\lambda_n}{l} Ax_n))\| \quad > \quad \frac{\mu}{\frac{\lambda_n}{l}} \|x_n - J^B_{\frac{\lambda_n}{l}} J^{-1}(Jx_n - \frac{\lambda_n}{l} Ax_n)\|.$$

Again by the L-Lipschitz continuity of A, we obtain $\lambda_n > \frac{\mu l}{L}$. This completes the proof. ■

Lemma 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then*

$$\phi(p, z_n) \leq \phi(p, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\Big)\phi(y_n, x_n), \;\; \forall p \in (A+B)^{-1}0,$$

where c and κ are the constants in Lemma 2.4.

Proof. From (3.30), we see that $\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_n}\|x_n - y_n\|$. By using the same arguments as in the proof of Lemma 3.1, we can show that this lemma holds. ■

Theorem 3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$.*

Proof. By using the same arguments as in the proof of Theorem 3.2, we immediately obtain the proof. ■

If we take $T = I$ in Theorem 3.6, then we obtain the following result regarding the monotone quasi-inclusion problem (1.1).

Corollary 3.7. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone operator. Assume that $(A+B)^{-1}0 \neq \emptyset$. Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in \big(0, \sqrt{\frac{c}{\kappa}}\big)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, u \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(Jy_n - \lambda_n(Ay_n - Ax_n))), \; \forall n \geq 1, \end{cases} \tag{3.33}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n\|Ax_n - Ay_n\| \leq \mu\|x_n - y_n\|.$$

Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.33) converges strongly to $x^* \in (A+B)^{-1}0$, where $x^* = \Pi_{(A+B)^{-1}0}(u)$.

4. Numerical Experiments

In this section, we provide numerical experiments to the signal recovery in compressed sensing by using our proposed algorithms. Moreover, we also compare the mentioned algorithms with Tseng's splitting algorithm (1.5). In signal recovery, compressed sensing can be modeled as the following under determinated linear equation system:

$$y = Cx + \varepsilon \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $C : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear observation operator. It is known that to solve (4.1) can be seen as solving the LASSO problem [5]:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Cx-y\|_2^2+\lambda\|x\|_1, \tag{4.2}$$

where $\lambda>0$. In this case, we set $A=\nabla f$ the gradient of f, where $f(x)=\frac{1}{2}\|Cx-y\|_2^2$ and $B=\partial g$ the subdifferential of g, where $g(x)=\lambda\|x\|_1$. Then the LASSO problem (4.2) can be considered as the monotone quasi-inclusion problem (1.1). It is known that $\nabla f(x)=C^t(Cx-y)$ and it is $\|C\|^2$-Lipschitz continuous and monotone (see [3]). Moreover, ∂g is maximal monotone (see [1]).

In this experiment, the sparse vector $x\in\mathbb{R}^N$ is generated from uniform distribution in the interval $[-2,2]$ with m nonzero elements. The matrix $C\in\mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio (SNR)=40. The restoration accuracy is measured by the mean squared error (MSE) as follows:

$$E_n=\frac{1}{N}\|x_n-x\|_2^2<10^{-5}, \tag{4.3}$$

where x_n is an estimated signal of x. In our numerical test, we compare our Algorithm 3 and Algorithm 3 ($T=I$) with Tseng's splitting algorithm (1.5).

We take $\alpha_n=\frac{1}{15(n+5)}$ and $\lambda_n=\frac{0.3}{\|C\|^2}$ in Algorithm 3 and take $\lambda_n=\frac{0.3}{\|C\|^2}$ in Tseng's splitting algorithm (1.5). For Alogorithm 3, we take $\alpha_n=\frac{1}{15(n+5)}$, $\gamma=5$, $\mu=0.5$, $l=0.3$. The point u is chosen to be $(1,1,1,\dots,1)\in\mathbb{R}^N$ and the starting point x_1 is randomly generated in $\mathbb{R}^N$. We perform the numerical test with the following four cases:

Case 1: $N=512$, $M=256$ and $m=10$;
Case 2: $N=1024$, $M=512$ and $m=30$;
Case 3: $N=2048$, $M=1024$ and $m=60$;
Case 4: $N=4096$, $M=2048$ and $m=100$.

The numerical results are reported as follows:

Table 1. The comparison of the proposed algorithms with Tseng's splitting algorithm

		Algorithm 3	Algorithm 3	Tseng's splitting algorithm
Case 1	No. of Iter.	1,850	4,864	5,689
Case 2	No. of Iter.	3,320	10,186	12,753
Case 3	No. of Iter.	7,126	19,076	24,666
Case 4	No. of Iter.	14,889	40,743	48,652

We next demonstrate the graphs of original signal and recovered signal by Algorithm 3, Algorithm 3 and Tseng's splitting algorithm. The number of iterations are reported in the Figures 1-8, respectively.

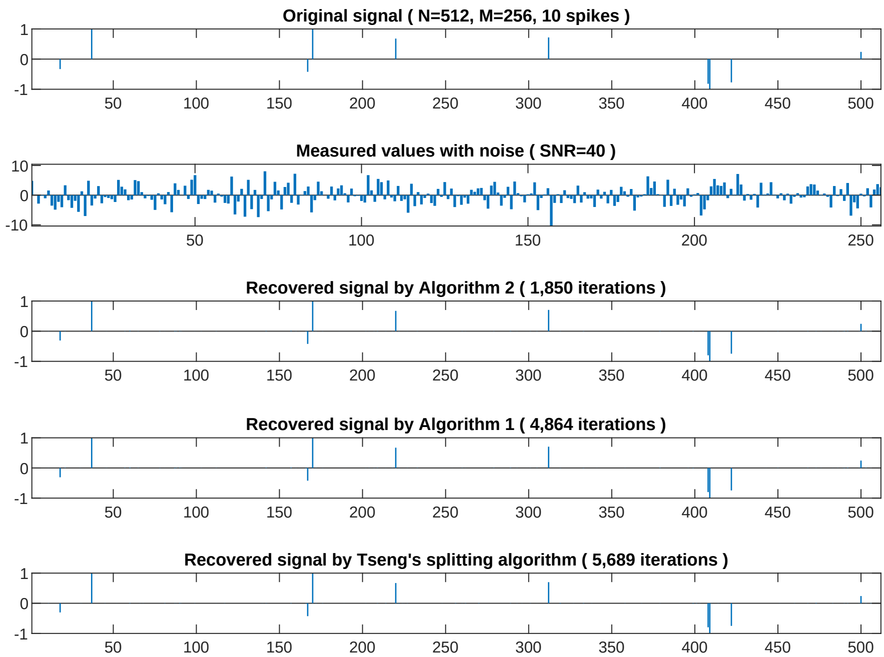

Figure 1: The comparison of recovered signal by using different algorithms in **Case 1**.

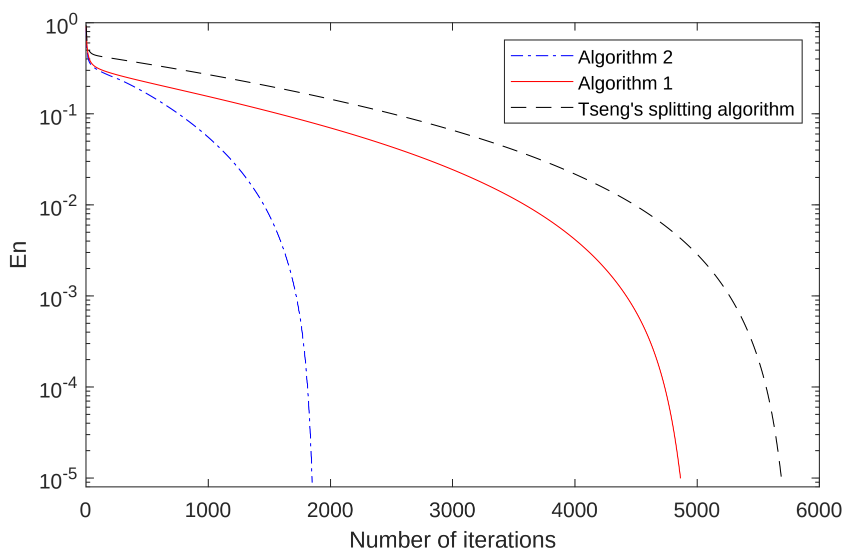

Figure 2: The plotting of MSE versus number of iterations in **Case 1**.

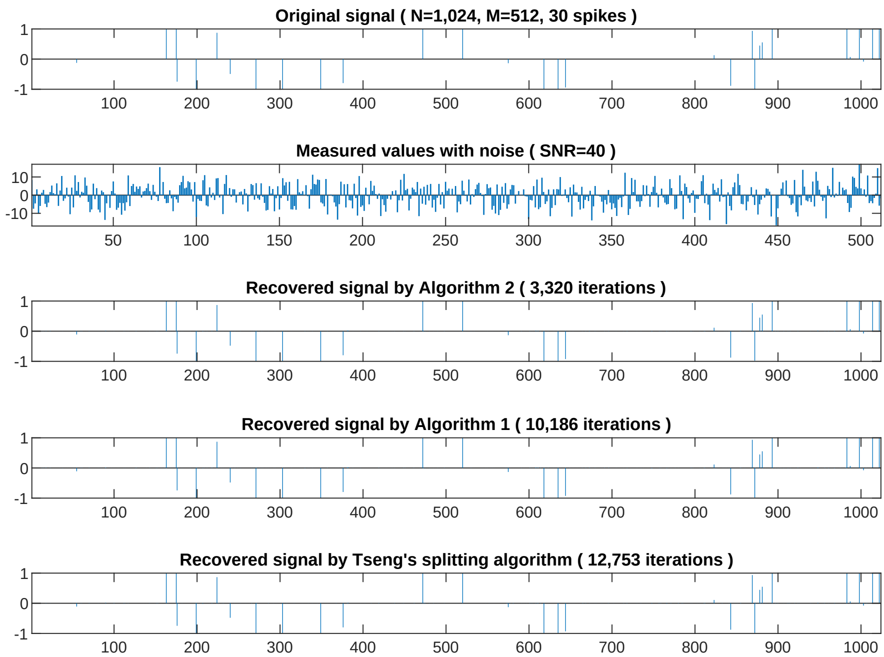

Figure 3: The comparison of recovered signal by using different algorithms in **Case 2**.

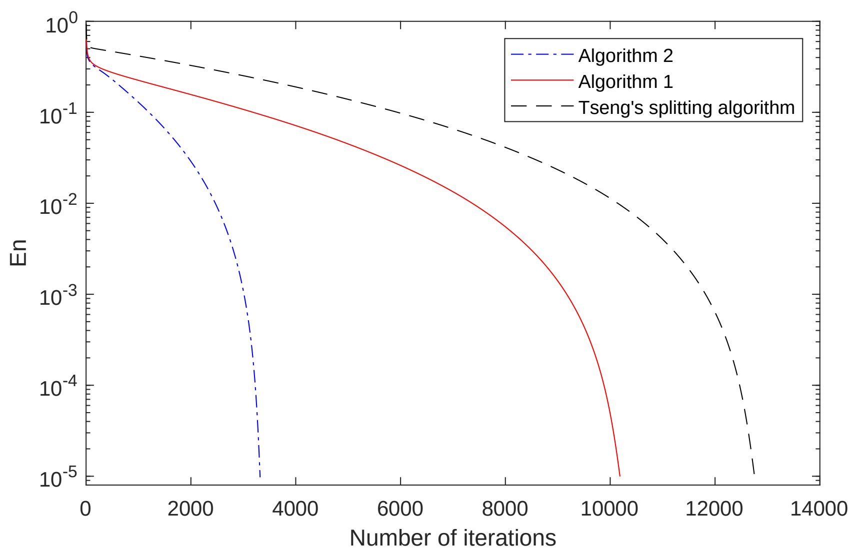

Figure 4: The plotting of MSE versus number of iterations in **Case 2**.

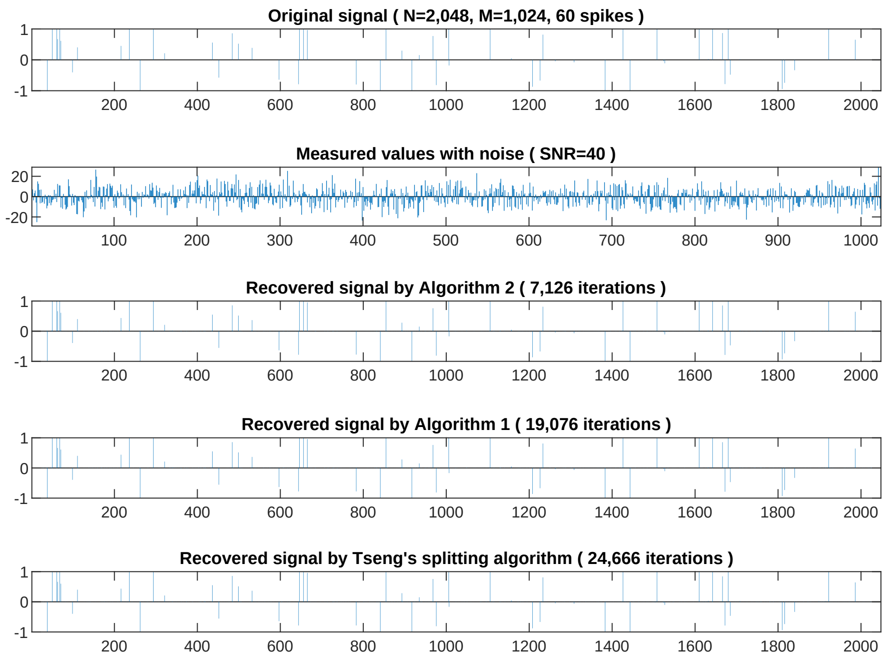

Figure 5: The comparison of recovered signal by using different algorithms in **Case 3**.

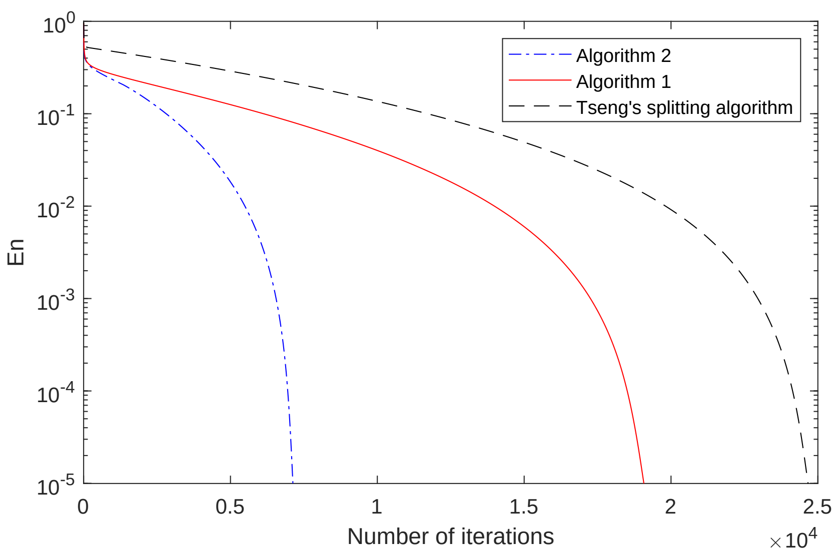

Figure 6: The plotting of MSE versus number of iterations in **Case 3**.

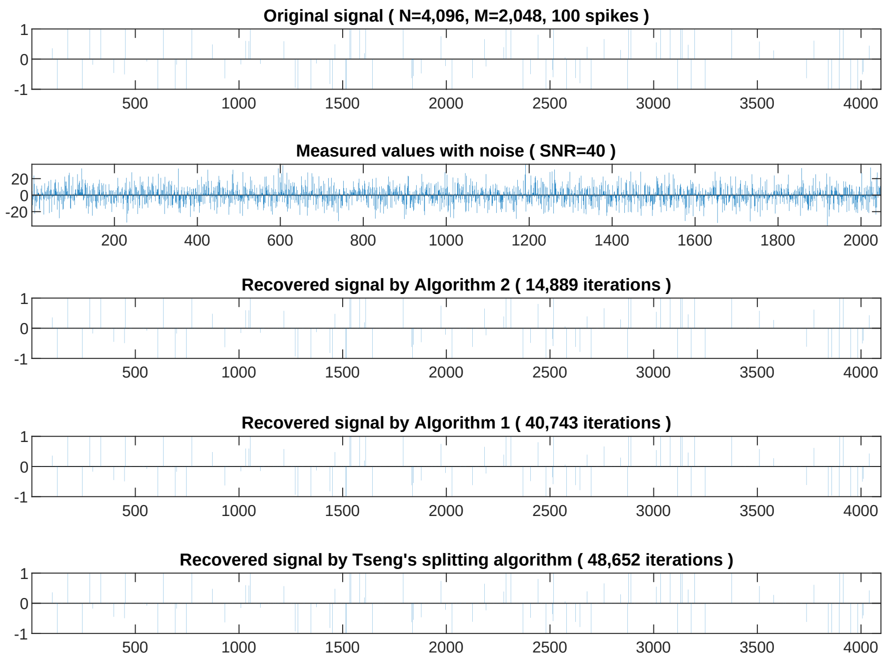

Figure7: The comparison of recovered signal by using different algorithms in **Case 4**.

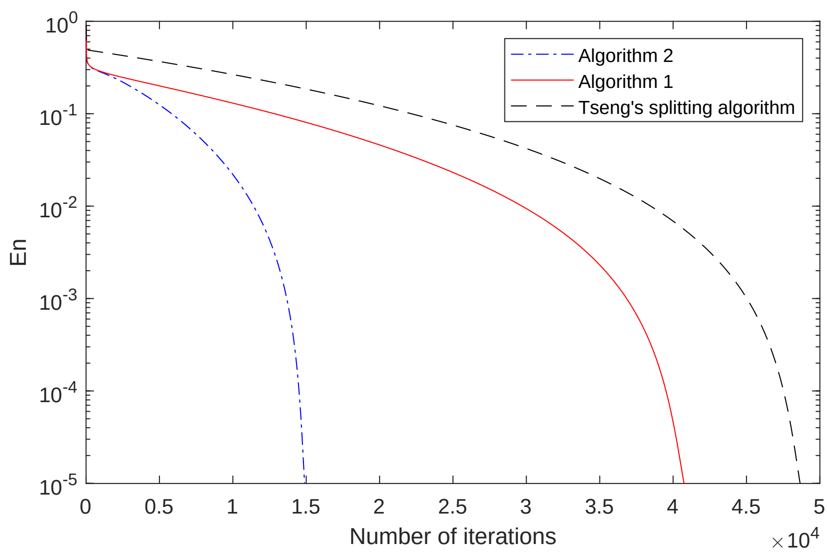

Figure 8: The plotting of MSE versus number of iterations in **Case 4**.

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ORIGINAL PAPER

An explicit parallel algorithm for solving variational inclusion problem and fixed point problem in Banach spaces

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Abstract
In this article, we introduce an explicit parallel algorithm for finding a common element of zeros of the sum of two accretive operators and the set of fixed point of a nonexpansive mapping in the framework of Banach spaces. We prove its strong convergence under some mild conditions. Finally, we provide some applications to the main result. The results presented in this paper extend and improve the corresponding results in the literature.

Keywords Variational inclusion · Banach space · Strong convergence · Accretive operator

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1 Introduction

Let X be a real Banach space, we consider the following so-called *variational inclusion problem*: find a point $x \in X$ such that

$$0 \in Ax + Bx, \tag{1.1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. The set of solutions of (1.1) is denoted by $(A + B)^{-1}0$.

Variational inclusions have been studied widely in applied sciences. They provide a unified framework for studying many real-world problems arising economics, structural analysis, mechanics, optimization problems, signal processing and image recovery. Furthermore, it is well known that this problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem.

In a Hilbert space H, a classical method for solving this problem is the forward-backward splitting method [9] which is defined by the following manner: $x_1 \in H$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \quad n \geq 1, \tag{1.2}$$

where $r > 0$. We see that each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of A and B. This method generalizes the proximal point algorithm [10] and the gradient method [11].

In 2010, Takahashi et al. [20] introduced the following iteration process for maximal monotone operators with nonlinear operator in Hilbert spaces H: $x, x_1 \in C$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n x + (1 - \alpha_n)J_{\lambda_n}(x_n - \lambda_n Ax_n)), \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{\lambda_n\}$ is a positive sequence, $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and S is a nonexpansive mapping on C. They showed that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point in $F(S) \cap (A + B)^{-1}0$ under some mild conditions.

For solving the problem (1.1) in Banach spaces X, López et al. [21] introduced the following Halpern-type forward-backward method. For an initial point $u, x_1 \in X$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n), \tag{1.4}$$

where J_r^B is the resolvent of B, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\}, \{b_n\}$ are error sequences in X. It was proved that the sequence $\{x_n\}$ generated by (1.4) strongly converges to a zero point of the sum of A and B under some appropriate conditions.

Recently, Yang et al. [25] introduced the following algorithm for solving the fixed point problem of a nonexpansive mapping S in a Hilbert space:

$$x_{n+1} = \beta Sx_n + (1 - \beta)P_C[(1 - \alpha_n)x_n], \quad n \geq 1. \tag{1.5}$$

It was proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the minimum-norm fixed point of S.

We know that, in many practical optimization problems with physical and engineering background, it is needed to find minimum-norm solution of given problems. We note that finding minimum-norm solutions attracted recently many researcher's interest, due to the fact that these algorithms have extensive applications in a variety applied areas such as inverse problems, partial differential equations, linear programming, image recovery, signal processing and so on (see [6,12,13]).

Motivated and inspired by the above results, we study an explicit parallel algorithm for solving variational inclusion problem for the sum of accretive and m-accretive operators and fixed point problems in the framework of q-uniformly smooth and uniformly convex Banach spaces. We prove its strong convergence under some mild conditions. Finally, we provide some applications to the main result. The results presented in this paper extend and improve the corresponding results in the literature.

2 Preliminaries

Throughout this paper, we denote by X and X^* the real Banach space and the dual space of X, respectively. Let C be a nonempty subset of a real Banach space X. A mapping $S : C \to C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If $L = 1$, then S is a nonexpansive mapping. We denote the fixed points set of the mapping S by $F(S)$, i.e., $F(S) = \{x \in C : x = Sx\}$.

Let $q > 1$ be a real number. The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ j_q(x) \in X^* : \langle x, j_q(x) \rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1} \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the *normalized duality mapping* and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If $X := H$ is a real Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q (see [19]).

We use the notation $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x and $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x. For $q > 1$, we say that a generalized duality mapping j_q is *weakly sequentially continuous* if for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, then $j_q(x_n) \rightharpoonup^* j_q(x)$.

The *modulus of convexity* of X is the function $\delta : (0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \tfrac{\|x+y\|}{2} : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \right\}.$$

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

The *modulus of smoothness* of X is the function $\rho : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho(t) = \sup\left\{ \tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1 \right\}.$$

A Banach space X is said to be *uniformly smooth* if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. Suppose that $1 < q \leq 2$, then X is said to be q-*uniformly smooth* if there exists $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. If X is q-uniformly smooth, then X is also uniformly smooth. For a Hilbert space H, L_p space and l_p space, we note that

$$H,\ L_p \text{ and } l_p \text{ are } \begin{cases} 2 - \text{uniformly smooth}, & \text{if } 2 \leq p < \infty, \\ p - \text{uniformly smooth}, & \text{if } 1 < p \leq 2. \end{cases}$$

Furthermore, the following facts are well known (see [1,23]).

(1) For $2 \leq p < \infty$, the spaces of L_p and l_p are 2-uniformly smooth with $K_q = K_2 = p - 1$.
(2) For $1 < p \leq 2$, the spaces of L_p and l_p are p-uniformly smooth with $K_q = K_p = (1 + t_p^{p-1})(1 + t_p)^{1-p}$, where t_p is the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0, \quad 0 < t < 1.$$

(3) Every Hilbert space is 2-uniformly smooth with $K_q = K_2 = 1$.
(4) For $1 < p < \infty$, the spaces of L_p and l_p are uniformly smooth and uniformly convex. More precisely, L_p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Let $A : X \to 2^X$ be a set-valued mapping. We denote the domain and range of an operator $A : X \to 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. Let $q > 1$. A set-valued mapping $A : D(A) \subset X \to 2^X$ is said to be *accretive* of order q if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0, \quad u \in Ax \quad \text{and} \quad v \in Ay.$$

An accretive operator A is said to be m-*accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$. For an accretive operator A, we can define a mapping $J_\lambda^A : R(I + \lambda A) \to D(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$. Such J_λ^A are called the *resolvents* of A for $\lambda > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone (see [19]).

Let $\alpha > 0$ and $q > 1$. A mapping $A : C \to X$ is said to be α-*inverse strongly accretive* (α-isa) of order q if for each $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^q.$$

If $X := H$ is a real Hilbert space, then $A : C \to H$ is called α-inverse strongly monotone (α-ism). It is obvious that if A is an α-ism then A is $1/\alpha$-Lipschitzian.

Lemma 2.1 [21] *Let X be a q-uniformly smooth Banach space. Let $A : C \to X$ be an α-isa of order q. Then the following inequality holds for all $x, y \in C$:*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^q \le \|x-y\|^q - \lambda(\alpha q - K_q\lambda^{q-1})\|Ax-Ay\|^q, \tag{2.1}$$

wherer κ_q is the q-uniformly smoothness coefficient of X. In particular, if $0<\lambda<\left(\frac{\alpha q}{K_q}\right)^{\frac{1}{q-1}}$, then $I-\lambda A$ is nonexpansive.

Let C be a nonempty, closed and convex subset of a real Banach space X and let D be a nonempty subset of C. A *retraction* from C to D is a mapping $Q : C \to D$ such that $Qx = x$ for all $x \in D$. A mapping $Q : C \to D$ is said to be *sunny* if

$$Q(tx + (1-t)Qx) = Qx,$$

whenever $tx + (1-t)Qx \in C$ for $x \in C$ and $t \ge 0$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive (see [19]). It is well known that if $X := H$ is a real Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from H onto C.

Lemma 2.2 [16] *Let C be a closed and convex subset of a smooth Banach space X. Let $Q : X \to C$ be a retraction. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive;*
(b) $\langle x - Qx, j_q(y-Qx)\rangle \le 0$ *for all $x \in X$ and $y \in C$.*

Lemma 2.3 [21] *Let X be a Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ be an m-accretive operator. Then we have*

$$F(J_\lambda^B(I-\lambda A)) = (A+B)^{-1}0.$$

Lemma 2.4 [23] *Let $q > 1$ and let X be a real normed space with the generalized duality mapping j_q. Then, for each $x, y \in X$, we have*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x+y)\rangle$$

for all $j_q(x+y) \in J_q(x+y)$.

Lemma 2.5 [23] *Let $p > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then the following are equivalent.*

(i) *X is uniformly convex;*
(ii) *There is a strictly increasing, continuous and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \le \|x\|^p - p\langle x, j_p(y)\rangle + (p-1)\|y\|^p, \quad \forall x, y \in B_r.$$

Lemma 2.6 [5] *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and $S : C \to C$ be a nonexpansive mapping. Then $I-S$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$ implies $x = Sx$.*

Lemma 2.7 [18] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Proposition 2.8 [14] *Let* $q > 1$. *Then the following inequality holds:*

$$a^q - b^q \le qa^{q-1}(a - b),$$

for arbitrary positive real numbers a, b.

Lemma 2.9 [24] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 (The Resolvent Identity [3]) *For* $\lambda, \mu > 0$ *and* $x \in X$, *we have*

$$J_\lambda^B x = J_\mu^B \left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^B x\right).$$

Using the resolvent identity, we have the following lemma.

Lemma 2.11 *For each* $\lambda, \mu > 0$ *then*

$$\|J_\lambda^B x - J_\mu^B x\| \le \left|\frac{\lambda - \mu}{\lambda}\right| \|J_\lambda^B x - x\| \quad \textit{for all } x \in X.$$

Lemma 2.12 *Let* X *be a real* q*-uniformly smooth Banach space. Let* B *be an* m*-accretive operator on* X *and let* J_λ^B *be a resolvent of* B *for* $\lambda > 0$. *Then we have*

$$\|J_\lambda^B x - J_\lambda^B y\|^q \le \langle x - y, j_q(J_\lambda^B x - J_\lambda^B y)\rangle, \quad \forall x, y \in X.$$

Proof For any $x, y \in X$ and $\lambda > 0$, we set $u = J_\lambda^B x$ and $v = J_\lambda^B y$. By definition of the accretive operator, we have $x - u \in \lambda Bu$ and $y - v \in \lambda Bv$. Since B is m-accretive, we have

$$\begin{aligned} 0 &\le \langle x - u - (y - v), j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \langle u - v, j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \|u - v\|^q, \end{aligned}$$

which implies that

$$\|u-v\|^q \leq \langle x-y, j_q(u-v)\rangle,$$

i.e.,

$$\|J_\lambda^B x - J_\lambda^B y\|^q \leq \langle x-y, j_q(J_\lambda^B x - J_\lambda^B y)\rangle, \quad \forall x, y \in X.$$

□

3 Main result

In this section, we now are in position to give a proof of our main result.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a q-uniformly smooth and uniformly convex Banach spaces X which admits a weakly sequentially continuous duality mapping $j_q : X \to X^*$. Let $A : C \to X$ be an α-isa of order q and let $B : D(B) \subset C \to 2^X$ be an m-accretive operator. Let $J_\lambda^B = (I+\lambda B)^{-1}$ be a resolvent of B for $\lambda > 0$ and let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap (A+B)^{-1}0 \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma J_{\lambda_n}^B((1-\alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\gamma \in (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(C2) $0 < a \leq \beta_n \leq b < 1$;
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < \left(\frac{\alpha q}{K_q}\right)^{\frac{1}{q-1}}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $x^* = Q_\Gamma 0$, where Q_Γ is a sunny nonexpansive retraction of C onto Γ.

Proof First, we will show that $\{x_n\}$ is bounded. Set $z_n = J_{\lambda_n}^B((1-\alpha_n)x_n - \lambda_n A x_n)$ for all $n \geq 1$. Taking $p \in \Gamma$, we obtain

$$p = Sp = J_{\lambda_n}^B(p - \lambda_n Ap) = J_{\lambda_n}^B\left(\alpha_n p + (1-\alpha_n)\left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right).$$

Since $J_{\lambda_n}^B$ and $I - \frac{\lambda_n}{1-\alpha_n}A$ are nonexpansive (see Lemma 2.1), we have

$$\begin{aligned} &\|z_n - p\| \\ &= \left\| J_{\lambda_n}^B\left((1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n\right)\right. \end{aligned}$$

$$
\begin{aligned}
&\quad -J_{\lambda_n}^B\left(\alpha_n p + (1-\alpha_n)\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right)\Big\| \\
&\le \left\|\alpha_n(-p) + (1-\alpha_n)\left[\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right]\right\| \\
&\le \alpha_n\|p\| + (1-\alpha_n)\left\|\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)p\right\| \\
&\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\|. \qquad (3.2)
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\|y_n - p\| &= \|(1-\gamma)(Sx_n - p) + \gamma(z_n - p)\| \\
&\le (1-\gamma)\|Sx_n - p\| + \gamma\|z_n - p\| \\
&\le (1-\gamma)\|x_n - p\| + \gamma\big[(1-\alpha_n)\|x_n - p\| + \alpha_n\|p\|\big] \\
&= (1-\alpha_n\gamma)\|x_n - p\| + \alpha_n\gamma\|p\|.
\end{aligned}
$$

Then, we see that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(y_n - p)\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\big[(1-\alpha_n\gamma)\|x_n - p\| + \alpha_n\gamma\|p\|\big] \\
&= \big[1 - (1-\beta_n)\alpha_n\gamma\big]\|x_n - p\| + (1-\beta_n)\alpha_n\gamma\|p\| \\
&\le \max\{\|x_n - p\|, \|p\|\}.
\end{aligned}
$$

By induction, we have

$$
\|x_n - p\| \le \max\{\|x_1 - p\|, \|p\|\}, \quad \forall n \ge 1.
$$

Hence, $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Set $z_n = J_{\lambda_n}^B u_n$, where $u_n = (1-\alpha_n)x_n - \lambda_n A x_n$. We observe that

$$
\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \|J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_n}^B u_n\| \\
&\le \|J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_{n+1}}^B u_n\| + \|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\| \\
&\le \|u_{n+1} - u_n\| + \|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\| \\
&= \|(1-\alpha_{n+1})x_{n+1} - \lambda_{n+1}Ax_{n+1} - ((1-\alpha_n)x_n - \lambda_n A x_n)\| \\
&\quad + \|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\| \\
&= \Big\|(\alpha_n - \alpha_{n+1})x_n + (1-\alpha_{n+1})\left[\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1}\right. \\
&\quad \left. -\left(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_n\right]
\end{aligned}
$$

$$
\begin{aligned}
&+(\lambda_n-\lambda_{n+1})Ax_n\Big\| + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\
&\le |\alpha_{n+1}-\alpha_n|\|x_n\| + (1-\alpha_{n+1})\Big\|\left(I-\frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_{n+1} \\
&-\left(I-\frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\right)x_n\Big\| \\
&+|\lambda_{n+1}-\lambda_n|\|Ax_n\| + \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \\
&\le (1-\alpha_{n+1})\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\|x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&+\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\|.
\end{aligned}
$$

On the other hand, by Lemma 2.11, we have

$$
\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| \le \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J^B_{\lambda_{n+1}}u_n - u_n\|.
$$

Then, it follows that

$$
\begin{aligned}
\|z_{n+1}-z_n\| \le (1-\alpha_{n+1})\|x_{n+1}-x_n\| &+ |\alpha_{n+1}-\alpha_n|\|x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&+\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\left\|J^B_{\lambda_{n+1}}u_n - u_n\right\|.
\end{aligned} \tag{3.3}
$$

Note that $x_{n+1}=\beta_n x_n + (1-\beta_n)y_n$, where $y_n=(1-\gamma)Sx_n+\gamma z_n$. Then, we have

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|(1-\gamma)(Sx_{n+1}-Sx_n)+\gamma(z_{n+1}-z_n)\| \\
&\le (1-\gamma)\|Sx_{n+1}-Sx_n\| + \gamma\|z_{n+1}-z_n\| \\
&\le (1-\gamma)\|x_{n+1}-x_n\| + \gamma\|z_{n+1}-z_n\|.
\end{aligned} \tag{3.4}
$$

Substituting (3.3) into (3.4), we have

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\le (1-\gamma)\|x_{n+1}-x_n\| + \gamma\Big[(1-\alpha_{n+1})\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\|x_n\| \\
&\quad +|\lambda_{n+1}-\lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|J^B_{\lambda_{n+1}}u_n-u_n\|\Big] \\
&\le (1-\alpha_{n+1}\gamma)\|x_{n+1}-x_n\| + \left(|\alpha_{n+1}-\alpha_n| + |\lambda_{n+1}-\lambda_n| + \frac{|\lambda_{n+1}-\lambda_n|}{a'}\right)M,
\end{aligned}
$$

where $M>0$ is an appropriate constant. From $(C1)$ and $(C3)$, we have

$$
\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0.
$$

So by Lemma 2.7, we conclude that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.5}$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|x_n - y_n\| = 0. \tag{3.6}$$

Next, we show that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. By the convexity of $\|\cdot\|^q$ for all $q > 1$ and 2.1, we see that

$$\begin{aligned}
&\|u_n - p\|^q \qquad (3.7)\\
&= \left\|(1-\alpha_n)\left[\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right] + \alpha_n(-p)\right\|^q\\
&\le (1-\alpha_n)\left\|\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right\|^q + \alpha_n\|p\|^q\\
&\le (1-\alpha_n)\left[\|x_n - p\|^q - \frac{\lambda_n}{1-\alpha_n}\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q\right] + \alpha_n\|p\|^q\\
&\le \|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\|p\|^q. \qquad (3.8)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|y_n - p\|^q\\
&= \|(1-\gamma)(Sx_n - p) + \gamma(J^B_{\lambda_n}u_n - p)\|^q\\
&\le (1-\gamma)\|Sx_n - p\|^q + \gamma\|J^B_{\lambda_n}u_n - p\|^q\\
&\le (1-\gamma)\|x_n - p\|^q + \gamma\|u_n - p\|^q\\
&\le (1-\gamma)\|x_n - p\|^q\\
&\quad + \gamma\left[\|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\|p\|^q\right]\\
&= \|x_n - p\|^q - \lambda_n\gamma\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\gamma\|p\|^q.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|x_{n+1} - p\|^q\\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|y_n - p\|^q\\
&\le \beta_n\|x_n - p\|^q\\
&\quad + (1-\beta_n)\left[\|x_n - p\|^q - \lambda_n\gamma\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\gamma\|p\|^q\right]
\end{aligned}$$

$$= \|x_n - p\|^q - \lambda_n(1-\beta_n)\gamma\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q$$
$$+ \alpha_n\gamma(1-\beta_n)\|p\|^q,$$

which implies by $(C2)$, $(C3)$ and Proposition 2.8 that

$$\begin{aligned}
&\lambda_n(1-\beta_n)\gamma\left(\alpha q - \frac{K_q\lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q \\
&\quad\le a'(1-b)\gamma\big(\alpha q - k_q(b')^{q-1}\big)\|Ax_n - Ap\|^q \\
&\quad\le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n\gamma(1-\beta_n)\|p\|^q \\
&\quad\le q\|x_n - p\|^{q-1}\big(\|x_n - p\| - \|x_{n+1} - p\|\big) + \alpha_n\gamma(1-\beta_n)\|p\|^q \\
&\quad\le q\|x_n - p\|^{q-1}\|x_{n+1} - x_n\| + \alpha_n\gamma(1-\beta_n)\|p\|^q.
\end{aligned}$$

Then, by $(C1)$, $(C3)$ and (3.6), we obtain

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.9}$$

On the other hand, by Proposition 2.12 and Lemma 2.5, we have

$$\begin{aligned}
&\|z_n - p\|^q \\
&\quad= \|J_{\lambda_n}^B((1-\alpha_n)x_n - \lambda_n Ax_n) - J_{\lambda_n}^B(p - \lambda_n Ap)\|^q \\
&\quad\le \langle (1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap), j_q(z_n - p)\rangle \\
&\quad\le \frac{1}{q}\Big[\|(1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^q + (q-1)\|z_n - p\|^q \\
&\qquad -g(\|(1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|)\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|z_n - p\|^q \\
&\quad\le \|(1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^q \\
&\qquad - g(\|(1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|) \\
&\quad\le \alpha_n\|p\|^q + \|x_n - p\|^q - g(\|(1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - z_n\|).
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
&\|y_n - p\|^q \\
&\quad\le (1-\gamma)\|Sx_n - p\|^q + \gamma\|z_n - p\|^q \\
&\quad\le (1-\gamma)\|x_n - p\|^q + \gamma\|z_n - p\|^q \\
&\quad\le (1-\gamma)\|x_n - p\|^q
\end{aligned}$$

$$+\gamma\Big[\alpha_n\|p\|^q+\|x_n-p\|^q-g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|)\Big]$$
$$=\|x_n-p\|^q+\alpha_n\gamma\|p\|^q-\gamma g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|). \tag{3.10}$$

Consequently,

$$\begin{aligned}
&\|x_{n+1}-p\|^q\\
&\quad\le\beta_n\|x_n-p\|^q+(1-\beta_n)\|y_n-p\|^q\\
&\quad\le\beta_n\|x_n-p\|^q\\
&\qquad+(1-\beta_n)\Big[\|x_n-p\|^q+\alpha_n\gamma\|p\|^q-\gamma g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|)\Big]\\
&\quad=\|x_n-p\|^q+\alpha_n\gamma(1-\beta_n)\|p\|^q\\
&\qquad-(1-\beta_n)\gamma g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|),
\end{aligned}$$

which implies from $(C2)$ that

$$\begin{aligned}
&(1-\beta_n)\gamma g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|)\\
&\quad\le(1-b)\gamma g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|)\\
&\quad\le\|x_n-p\|^q-\|x_{n+1}-p\|^q-\alpha_n\gamma(1-\beta_n)\|x_n-p\|^q+\alpha_n\gamma(1-\beta_n)\|p\|^q\\
&\quad\le q\|x_n-p\|^{q-1}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)-\alpha_n\gamma(1-\beta_n)\|x_n-p\|^q\\
&\qquad+\alpha_n\gamma(1-\beta_n)\|p\|^q\\
&\quad\le q\|x_n-p\|^{q-1}\|x_n-x_{n+1}\|-\alpha_n\gamma(1-\beta_n)\|x_n-p\|^q+\alpha_n\gamma(1-\beta_n)\|p\|^q.
\end{aligned}$$

Then, by $(C1)$, $(C3)$ and (3.6), we have

$$\lim_{n\to\infty}g(\|(1-\alpha_n)x_n-\lambda_n(Ax_n-Ap)-z_n\|)=0.$$

Since g is a continuous function, by (3.9), we obtain that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.11}$$

Since $y_n=(1-\gamma)Sx_n+\gamma z_n$, we have $(1-\gamma)(x_n-Sx_n)=x_n-y_n+\gamma(z_n-x_n)$. From (3.5) and (3.11), we obtain

$$(1-\gamma)\|x_n-Sx_n\|\le\|x_n-y_n\|+\gamma\|x_n-z_n\|\to 0\ \text{ as }\ n\to\infty. \tag{3.12}$$

Next, we show that

$$\limsup_{n\to\infty}\langle x^*,j_q(x^*-z_n)\rangle\le 0,$$

where $x^* = Q_\Gamma 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^*, j_q(x^*-x_n)\rangle = \lim_{i\to\infty}\langle x^*, j_q(x^*-x_{n_i})\rangle.$$

By the reflexivity of X and the boundedness of $\{x_n\}$, without loss of generality, we may assume that $x_{n_i} \rightharpoonup z \in C$ as $i \to \infty$. By (3.12) and Lemma 2.6, we get $z \in F(S)$. Further, we show that $z \in (A+B)^{-1}0$. Let $v \in Bu$. Note that

$$z_n = J_\lambda^B((1-\alpha_n)x_n - \lambda_n Ax_n).$$

Then, we have

$$(1-\alpha_n)x_n - \lambda_n Ax_n \in (I+\lambda B)z_n \Longleftrightarrow \frac{1}{\lambda_n}\big((1-t_n)x_n - \lambda_n Ax_n - z_n\big) \in Bz_n.$$

Since B is m-accretive, we have for all $(u, v) \in B$,

$$\begin{aligned}&\left\langle \frac{1}{\lambda_n}\big((1-\alpha_n)x_n - \lambda_n Ax_n - z_n\big) - v, j_q(z_n-u)\right\rangle \geq 0\\ &\quad\Longleftrightarrow \langle (1-\alpha_n)x_n - \lambda_n Ax_n - z_n - \lambda_n v, j_q(z_n-u)\rangle \geq 0.\end{aligned}$$

It follows that

$$\begin{aligned}\langle Ax_n + v, j_q(z_n-u)\rangle &\leq \frac{1}{\lambda_n}\langle x_n - z_n, j_q(z_n-u)\rangle + \frac{\alpha_n}{\lambda_n}\langle -x_n, j_q(z_n-u)\rangle\\ &\leq \frac{1}{\lambda_n}\|x_n-z_n\|\|z_n-u\|^{q-1} + \frac{\alpha_n}{\lambda_n}\|x_n\|\|z_n-u\|^{q-1}\\ &\leq \big(\|x_n-z_n\| + \alpha_n\big)K,\end{aligned}$$

where $K > 0$ is an appropriate constant. Since $x_n - z_n \to 0$, $x_n \rightharpoonup z$, A is Lipschitz continuous and j_q is weakly sequentially continuous, we obtain $\langle Az+v, j_q(z-u)\rangle \leq 0$, that is $\langle -Az-v, j_q(z-u)\rangle \geq 0$, this implies that $-Az \in Bz$, that is $z \in (A+B)^{-1}0$. Hence $z \in \Gamma := F(S)\cap(A+B)^{-1}0$. From (3.11), we get

$$\begin{aligned}\limsup_{n\to\infty}\langle x^*, j_q(x^*-z_n)\rangle &= \limsup_{n\to\infty}\langle x^*, j_q(x^*-x_n)\rangle\\ &= \lim_{i\to\infty}\langle x^*, j_q(x^*-x_{n_i})\rangle\\ &= \langle x^*, j_q(x^*-z)\rangle \leq 0. \end{aligned}\tag{3.13}$$

Finally, we show that $x_n \to x^*$. From Lemma 2.4, we have

$$
\begin{aligned}
&\|y_n - x^*\|^q \\
&\le (1-\gamma)\|Sx_n - x^*\|^q + \gamma\|z_n - x^*\|^q \\
&\le (1-\gamma)\|x_n - x^*\|^q \\
&\quad + \gamma\left\|(1-\alpha_n)\left[\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*\right] + \alpha_n(-x^*)\right\|^q \\
&\le (1-\gamma)\|x_n - x^*\|^q + \gamma(1-\alpha_n)^q\left\|\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*\right\|^q \\
&\quad + q\alpha_n\gamma\langle -x^*, j_q(z_n - x^*)\rangle \\
&\le (1-\gamma)\|x_n - x^*\|^q + \gamma(1-\alpha_n)^q\|x_n - x^*\|^q + q\alpha_n\gamma\langle x^*, j_q(x^* - z_n)\rangle \\
&\le (1-\alpha_n\gamma)\|x_n - x^*\|^q + q\alpha_n\gamma\langle x^*, j_q(x^* - z_n)\rangle.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&= \beta_n\|x_n - x^*\|^q + (1-\beta_n)\|y_n - x^*\|^q \\
&\le \beta_n\|x_n - x^*\|^q + (1-\beta_n)\Big[(1-\alpha_n\gamma)\|x_n - x^*\|^q + q\alpha_n\gamma\langle x^*, j_q(x^* - z_n)\rangle\Big] \\
&\le \big(1 - \alpha_n\gamma(1-\beta_n)\big)\|x_n - x^*\|^q + q\alpha_n\gamma(1-\beta_n)\langle x^*, j_q(x^* - z_n)\rangle.
\end{aligned}
$$

Therefore, by Lemma 2.9, we conclude that $x_n \to x^*$. □

As a direct consequence of Theorem 3.1, we get the following results:

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a 2-uniformly smooth and uniformly convex Banach spaces X which admits a weakly sequentially continuous duality mapping. Let $A : C \to X$ be an α-isa and let $B : D(B) \subset C \to 2^X$ be an m-accretive operator. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent of B for $\lambda > 0$ and let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap (A+B)^{-1}0 \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$
\begin{cases}
y_n = (1-\gamma)Sx_n + \gamma J_{\lambda_n}^B((1-\alpha_n)x_n - \lambda_n Ax_n), \\
x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \ge 1,
\end{cases}
\tag{3.14}
$$

where $\gamma \in (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$;
(C3) $0 < a' \le \lambda_n < \frac{\lambda_n}{1-\alpha_n} \le b' < \frac{\alpha}{K^2}$ *and* $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (3.14) converges strongly to a point $x^* = Q_\Gamma 0$, where Q_Γ is a sunny nonexpansive retraction of C onto Γ.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $A : C \to H$ be an α-ism and let $B : D(B) \subset C \to 2^H$ be a maximal monotone operator. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be a resolvent of B for $\lambda > 0$ and let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap (A + B)^{-1}0 \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma J_{\lambda_n}^B((1-\alpha_n)x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{3.15}$$

where $\gamma \in (0, 1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a \leq \beta_n \leq b < 1$*;*
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (3.15) converges strongly to a point $x^* = P_\Gamma 0$, where P_Γ is a metric projection of C onto Γ.

4 Applications

In this section, we shall utilize Theorem 3.1 to give some applications in the framework of Hilbert spaces. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Consider the following particular variational inequality problem: find $\hat{x} \in C$ such that

$$\langle \hat{x}, \hat{x} - x \rangle \leq 0, \quad \forall x \in C. \tag{4.1}$$

It is well known that (4.1) is equivalent to solve minimization problem: find $\bar{x} \in C$ such that

$$\|\hat{x}\| = \min_{x \in C} \|x\|. \tag{4.2}$$

That is, $\hat{x}$ is the minimum-norm solution. In other words, $\hat{x}$ is the metric projection of the origin onto C, *i.e.*, $\hat{x} = P_C 0$, where P_C is the metric projection from H onto C.

4.1 Applications to variational inequality problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear monotone operator. The *classical variational inequality problem* is to find $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - x \rangle \geq 0, \quad \forall y \in C. \tag{4.3}$$

The set of solutions of problem (4.3) is denoted by $VI(C, A)$. Variational inequality problem which was first introduced by Stampacchi [17] in 1964 in the field of potential theory.

Let $g : H \to (-\infty, \infty]$ be a proper convex lower semi-continuous function. The *subdifferential* ∂g of g is defined by

$$\partial g(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y\rangle, \quad \forall z \in H\}, \quad \forall x \in H.$$

It is known that ∂g is maximal monotone (see [15]). Let i_C be the indicator function of C defined by

$$i_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & x \notin C. \end{cases} \tag{4.4}$$

Since i_C is a proper lower semicontinuous convex function on H, the subdifferential ∂i_C of i_C is a maximal monotone operator. So, we can define the resolvent $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C} x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. Then, we have for each $x \in H$ and $u \in C$, $u = J_\lambda^{\partial i_C} x \Longleftrightarrow u = P_C x$ and $(A + \partial i_C)^{-1} 0 = VI(C, A)$ (see [20]). Put $B = \partial i_C$ in Theorem 3.1, we obtained the following result.

Theorem 4.1 *Let $A : C \to H$ be an α-ism and let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap VI(C, A) \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1 - \gamma)Sx_n + \gamma P_C((1 - \alpha_n)x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\gamma \in (0, 1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a \leq \beta_n \leq b < 1$*;*
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (4.5) converges strongly to the minimum-norm common element of Γ.

4.2 Applications to equilibrium problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of all real numbers. The *equilibrium problem* is to find $\hat{x} \in C$ such that

$$G(\hat{x}, y) \geq 0, \tag{4.6}$$

for all $y \in C$. The set of solutions of problem (4.6) is denoted by $EP(G)$. Numerous problems in physics, economics, optimization and applied sciences can be reduced to find solutions of equilibrium problems. For solving the equilibrium problem, let us assume that a bifunction $G : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $G(x, x) = 0$ for all $x \in C$;
(A2) G is monotone, *i.e.*, $G(x, y) + G(y, x) \leq 0$ for all $x \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)y, y) \leq G(x, y)$;
(A4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semi-continuous.

Lemma 4.2 [4] *Let $G : C \times C \to \mathbb{R}$ satisfying the conditions $(A1) - (A4)$. Let $\lambda > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 4.3 [8] *Assume that $G : C \times C \to \mathbb{R}$ satisfies the conditions $(A1) - (A4)$. For $\lambda > 0$ and $x \in H$, define a mapping $T_\lambda : H \to C$ as follows:*

$$T_\lambda(x) = \left\{z \in C : G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}, \quad \forall x \in H.$$

Then, the following hold:

(1) *T_λ is single-valued;*
(2) *T_λ is firmly nonexpansive, i.e., for each $x, y \in H$,*

$$\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y\rangle;$$

(3) *$F(T_\lambda) = EP(G)$;*
(4) *$EP(G)$ is closed and convex.*

We call such T_λ the resolvent of G for $\lambda > 0$.

Lemma 4.4 [20] *Let $G : C \times C \to \mathbb{R}$ be a bifunction that satisfies the conditions $(A1) - (A4)$. Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & x \in C; \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(G) = A_G^{-1}0$ and A_G is a maximal monotone operator with $D(A_G) \subset C$. Further, for any $x \in H$ and $\lambda > 0$, the resolvent T_λ of G coincides with the resolvent of A_G, that is,

$$T_\lambda x = (I + \lambda A_G)^{-1} x.$$

Put $B = A_G$ in Theorem 3.1, we obtained the following result.

Theorem 4.5 *Let $A : C \to H$ be an α-ism. Let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies the conditions $(A1)-(A4)$. Let T_λ be the resolvent of G for $\lambda > 0$ and let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap EP(G) \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma T_{\lambda_n}((1-\alpha_n)x_n - \lambda_n Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{4.7}$$

where $\gamma \in (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a \leq \beta_n \leq b < 1$*;*
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (4.7) converges strongly to the minimum-norm common element of Γ.

4.3 Applications to convex minimization problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $f : H \to \mathbb{R}$ be a convex smooth function and $g : H \to \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. The *convex minimization problem* is to find $\hat{x} \in C$ such that

$$f(\hat{x}) + g(\hat{x}) \leq \min_{x \in C}\{f(x) + g(x)\}. \tag{4.8}$$

The set of solutions of problem (4.8) is denoted by $CMP(f, g)$. By Fermat's rule, it is known that the problem (4.8) is equivalent to the problem of finding $\hat{x} \in C$ such that

$$0 \in \nabla f(\hat{x}) + \partial g(\hat{x}),$$

where ∇f is a gradient of f and ∂g is a subdifferential of g. It is also known that if ∇f is $(1/L)$-Lipschitzian, then it is L-ism (see [2]). Further, ∂g is maximal monotone (see [15]). In fact, set $A = \nabla f$ and $B = \partial g$ in Theorem 3.1, we obtained the following result.

Theorem 4.6 *Let $f : H \to \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇f and $g : H \to \mathbb{R}$ be a convex and lower semi-continuous function such that $D(\partial g) \subset C$. Let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap CMP(f, g) \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma J_{\lambda_n}^{\partial g}((1-\alpha_n)x_n - \lambda_n \nabla f(x_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{4.9}$$

where $\gamma \in (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$;
(C3) $0 < a' \le \lambda_n < \frac{\lambda_n}{1-\alpha_n} \le b' < \frac{2}{L}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ defined by (4.9) converges strongly to the minimum-norm common element of Γ.

4.4 Applications to split feasibility problem

Let C and Q be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : C \subset H_1 \to H_2$ be a linear bounded operator. The *split feasibility problem* is to find

$$\hat{x} \in C \quad \text{such that} \quad A\hat{x} \in Q. \tag{4.10}$$

The set of solutions of problem (4.10) is denoted by $\Omega := C \cap A^{-1}(Q) = \{x \in C : Ax \in Q\}$. The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [7] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Consider the proximity function

$$f(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2, \quad \forall x \in H.$$

The function f is continuously differentiable with gradient given by [22]

$$\nabla f(x) = A^*(I - P_Q)Ax,$$

where A^* is the adjoint operator of A. Further, ∇f is $\|A\|^2$-Lipschitz continuity this implies the gradient operator ∇f is $1/\|A\|^2$-ism. It is observed that $\hat{x} \in C$ is a solution of (4.10) if and only if $0 \in \nabla f(\hat{x}) = A^*(I - P_Q)A\hat{x}$. Set $A = \nabla f$ and $B = 0$ in Theorem 3.1, we obtain the following result.

Theorem 4.7 *Let $A : H_1 \to H_2$ be a bounded linear operator and let $S : C \to C$ be a nonexpansive mapping such that $\Gamma := F(S) \cap \Omega \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma P_C[(1-\alpha_n)x_n - \lambda_n A^*(I - P_Q)Ax_n], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \ge 1, \end{cases} \tag{4.11}$$

where $\gamma \in (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\}$ is a real positive sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$;

(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < \frac{2}{\|A\|^2}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ *defined by* (4.11) *converges strongly to the minimum-norm common element of* Γ.

4.5 Applications to linear inverse problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : H \to H$ be a bounded linear operator and $d \in H$. The *constrained linear inverse problem* is to find

$$\hat{x} \in C \text{ such that } T\hat{x} = d. \tag{4.12}$$

It is well known that the problem (4.12) is equivalent to the following minimization problem:

$$\min_{x\in C} \frac{1}{2}\|Tx - d\|^2.$$

The set of solutions of problem (4.12) is denoted by $\Omega = \{x \in C : x = T^{-1}d\}$. Consider the functional

$$f(x) = \frac{1}{2}\|Tx - d\|^2.$$

From [6], it is known that

$$\nabla f(x) = T^*(Tx - d).$$

Further, ∇f is $\|T\|^2$-Lipschitzian and hence ∇f is also $1/\|T\|^2$-ism. It is observed that $\hat{x} \in C$ is a solution of (4.12) if and only if $0 \in \nabla f(\hat{x}) = T^*(T\hat{x} - d)$. Set $A = \nabla f$ and $B = 0$ in Theorem 3.1, we obtain the following result.

Theorem 4.8 *Let* $T : H \to H$ *be a bounded linear operator and* $d \in H$. *Let* $S : C \to C$ *be a nonexpansive mapping such that* $\Gamma := F(S) \cap \Omega \neq \emptyset$. *For given* $x_1 \in C$*, let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} y_n = (1-\gamma)Sx_n + \gamma P_C((1-\alpha_n)x_n - \lambda_n T^*(Tx_n - d)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{4.13}$$

where $\gamma \in (0, 1)$*,* $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ *and* $\{\lambda_n\}$ *is a real positive sequence satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a \leq \beta_n \leq b < 1$*;*
(C3) $0 < a' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq b' < \frac{2}{\|T\|^2}$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, the sequence $\{x_n\}$ *defined by* (4.13) *converges strongly to the minimum-norm common element of* Γ.

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A modified extragradient method for variational inclusion and fixed point problems in Banach spaces

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ABSTRACT
In this work, we introduce a modified extragradient method for solving the fixed point problem of a nonexpansive mapping and the variational inclusion problem for two accretive operators in the framework of Banach spaces. We then prove its strong convergence under certain assumptions imposed on the parameters. As applications, we apply our main result to the variational inequality problem, split feasibility problem and the LASSO problem.

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1. Introduction

Variational inequality theory has been studied widely in several branches of pure and applied sciences. In particular, applications of variational inequalities span as diverse disciplines as differential equations, time-optimal control, optimization, mathematical programming, mechanics, economic and other applied science problems. Let C be a nonempty, closed and convex subset of a real Hilbert space H. The *variational inequality problem* is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1}$$

where $A : C \to H$ is a mapping. The set of solutions of problem (1) is denoted by $VI(C, A)$. In recent years, several methods have been invented and modified for solving the variational inequality problem.

A simple method for solving problem (1) is *projection method* which is defined by the following manner: For a given $x_0 \in C$ and

$$x_{n+1} = P_C(x_n - \lambda Ax_n), \ \forall n \geq 0, \tag{2}$$

where P_C is the metric projection of H into C, λ is a positive real number. In fact, this method requires a slightly strong assumption that operators are strongly monotone or inverse strongly monotone [1].

To avoid this assumption, Korpelevich [2] (see also [3]) introduced the *extragradient method* for solving saddle point problems, and later, this method was successfully studied and extended for

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solving variational inequalities both Euclidean and Hilbert spaces. More precisely, the extragradient method is defined by the following way: For a given $x_1 \in C$ and

$$y_n = P_C(x_n - \lambda A x_n),$$
$$x_{n+1} = P_C(x_n - \lambda A y_n), \quad \forall n \geq 1. \tag{3}$$

In fact, the convergence of the extragradient method only requires that the operator A is monotone and Lipschitz continuous. However, this method has only weak convergence. In recent years, some modifications of an extragradient method have been constructed so that it generates a strongly convergent sequence (see, e.g. [4–8]and the references cited therein).

We next consider the *variational inclusion problem* which is the problem of finding:

$$\text{find } x^* \in H \text{ such that } 0 \in (A+B)x^*, \tag{4}$$

where $A : H \to H$ and $B : H \multimap H$ are single and multi-valued mappings, respectively, and 0 is a zero vector in H. The set of solutions of problem (4) is denoted by $(A+B)^{-1}0$. This problem is a fundamental problem in optimization theory and it is a generalization of many mathematical problems such as convex programming, variational inequalities, split feasibility problem and minimization problem [9–11]. Moreover, it has wide applications in machine learning, image processing, statistical regression and linear inverse problem.

A one classical method for solving the problem (4) is the *forward–backward algorithm* [12,13] which is defined by the following manner: for any fixed $x_1 \in H$,

$$x_{n+1} = J_\lambda^B(x_n - \lambda A x_n), \quad \forall n \geq 1, \tag{5}$$

where $J_\lambda^B := (I + \lambda B)^{-1}$ is a resolvent of B for $\lambda > 0$. It is known that if A is inverse strongly monotone, then the sequence $\{x_n\}$ defined by (5) converges weakly to a solution of the problem (4). This method includes, in particular, the proximal point algorithm [14–16] and the gradient method [17,18].

In order to obtain strong convergence, Takahashi et al. [19] introduced the following Halpern-type iterative method for a maximal monotone operator B on H: for given $x_1, u \in H$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n) J_{\lambda_n}^B(x_n - \lambda_n A x_n)), \quad \forall n \geq 1, \tag{6}$$

where A is an α-inverse strongly monotone mapping on H. They also proved the strong convergence of algorithm (6) to a solution of problem (4).

For solving problem (4) in the framework of Banach spaces E, López et al. [9] introduced the following Halpern-type forward–backward method: $x_1, u \in E$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(J_{\lambda_n}^B(x_n - r_n(Ax_n + a_n)) + b_n), \quad \forall n \geq 1, \tag{7}$$

where J_λ^B is the resolvent of B, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\}, \{b_n\}$ are error sequences in E. It was proved that the sequence $\{x_n\}$ generated by (7) strongly converges to a zero point of the sum of A and B under some appropriate conditions.

Recently, Pholasa et al. [20] extended the result of Takahashi et al. [19] to Banach spaces. It was proved that $\{x_n\}$ converges strongly to a point in $(A+B)^{-1}0$.

Due to the importance and interesting of such a problem, many researchers have developed iterative methods for solving problem (4) in several approaches (see [21–28]and the references cited therein).

On the other hand, we consider the fixed point problem which is problem of finding a point

$$x^* \in C \quad \text{such that } x^* = Sx^*, \tag{8}$$

where $S : C \to C$ is a nonlinear mapping. The set of solutions of problem (8) is denoted by $F(S)$. In real life, many mathematical models have been formulated as this problem. Currently, many mathematicians are interested in finding a common solution of the fixed point problem (8) and the variational inclusion problem (4). We aim to find a point x^* such that

$$x^* \in F(S) \cap (A+B)^{-1}0. \tag{9}$$

In the case of $A : C \to H$ is an inverse strongly monotone mapping and $S : C \to C$ is a nonexpansive mapping, Manaka and Takahashi [29] introduced the following iteration process in Hilbert spaces H: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S J_{\lambda_n}(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{10}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\}$ is a positive sequence, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator. They showed that the sequence $\{x_n\}$ generated by (10) converges weakly to a point in $F(S) \cap (A+B)^{-1}0$ under some mild conditions.

Recently, Takahashi et al. [11] introduced an iterative scheme for finding a common element of $F(S) \cap (A+B)^{-1}0$ and obtained the following strong convergence theorem.

Theorem T: *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S(\alpha_n x + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)), \quad \forall n \geq 1, \tag{11}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < a \leq \lambda_n \leq b < 2\alpha$;
(iii) $0 < c \leq \beta_n \leq d < 1$.

Then, $\{x_n\}$ converges strongly to a point of $F(S) \cap (A+B)^{-1}0$ provided asymptotically regular assumption on $\{\lambda_n\}$.

In this paper, motivated and inspired by the idea of Korpelevich's extragradient method, we introduce an iterative method for solving the variational inclusion problem for two accretive operators and fixed point problem of a nonexpansive mapping in the framework of Banach spaces. We then prove its strong convergence under some mild assumption on the control conditions. As applications, we apply our main result to the variational inequality problem, split feasibility problem and the LASSO problem. The results presented in this paper extend and generalize the corresponding results in the literature.

2. Preliminaries

In this section, we collect some definitions and lemmas which will be used in the sequel. In what follows, we shall use the following notations: $x_n \to x$ mean that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ mean that $\{x_n\}$ converges weakly to x. Let E and E^* be real Banach spaces and the dual space of E, respectively. Let C be a subset of E and S be a self-mapping of C. We use $F(S)$ to denote the fixed points of S. Recall that a mapping $S : C \to C$ is said to be *nonexpansive*, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The *modulus of convexity* of E is the function $\delta : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \right\}.$$

The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}.$$

Definition 2.1: A Banach space E is said to be

- *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$;
- *uniformly smooth* if $\lim_{\tau \to 0} \rho_E(\tau)/\tau = 0$;
- *q-uniformly smooth* if there exists a $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q \tau^q$ for all $\tau > 0$.

If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth and if E is uniformly convex Banach space (uniformly smooth Banach space), then E is also reflexive and strictly convex [30]. It is known that Hilbert space H is 2-uniformly smooth. Further, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [31].

Definition 2.2: Let $q > 1$. The *generalized duality mapping* $J_q : E \multimap E^*$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*.

In particular, $J_2 := J$ is called the *normalized duality mapping*. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for $x \neq 0$ and that J_q is the subdifferential of the functional $(1/q)\| \cdot \|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued, which is denoted by j_q. Furthermore, j_q satisfies $j_q = j_p^{-1}$, where $j_q = j_p^{-1}$ is the generalized duality mapping of E^* with $1/p + 1/q = 1$ (see [32] for more details). If E is a Hilbert space H, then j is the identity operator I, which is linear in Hilbert spaces. Inversely, if the operator j is linear in E, then E is a Hilbert space.

The following examples of generalized duality mapping can be found in [30,33].

Example 2.3: Let $x = (x_1, x_2, \ldots) \in \ell_p$ $(1 < p < \infty)$. Then the generalized duality mapping j_p in ℓ_p is given by

$$j_p(x) = (|x_1|^{p-1}\mathrm{sgn}(x_1), |x_2|^{p-1}\mathrm{sgn}(x_2), \ldots) \in \ell_q,$$

where $1/p + 1/q = 1$.

Example 2.4: Let $x \in L_p(G)$ $(1 < p < \infty)$. Then the generalized duality mapping j_p in L_p is given by

$$j_p(x) = |x(t)|^{p-1}\mathrm{sgn}(x(t)) \in L_q(G), \quad t \in G,$$

where $1/p + 1/q = 1$.

Example 2.5: For the Sobolev space $W_p^m(G)$. Let $x \in W_p^m(G)$ $(1 < p < \infty)$. Then the generalized duality mapping j_p in $W_p^m(G)$ is given by

$$j_p(x) = \sum_{|\alpha| \leq m} (-1)^\alpha D^\alpha \left(|D^\alpha x(t)|^{p-1} \operatorname{sgn}(D^\alpha x(t))\right) \in W_q^{-m}, \quad m > 0,\ t \in G,$$

where $1/p + 1/q = 1$.

Lemma 2.6 ([34]): *Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in E$, we have*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle, \tag{13}$$

where $j_q(x + y) \in J_q(x + y)$.

The following lemma can be obtained from the result in [31] (see also Theorem 2.8.17 of [33]).

Lemma 2.7: *Let $p > 1$ and $r > 0$ be two fixed real numbers and E be a uniformly convex Banach space. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ and $h(0) = 0$ such that*

(i) $\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \lambda(1 - \lambda)g(\|x - y\|)$;
(ii) $h(\|x - y\|) \leq \|x\|^p - p\langle x, j_p(y)\rangle + (p - 1)\|y\|^p$

for all $x, y \in B_r$ and $j_p(y) \in J_p(y)$, where $B_r := \{x \in E : \|x\| \leq r\}$.

The following lemma is an analogue of Lemma 2.7 (i).

Lemma 2.8: *Let $p > 1$ and $r > 0$ be two fixed real numbers and E be a uniformly convex Banach space. Then there exist strictly increasing, continuous and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^p \leq \lambda\|x\|^p + \mu\|y\|^p + \gamma\|z\|^p - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Definition 2.9: Let C be a nonempty, closed and convex subset of E and Q be a mapping of E onto C. Then Q is said to be

(1) *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$;
(2) *retraction* if $Qx = x$ for all $x \in C$;
(3) a sunny nonexpansive retraction if Q is sunny, nonexpansive and a retraction from E onto C.

It is known that if $E := H$ is a real Hilbert space, then a sunny nonexpansive retraction Q is coincident with the metric projection from E onto C. Moreover, if E is uniformly smooth and S is a nonexpansive mapping of C into itself with $F(S) \neq \emptyset$, then $F(S)$ is a sunny nonexpansive retract from E onto C [35]. We know that in a uniformly smooth Banach space E, a retraction $Q : E \to C$ is sunny and nonexpansive, if and only if $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$ for all $x \in E$ and $y \in C$ [36].

Let $A : E \multimap E$ be a set-valued mapping. We denote the domain and range of an operator $A : E \multimap E$ by $\mathcal{D}(A) = \{x \in E : Ax \neq \emptyset\}$ and $\mathcal{R}(A) = \bigcup\{Az : z \in \mathcal{D}(A)\}$, respectively. Let $q > 1$. A set-valued mapping $A : \mathcal{D}(A) \subset E \multimap E$ is said to be *accretive* of order q if for each $x, y \in \mathcal{D}(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle u - v, j_q(x-y)\rangle \geq 0, \quad u \in Ax \quad \text{and} \quad v \in Ay.$$

An accretive operator A is said to be *m-accretive* if $\mathcal{R}(I + \lambda A) = E$ for all $\lambda > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone [37].

Let $\alpha > 0$ and $q > 1$. A mapping $A : C \to E$ is said to be *α-inverse strongly accretive* (α-isa) of order q if for each $x, y \in E$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \alpha \|Ax - Ay\|^q.$$

If $E := H$ is a real Hilbert space, then $A : C \to H$ is called α-inverse strongly monotone (α-ism).

Let A be an m-accretive operator on E, we use $A^{-1}0$ to denote the set of all zeros of A, *i.e.*, $A^{-1}0 = \{x \in \mathcal{D}(A) : 0 \in Ax\}$. For an accretive operator A, we can define a single valued operator $J_\lambda^A : \mathcal{R}(I + \lambda A) \to \mathcal{D}(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$, which is called the *resolvent* of A for λ. It is known that the resolvent operator J_λ^A is single-valued nonexpansive with $F(J_\lambda^A) = A^{-1}0$ [37]. Further, J_λ^A is firmly nonexpansive [27], *i.e.*,

$$\|J_\lambda^A x - J_\lambda^A y\|^q \leq \langle x - y, j_q(J_\lambda^A x - J_\lambda^A y)\rangle, \quad \forall x, y \in E.$$

Lemma 2.10 ([9]): *Let C be a subset of a real q-uniformly smooth Banach space E and $A : C \to E$ be an α-isa of order q. Then the following inequality holds for all $x, y \in C$:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax - Ay\|^q, \tag{14}$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 < \lambda \leq (\alpha q/\kappa_q)^{1/(q-1)}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.11 ([9]): *If $T_\lambda^{A,B} := J_\lambda^B(I - \lambda A)$, then the following statements hold:*

(i) *For $\lambda > 0$, $F(T_\lambda^{A,B}) = (A+B)^{-1}0$.*
(ii) *For $0 < \lambda \leq r$ and $x \in E$, $\|x - T_\lambda^{A,B}x\| \leq 2\|x - T_r^{A,B}x\|$.*

We now give the following examples of a resolvent operator.

Example 2.12 ([25]): Let $E = \ell_3$ with norm $\|x\| = (\sum_{k=1}^{\infty} |x_k|^3)^{1/3}$ for $x = (x_1, x_2, x_3, \dots) \in \ell_3$. Let $A, B : \ell_3 \to \ell_3$ be defined

$$Ax = 2x + (1,1,1,0,0,0,0,\dots) \quad \text{and} \quad Bx = 5x \text{ for } x \in \ell_3.$$

It is to see that A is 1/2-isa of order 2 and B is an m-accretive operator with $\mathcal{R}(I + \lambda B) = \ell_3$ for all $\lambda > 0$. Moreover,

$$J_\lambda^B(x - \lambda Ax) = \frac{1 - 2\lambda}{1 + 5\lambda}x - \frac{\lambda}{1 + 5\lambda}(1,1,1,0,0,0,0,\dots),$$

for all $x \in \ell_3$.

Example 2.13: Let $E=\mathbb{R}^+ := [0,\infty)$ with the absolute-value norm. Let $A,B:\mathbb{R}^+\to\mathbb{R}^+$ be defined as

$$Ax = 3\ln(x+1) \quad \text{and} \quad Bx = 2x \text{ for } x\in\mathbb{R}^+.$$

By Mean Value Theorem, it is easy to show that $|\ln(x+1)-\ln(y+1)|\le|x-y|$ for all $x,y\in\mathbb{R}^+$. Without loss of generality, we may assume that $x\ge y$. Then, for each $x,y\in\mathbb{R}^+$

$$\begin{aligned}\langle Ax-Ay,x-y\rangle &= \big(3\ln(x+1)-3\ln(y+1)\big)(x-y)\\ &\ge 3\big(\ln(x+1)-\ln(y+1)\big)^2\\ &= \frac{1}{9}\big(3\ln(x+1)-3\ln(y+1)\big)^2\\ &= \frac{1}{9}|Ax-Ay|^2,\end{aligned}$$

i.e. A is a 1/9-ism. Moreover, we also have

$$\langle Bx-By,x-y\rangle = (2x-2y)(x-y) = 2|x-y|^2\ge 0$$

and $\mathcal{R}(I+\lambda B)=\mathbb{R}^+$ for all $\lambda>0$. Then, we can write the explicit resolvent of B for $\lambda>0$ in the following form:

$$\begin{aligned}J_\lambda^B(x-\lambda Ax) &= (I+\lambda B)^{-1}(x-\lambda Ax)\\ &= \frac{1}{1+2\lambda}x-\frac{3\lambda}{1+2\lambda}\ln(x+1),\end{aligned}$$

for all $x\in\mathbb{R}^+$ with $(A+B)^{-1}0=\{0\}$.

Lemma 2.14 ([38]): *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and $S:C\to C$ be a nonexpansive mapping. Then $I-S$ is demiclosed at zero, i.e. $x_n\rightharpoonup x$ and $x_n-Sx_n\to 0$, we have $x=Sx$.*

Lemma 2.15 ([39]): *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space E. Let $T:C\to C$ be a nonexpansive self-mapping such that $F(T)\ne\emptyset$ and $f:C\to C$ be a contraction with coefficient $\rho\in(0,1)$. Let $\{z_t\}$ be a net sequence defined by*

$$z_t = tf(z_t)+(1-t)Tz_t, \quad \forall t\in(0,1).$$

Then, $\{z_t\}$ converges strongly as $t\to 0$ to a point $x^*\in F(T)$.

Proposition 1 ([40]): *Let $q>1$. Then the following inequality holds:*

$$a^q-b^q\le qa^{q-1}(a-b), \tag{15}$$

for arbitrary positive real numbers a, b.

Lemma 2.16 ([41]): *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\le(1-\theta_n)a_n+\theta_n\delta_n,$$

where $\{\theta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\theta_n = \infty$;
(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\theta_n\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.17 ([42]): *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Define the sequence* $\{\tau(n)\}_{n\geq n_0}$ *of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ *such that* $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. *Then, the following hold:*

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Main result

Throughout this paper, we assume that E be a q-uniformly smooth and uniformly convex Banach space. Let $f : E \to E$ be a contraction with constant $\rho \in (0,1)$ and $S : E \to E$ be a nonexpansive mapping. Let $A : E \to E$ be an α-isa of order q and $B : E \multimap E$ be an m-accretive operator. Assume that $\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset$.

Algorithm 3.1: *Modified viscosity-type extragradient method for variational inclusion and fixed point problems*

Initialization: Given $x_1 \in E$ *be arbitrary.*

Iterative Steps: Given the current iterate x_n, *calculate* x_{n+1} *as follows:*

Step 1: Compute

$$y_n = J_{\lambda_n}^B(x_n - \lambda_n A x_n)$$

Step 2: Compute

$$z_n = J_{\lambda_n}^B(x_n - \lambda_n A y_n + r_n(y_n - x_n))$$

Step 3: Compute

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n,$$

where $J_{\lambda_n}^B = (I + \lambda_n B)^{-1}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$.

Set n: $=$ *n* + *1 and go to Step 1.*

Lemma 3.2: *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 3.1, *then* $\{x_n\}$ *is bounded.*

Proof: Let $z \in \Omega := F(S) \cap (A+B)^{-1}0$, it is observed that

$$z = Sz = J_{\lambda_n}^B(z - \lambda_n A z) = J_{\lambda_n}^B\left((1-r_n)z + r_n\left(z - \frac{\lambda_n}{r_n}Az\right)\right).$$

From (14), we have

$$\begin{aligned}\|y_n - z\|^q &= \|J^B_{\lambda_n}(x_n - \lambda_n Ax_n) - J^B_{\lambda_n}(z - \lambda_n Az)\|^q \\ &\le \|x_n - z\|^q - \lambda_n(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax_n - Az\|^q. \end{aligned} \tag{16}$$

This implies that

$$\|y_n - z\| \le \|x_n - z\|.$$

By the convexity of $\|\cdot\|^q$ for all $q > 1$ and (16), we see that

$$\begin{aligned}&\|z_n - z\|^q \\ &= \left\| J^B_{\lambda_n}\left((1-r_n)x_n + r_n\left(y_n - \frac{\lambda_n}{r_n}Ay_n\right)\right) - J^B_{\lambda_n}\left((1-r_n)z + r_n\left(z - \frac{\lambda_n}{r_n}Az\right)\right)\right\|^q \\ &\le (1-r_n)\|x_n - z\|^q + r_n\left\|\left(I - \frac{\lambda_n}{r_n}A\right)y_n - \left(I - \frac{\lambda_n}{r_n}A\right)z\right\|^q \\ &\le (1-r_n)\|x_n - z\|^q + r_n\left[\|y_n - z\|^q - \frac{\lambda_n}{r_n}\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Az\|^q\right] \\ &\le (1-r_n)\|x_n - z\|^q + r_n\left[\|x_n - z\|^q - \lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Az\|^q\right. \\ &\quad \left. - \frac{\lambda_n}{r_n}\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Az\|^q\right] \\ &= \|x_n - z\|^q - r_n\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Az\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Az\|^q. \end{aligned} \tag{17}$$

This implies that

$$\|z_n - z\| \le \|x_n - z\|.$$

Then, we have

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + \beta_n(x_n - z) + \gamma_n(Sz_n - z)\| \\ &\le \alpha_n\|f(x_n) - z\| + \beta_n\|x_n - z\| + \gamma_n\|Sz_n - z\| \\ &\le \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + \beta_n\|x_n - z\| + \gamma_n\|z_n - z\| \\ &\le (1 - \alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\le \max\{\|x_n - z\|, \frac{1}{1-\rho}\|f(z) - z\|\}. \end{aligned}$$

By induction, we have

$$\|x_n - z\| \le \max\left\{\|x_1 - z\|, \frac{1}{1-\rho}\|f(z) - z\|\right\}, \quad \forall n \ge 1.$$

Thus $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Sz_n\}$ and $\{Ax_n\}$. ■

Theorem 3.3: *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < a \le \beta_n \le b < 1$;
(C3) $0 < \lambda \le \lambda_n < \lambda_n/r_n \le \mu < (\alpha q/\kappa_q)^{1/(q-1)}$ *and* $0 < r \le r_n < 1$.

Then, the sequence $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to an element* $x^* = Q_\Omega f(x^*)$, *where* Q_Ω *is the sunny nonexpansive retraction of E onto* Ω.

Proof: Let $x^* \in \Omega$. From (13) and (17), we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^q \\
&\quad = \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(Sz_n - x^*)\|^q \\
&\quad = \|\alpha_n(f(x_n) - f(x^*)) + \alpha_n(f(x^*) - x^*) + \beta_n(x_n - x^*) + \gamma_n(Sz_n - x^*)\|^q \\
&\quad \le \|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(Sz_n - x^*)\|^q + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le \alpha_n\|f(x_n) - f(x^*)\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|Sz_n - x^*\|^q - \beta_n\gamma_n g(\|x_n - Sz_n\|) \\
&\qquad + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le \alpha_n\rho\|x_n - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\Bigg[\|x_n - x^*\|^q - r_n\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Ax^*\|^q \\
&\qquad - \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\Bigg] - \beta_n\gamma_n g(\|x_n - Sz_n\|) + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad = (1 - (1-\rho)\alpha_n)\|x_n - x^*\|^q - \gamma_n\Bigg[r_n\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Ax^*\|^q \\
&\qquad + \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\Bigg] \\
&\qquad - \beta_n\gamma_n g(\|x_n - Sz_n\|) + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle.
\end{aligned}
\tag{18}
$$

For each $n \ge 1$, we put

$$
\begin{aligned}
\Gamma_n &= \|x_n - x^*\|^q, \\
\theta_n &= (1-\rho)\alpha_n, \\
\eta_n &= \gamma_n\Bigg[r_n\lambda_n(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Ax^*\|^q + \lambda_n\left(\alpha q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\Bigg] \\
&\quad + \beta_n\gamma_n g(\|x_n - Sz_n\|), \\
\delta_n &= q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle.
\end{aligned}
$$

Then (18) becomes to the following formulae:

$$\Gamma_{n+1} \le (1-\theta_n)\Gamma_n - \eta_n + \delta_n, \quad \forall n \ge 1, \tag{19}$$

and

$$\Gamma_{n+1} \le (1-\theta_n)\Gamma_n + \delta_n, \quad \forall n \ge 1. \tag{20}$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}$ is non-increasing. Then

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

From (19), we have

$$0 \le \eta_n \le \Gamma_n - \Gamma_{n+1} + \delta_n - \theta_n \Gamma_n.$$

Since $\theta_n \to 0$ and $\delta_n \to 0$, we have $\eta_n \to 0$. This implies that

$$\lim_{n\to\infty} \|Ax_n - Ax^*\| = 0 \tag{21}$$

and

$$\lim_{n\to\infty} \|Ay_n - Ax^*\| = 0. \tag{22}$$

By the property of g, we also have

$$\lim_{n\to\infty} \|x_n - Sz_n\| = 0. \tag{23}$$

Since $J^B_{\lambda_n}$ is firmly nonexpansive and by Lemma 2.8 (*ii*), we have

$$\begin{aligned}
&\|y_n - x^*\|^q \\
&\quad = \|J^B_{\lambda_n}(x_n - \lambda_n Ax_n) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\
&\quad \le \langle x_n - \lambda_n Ax_n - (x^* - \lambda_n Ax^*), j_q(y_n - x^*)\rangle \\
&\quad \le \frac{1}{q}[\|x_n - \lambda_n Ax_n - (x^* - \lambda_n Ax^*)\|^q + (q-1)\|y_n - x^*\|^q - h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - y_n\|)],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n - x^*\|^q &\le \|x_n - \lambda_n Ax_n - (x^* - \lambda_n Ax^*)\|^q - h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - y_n\|) \\
&\le \|x_n - x^*\|^q - h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - y_n\|). \qquad (24)
\end{aligned}$$

Combining (17) and (24), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|Sz_n - x^*\|^q \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\big[(1-r_n)\|x_n - x^*\|^q + r_n\|y_n - x^*\|^q\big] \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\big[(1-r_n)\|x_n - x^*\|^q \\
&\qquad + r_n(\|x_n - x^*\|^q - h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - y_n\|))\big] \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n r_n h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - y_n\|),
\end{aligned}$$

which implies that

$$\gamma_n r_n h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - y_n\|) \le \Gamma_n - \Gamma_{n+1} + \alpha_n\|f(x_n) - x^*\|^q.$$

Since h_1 is a continuous function, by (21), we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{25}$$

In a similar way, we have

$$\begin{aligned}
&\|z_n - x^*\|^q \\
&\quad = \|J^B_{\lambda_n}(x_n - \lambda_n Ay_n + r_n(y_n - x_n)) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\
&\quad \le \langle x_n - \lambda_n Ay_n + r_n(y_n - x_n) - (x^* - \lambda_n Ax^*), j_q(z_n - x^*)\rangle \\
&\quad \le \frac{1}{q}\Big[\|x_n - \lambda_n Ay_n + r_n(y_n - x_n) - (x^* - \lambda_n Ax^*)\|^q + (q-1)\|z_n - x^*\|^q \\
&\qquad - h_2(\|x_n + r_n(y_n - x_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - x^*\|^q &\le \|x_n - \lambda_n Ay_n + r_n(y_n - x_n) - (x^* - \lambda_n Ax^*)\|^q \\
&\quad - h_2(\|x_n + r_n(y_n - x_n) - \lambda_n(Ay_n - Ax^*) - z_n\|) \\
&\le \|x_n - x^*\|^q - h_2(\|x_n + r_n(y_n - x_n) - \lambda_n(Ay_n - Ax^*) - z_n\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|Sz_n - x^*\|^q \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\big[\|x_n - x^*\|^q \\
&\qquad - h_2(\|x_n + r_n(y_n - x_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)\big] \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n h_2(\|x_n + r_n(y_n - x_n) - \lambda_n(Ay_n - Ax^*) - z_n\|),
\end{aligned}$$

which implies that

$$\gamma_n h_2(\|x_n + r_n(y_n - x_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)) \le \Gamma_n - \Gamma_{n+1} + \alpha_n\|f(x_n) - x^*\|^q.$$

Since h_2 is a continuous function, by (22) and (25), we get

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{26}$$

From (23) and (26), we obtain

$$\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - Sz_n\| + \|Sz_n - Sx_n\| \\
&\le \|x_n - Sz_n\| + \|z_n - x_n\| \\
&\to 0.
\end{aligned}\tag{27}$$

For each $n \ge 1$, we put $T_n := J^B_{\lambda_n}(I - \lambda_n A)$. Then from (25), we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{28}$$

Since $0 < \lambda \le \lambda_n$ for all $n \ge 1$. By Lemma 2.11 (*ii*), we have

$$\begin{aligned}
\|T^{A,B}_{\lambda}x_n - x_n\| &\le 2\|T_n x_n - x_n\| \\
&\to 0.
\end{aligned}\tag{29}$$

Define a mapping $W : E \to E$ by $Wx := ST_\lambda^{A,B}x$ for all $x \in E$. We see that $F(W) = F(S) \cap (A + B)^{-1}0$. Then, we have

$$\begin{aligned}\|x_n - ST_\lambda^{A,B}x_n\| &\le \|x_n - Sx_n\| + \|Sx_n - ST_\lambda^{A,B}x_n\| \\ &\le \|x_n - Sx_n\| + \|x_n - T_\lambda^{A,B}x_n\| \\ &\to 0. \end{aligned} \tag{30}$$

Let $z_t = tf(z_t) + (1-t)Wz_t,\ \forall t \in (0,1)$. Then it follows from Lemma 2.15 that $\{z_t\}$ converges strongly to a fixed point $x^* \in F(T_\lambda)$. From Lemma 2.6, we have

$$\begin{aligned}\|z_t - x_n\|^q &= \|t(f(z_t) - x_n) + (1-t)(Wz_t - x_n)\|^q \\ &\le (1-t)^q\|Wz_t - x_n\|^q + qt\langle f(z_t) - x_n, j_q(z_t - x_n)\rangle \\ &= (1-t)^q\|Wz_t - x_n\|^q + qt\langle f(z_t) - z_t, j_q(z_t - x_n)\rangle + qt\langle z_t - x_n, j_q(z_t - x_n)\rangle \\ &\le (1-t)^q\big(\|Wz_t - T_\lambda x_n\| + \|Wx_n - x_n\|\big)^q \\ &\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q \\ &\le (1-t)^q\big(\|z_t - x_n\| + \|Wx_n - x_n\|\big)^q + qt\langle f(z_t) - z_t, j_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q,\end{aligned}$$

which implies that

$$\langle f(z_t) - z_t, j_q(x_n - z_t)\rangle \le \frac{(1-t)^q}{qt}\big(\|z_t - x_n\| + \|Wx_n - x_n\|\big)^q + \frac{qt-1}{qt}\|z_t - x_n\|^q.$$

From (24), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle f(z_t) - z_t, j_q(x_n - z_t)\rangle &\le \frac{(1-t)^q}{qt}M + \frac{qt-1}{qt}M \\ &= \left(\frac{(1-t)^q + qt - 1}{qt}\right)M, \end{aligned} \tag{31}$$

where $M = \limsup_{n\to\infty}\|z_t - x_n\|^q$, $t \in (0,1)$. We see that $((1-t)^q + qt - 1)/qt \to 0$ as $t \to 0$. Since j_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_t \to x^*$, we have

$$\|j_q(x_n - z_t) - j_q(x_n - x^*)\| \to 0 \quad \text{as } t \to 0.$$

So we have

$$\begin{aligned}&|\langle f(z_t) - z_t, j_q(x_n - z_t)\rangle - \langle f(x^*) - x^*, j_q(x_n - x^*)\rangle| \\ &\quad = |\langle f(z_t) - f(x^*) + f(x^*) - x^* + x^* - z_t, j_q(x_n - z_t)\rangle - \langle f(x^*) - x^*, j_q(x_n - x^*)\rangle| \\ &\quad = |\langle f(z_t) - f(x^*), j_q(x_n - z_t)\rangle + \langle f(x^*) - x^*, j_q(x_n - z_t)\rangle + \langle x^* - z_t, j_q(x_n - z_t)\rangle \\ &\qquad - \langle f(x^*) - x^*, j_q(x_n - x^*)\rangle| \\ &\quad \le |\langle f(x^*) - x^*, j_q(x_n - z_t) - j_q(x_n - x^*)\rangle| + |\langle f(z_t) - f(x^*), j_q(x_n - z_t)\rangle| \\ &\qquad + |\langle x^* - z_t, j_q(x_n - z_t)\rangle| \\ &\quad \le \|f(x^*) - x^*\|\|j_q(x_n - z_t) - j_q(x_n - x^*)\| + (1+\theta)\|z_t - x^*\|\|x_n - z_t\|^{q-1}.\end{aligned}$$

Hence, as $t \to 0$, we have

$$\langle f(z_t) - z_t, j_q(x_n - z_t)\rangle \to \langle f(x^*) - x^*, j_q(x_n - x^*)\rangle.$$

From (31), as $t \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_q(x_n - x^*)\rangle \le 0. \tag{32}$$

By $(C1)$ and (23), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)Sz_n - x_n\|\\ &= \|\alpha_n(f(x_n) - Sz_n) + (1-\beta_n)(Sz_n - x_n)\|\\ &\le \alpha_n\|f(x_n) - Sz_n\| + (1-\beta_n)\|Sz_n - x_n\|\\ &\to 0.\end{aligned} \tag{33}$$

Combining (32) and (33), we get

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \le 0. \tag{34}$$

By Lemma 2.16 and (34), we can conclude that $\Gamma_n \to 0$. Hence, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then, by Lemma 2.17, we have

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n = \|x_n - x^*\|^q$ for all $n \in \mathbb{N}$. Following the proof line in *Case 1*, we can show that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{35}$$

and

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_q(x_{\tau(n)+1} - x^*)\rangle \le 0. \tag{36}$$

Since $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$, by (20), we have

$$\|x_{\tau(n)} - x^*\|^q \le \frac{q}{1-\rho}\langle f(x^*) - x^*, j_q(x_{\tau(n)+1} - x^*)\rangle$$

and hence

$$\limsup_{n\to\infty}\|x_{\tau(n)} - x^*\|^q \le 0.$$

So, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^q = 0.$$

By Proposition 1 and (35), we see that

$$\begin{aligned}\|x_{\tau(n)+1} - x^*\|^q - \|x_{\tau(n)} - x^*\|^q &\leqq \|x_{\tau(n)} - x^*\|^{q-1}(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|)\\ &\leqq \|x_{\tau(n)} - x^*\|^{q-1}\|x_{\tau(n)+1} - x_{\tau(n)}\|\\ &\to 0.\end{aligned}$$

Since $\Gamma_n \le \Gamma_{\tau(n)+1}$. So, we have

$$\|x_n - x^*\|^q \le \|x_{\tau(n)+1} - x^*\|^q = \|x_{\tau(n)} - x^*\|^q + (\|x_{\tau(n)+1} - x^*\|^q - \|x_{\tau(n)} - x^*\|^q) \to 0.$$

Hence, $x_n \to x^*$ as $n \to \infty$. This completes the proof. ∎

We also obtain the convergence theorem of a modified viscosity-type extragradient method in a real Hilbert space. It is well known that $\kappa_2 = 1$ [31]. Then, from Theorem 3.3, we have the following result.

Corollary 3.4: *Let H be a real Hilbert space. Let $S : H \to H$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $A : H \to H$ be an α-ism and let $B : H \multimap H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Assume that $\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Choose an initial guess $x_1 \in H$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by*

$$\begin{aligned} y_n &= J_{\lambda_n}^B(x_n - \lambda_n A x_n), \\ z_n &= J_{\lambda_n}^B(x_n - \lambda_n A y_n + r_n(y_n - x_n)), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n, \quad \forall n \geq 1, \end{aligned} \tag{37}$$

where $J_{\lambda_n}^B = (I + \lambda_n B)^{-1}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\} \subset (0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_n\} \subset (0, \infty)$. Suppose that conditions (C1), (C2) hold and λ_n and r_n satisfy

$$0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < 2\alpha \quad \text{and} \quad 0 < r \leq r_n < 1.$$

Then, the sequence $\{x_n\}$ generated by (37) converges strongly to an element $x^* = P_\Omega u'$, where P_Ω is the metric projection of H onto Ω.

Remark 3.5: (1) Our result is proved with a new technique and new assumption on the control conditions. In addition, we can remove asymptotically regular assumption on $\{\lambda_n\}$.

(2) Our result is applicable for the family of nonexpansive mappings, for example, W_n-mapping, a countable family of nonexpansive mappings and nonexpansive semigroups.

(3) Our result holds in ℓ_p and L_p spaces with $\kappa_2 = p - 1$ for $2 \leq p < \infty$ and $\kappa_p = (1 + t_p^{p-1})(1 + t_p)^{1-p}$ for $1 < p \leq 2$, where t_p is the unique solution of the equation $(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0$, $0 < t < 1$ [31].

4. Some applications

In this section, we give some applications of Theorem 3.3 to important mathematical problems in the framework of Hilbert spaces.

4.1. Application to variational inequality problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear monotone operator. The following problem so-called *variational inequality problem* is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{38}$$

The set of solutions of problem (38) is denoted by $VI(C, A)$. Note that if the variational inequality problem (38) is consistent, it is easy to see that x^* solves problem (38) if and only if it solves the fixed point equation

$$x^* = P_C(x^* - \lambda Ax^*), \tag{39}$$

where P_C is a metric projection from H onto C and $\lambda > 0$. Let i_C be the indicator function of C defined by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases} \tag{40}$$

Denote N_C by the normal cone of C, i.e.

$$N_C(u) = \{z \in H : \langle z, v - u\rangle \leq 0, \quad \forall v \in C\}.$$

It is known that i_C is a proper, convex and lower semi-continuous function and sub-differential ∂i_C is a maximal monotone operator [10]. We define the resolvent operator $J_\lambda^{\partial i_C}$ of i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x), \quad \forall x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{z \in H : i_C(v) + \langle z, v - u\rangle \leq i_C(u), \quad \forall u \in H\} \\ &= \{z \in H : \langle z, v - u\rangle \leq 0, \quad \forall v \in C\} = N_C(u),\ u \in C. \end{aligned}$$

So, we have

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) &\Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, v - u\rangle \leq 0, \quad \forall v \in C \\ &\Leftrightarrow u = P_C(x), \end{aligned}$$

where P_C is the metric projection from H onto C. Further, we also have $(A + \partial i_C)^{-1}0 = VI(C, A)$ [11].

Then, we can set $B = \partial i_C$ in Algorithm 3.1. So we obtain the following Algorithm 4.1.

Algorithm 4.1: *Modified viscosity-type extragradient method for variational inequality and fixed point problems*

Initialization: Given $x_1 \in C$ be arbitrary.

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1: Compute

$$y_n = P_C(x_n - \lambda_n Ax_n)$$

Step 2: Compute

$$z_n = P_C(x_n - \lambda_n Ay_n + r_n(y_n - x_n))$$

Step 3: Compute

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sz_n,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$.

Set $n := n + 1$ and go to Step 1.

Theorem 4.2: *Suppose that conditions* (C1), (C2) *hold and λ_n and r_n satisfy*

$$0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq 2\alpha \quad \text{and} \quad 0 < r \leq r_n < 1.$$

Then, the sequence $\{x_n\}$ generated by Algorithm 4.1 *converges strongly to a common element of $F(S) \cap VI(C, A)$.*

4.2. Application to split feasibility problem

Let H_1 and H_2 be two Hilbert spaces. The *split feasibility problem* (SFP) is to find

$$\hat{x} \in C \quad \text{such that} \quad T\hat{x} \in Q, \tag{41}$$

where C and Q are closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $T : H_1 \to H_2$ is a bounded linear operator with its adjoint T^*. The set of solutions of SFP is denoted by $\Gamma := C \cap T^{-1}(Q) = \{x \in C : Tx \in Q\}$. The SFP was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [43] in 1994 for modelling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction [44].

It is known that $\hat{x}$ solves the SFP (41) if and only if $\hat{x}$ is the solution of the following minimization problem [45]:

$$\min_{x \in C} g(x) := \tfrac{1}{2}\|Tx - P_Q Tx\|^2.$$

Note that the function g is differentiable convex and has a Lipschitz gradient given by $\nabla g = T^*(I - P_Q)T$. Further, ∇g is $1/\|T\|^2$-ism, where $\|T\|^2$ is the spectral radius of T^*T [46]. Hence, x^* solves the SFP if and only if x^* solves the variational inclusion problem of finding $x^* \in H_1$ such that

$$\begin{aligned}
0 \in \nabla g(x^*) + \partial i_C(x^*) &\Leftrightarrow 0 \in x^* + \lambda \partial i_C(x^*) - (x^* - \lambda \nabla g(x^*)) \\
&\Leftrightarrow x^* - \lambda \nabla g(x^*) \in x^* + \lambda \partial i_C(x^*) \\
&\Leftrightarrow x^* = (I + \lambda \partial i_C)^{-1}(x^* - \lambda \nabla g(x^*)) \\
&\Leftrightarrow x^* = P_C(x^* - \lambda \nabla g(x^*)).
\end{aligned}$$

For solving the SFP, Byrne [44] introduces the following so-called *CQ-iterative procedure* for approximating a solution of SFP, which is defined by

$$x_{n+1} = P_C(x_n - \lambda T^*(I - P_Q)Tx_n), \quad \forall n \geq 1, \tag{42}$$

where $0 < \lambda < 2\alpha$ with $\alpha = 1/\|T\|^2$. Here, $\|T\|^2$ is the spectral radius of T^*T. It was shown that the sequence $\{x_n\}$ converges weakly to a solution of the SFP.

In fact, we set $A = \nabla g$ and $B = \partial i_C$ in Algorithm 3.1. So we obtain the following Algorithm 4.3.

Algorithm 4.3: *Modified viscosity-type extragradient method for split feasibility and fixed point problems*

Initialization: Given $x_1 \in C$ be arbitrary.

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1: Compute

$$y_n = P_C(x_n - \lambda_n T^*(I - P_Q)Tx_n)$$

Step 2: Compute

$$z_n = P_C(x_n - \lambda_n T^*(I - P_Q)Ty_n + r_n(y_n - x_n))$$

Step 3: Compute

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$.

Set n: $= n + 1$ and go to Step 1.

Theorem 4.4: *Suppose that conditions* (C1), (C2) *hold and λ_n and r_n satisfy*

$$0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < \frac{2}{\|T\|^2} \quad \text{and} \quad 0 < r \leq r_n < 1.$$

Then, the sequence $\{x_n\}$ generated by Algorithm 4.3 *converges strongly to a common element of $F(S) \cap \Gamma$.*

4.3. Application to LASSO problem

The LASSO is abbreviation for the least absolute shrinkage and selection operator [47], which can be formulated as a convex constrained optimization problem:

$$\min_{x\in H} \frac{1}{2}\|Tx-b\|_2^2 \quad \text{subject to } \|x\|_1 \leq t, \tag{43}$$

where $T : H \to H$ is a bounded operator on H, b is a given vector in H and $t > 0$. Let Γ be the solution set of LASSO (43). The LASSO has received much attention due to the involvement of the ℓ_1 norm which promotes sparsity, phenomenon of many practical problems arising in statics model, image compression, compressed sensing and signal processing theory.

According to the optimization theory, it is known that the solution to the LASSO problem (43) is a minimizer of the following convex unconstrained minimization problem so-called *Basis Pursuit denoising problem*:

$$\min_{x\in H} g(x) + h(x), \tag{44}$$

where $g(x) := (1/2)\|Tx-b\|_2^2$, $h(x) := \lambda\|x\|_1$ and $\lambda \geq 0$ is a regularization parameter. We know that $\nabla g(x) = T^*(Tx-b)$ is $1/\|T^*T\|$-ism. So, we have x^* solves the LASSO if and only if x^* solves the variational inclusion problem of finding $x^* \in H$ such that

$$\begin{aligned}
0 \in \nabla g(x^*) + \partial h(x^*) &\Leftrightarrow 0 \in x^* + \lambda\partial h(x^*) - (x^* - \lambda\nabla g(x^*))\\
&\Leftrightarrow x^* - \lambda\nabla g(x^*) \in x^* + \lambda\partial h(x^*)\\
&\Leftrightarrow x^* = (I + \lambda\partial h)^{-1}(x^* - \lambda\nabla g(x^*))\\
&\Leftrightarrow x^* = \text{prox}_h(x^* - \lambda\nabla g(x^*)),
\end{aligned}$$

where $\text{prox}_h(x)$ is the proximal of $h(x) := \lambda\|x\|_1$ is given by

$$\text{prox}_h(x) = \text{argmin}_{u\in H}\left\{\lambda\|u\|_1 + \frac{1}{2}\|u-x\|_2^2\right\}, \ \forall x \in H,$$

which is separable in indices. Then, for $x \in H$,

$$\begin{aligned}
\text{prox}_h(x) &= \text{prox}_{\lambda\|\cdot\|_1}(x)\\
&= \left(\text{prox}_{\lambda|\cdot|}(x_1), \text{prox}_{\lambda|\cdot|}(x_2), \ldots, \text{prox}_{\lambda|\cdot|}(x_n)\right),
\end{aligned}$$

where $\text{prox}_{\lambda|\cdot|}(x_k) = \text{sgn}(x_k)\max\{|x_k| - \lambda, 0\}$ for $k = 1, 2, \ldots, n$.

In order to solve the LASSO problem, Xu [48] (see also [49]) proposed the following proximal-gradient algorithm (PGA):

$$x_{n+1} = \text{prox}_h(x_n - \lambda_n T^*(Tx_n - b)). \tag{45}$$

He proved that the PGA (45) converges weakly to a solution of the LASSO problem (43).

In fact, we set $A = \nabla g$ and $B = \partial h$ in Algorithm 3.1, we obtain the following Algorithm 4.5.

Algorithm 4.5: *Modified viscosity-type extragradient method for LASSO and fixed point problems*

Initialization: Given $x_1 \in H$ be arbitrary.

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1: Compute

$$y_n = \text{prox}_h(x_n - \lambda_n T^*(Tx_n - b))$$

Step 2: Compute

$$z_n = \text{prox}_h(x_n - \lambda_n T^*(Ty_n - b) + r_n(y_n - x_n))$$

Step 3: Compute

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$.

Set $n := n+1$ *and go to Step 1.*

Theorem 4.6: *Suppose that conditions* (C1), (C2) *hold and* λ_n *and* r_n *satisfy*

$$0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \mu < \frac{2}{\|T^*T\|} \quad \text{and} \quad 0 < r \leq r_n < 1.$$

Then, the sequence $\{x_n\}$ *generated by Algorithm* 4.5 *converges strongly to a common element of* $F(S) \cap \Gamma$.

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An iterative method with residual vectors for solving the fixed point and the split inclusion problems in Banach spaces

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Abstract

In this paper, we propose an iterative technique with residual vectors for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of a split inclusion problem (SIP) with a way of selecting the stepsizes without prior knowledge of the operator norm in the framework of p-uniformly convex and uniformly smooth Banach spaces. Then strong convergence of the proposed algorithm to a common element of the above two sets is proved. As applications, we apply our result to find the set of common fixed points of a family of mappings which is also a solution of the SIP. We also give a numerical example and demonstrate the efficiency of the proposed algorithm. The results presented in this paper improve and generalize many recent important results in the literature.

Keywords Resolvent operator · Relatively nonexpansive mapping · Strong convergence · Iterative methods · Banach spaces

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1 Introduction

Let H_1 and H_2 be two Hilbert spaces. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be two set-valued maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. Moudafi (2011) introduced the following so-called *split inclusion problem* (SIP):

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by Γ, *i.e.*, $\Gamma := \{x^* \in H_1 : x^* \in B_1^{-1}(0) \text{ and } Ax^* \in B_2^{-1}(0)\}$. In fact, we know that the split inclusion problem is a generalization of the inclusion problem and the split feasibility problem. Next, we provide some special cases of SIP (1.1).

- Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex functions. If we take $B_1 = \partial f$ and $B_2 = \partial g$, where ∂f and ∂g are the subdifferential of f and g, then the SIP (1.1) becomes the following so-called *proximal split feasibility problem*:

$$\text{Find } x^* \in \operatorname{argmin} f \text{ such that } Ax^* \in \operatorname{argmin} g, \tag{1.2}$$

  where $\operatorname{argmin} f = \{x \in H_1 : f(x) \leq f(y),\ \forall y \in H_1\}$ and $\operatorname{argmin} g = \{x \in H_2 : g(x) \leq g(y),\ \forall y \in H_2\}$. In particular, if we take $f(x) = \frac{1}{2}\|Mx - b\|^2$ and $g(x) = \frac{1}{2}\|Nx - c\|^2$, where M and N are matrices, and $b, c \in H_1$, then the SIP (1.2) becomes the least square problem. This problem has been intensively studied, especially, in Hilbert spaces; see for instance (Moudafi and Thakur 2014).
- Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. If $B_1 = N_C$, $B_2 = N_Q$, where N_C and N_Q are the normal cones of C and Q, respectively, then the SIP (1.2) becomes the following so-called *split feasibility problem*:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.3}$$

This problem was first introduced, in a finite dimensional Hilbert space, by Censor and Elfving (1994) for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction, especially intensity modulated therapy (Censor et al. 2006).

To solve the SIP (1.1), Byrne et al. (2011) gave the following convergence theorem in infinite dimensional Hilbert spaces:

Theorem 1.1 *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be set-valued maximal monotone mappings, $\lambda > 0$ and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. Suppose that $\Gamma \neq \emptyset$. For given $x_1 \in H_1$, let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad \forall n \geq 1. \tag{1.4}$$

Then $\{x_n\}$ converges weakly to an element $x^* \in \Gamma$.

In order to obtain strong convergence, Kazmi and Rizvi (2014) proposed an algorithm for solving SIP (1.1) with fixed points of a nonexpansive mapping T. They obtained the following result:

Theorem 1.2 *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : H_1 \to H_1$ be a contraction mapping with a constant $\alpha \in (0, 1)$. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be set-valued maximal monotone mappings, $\lambda > 0$. Let $T : H_1 \to H_1$ be a nonexpansive mapping such that $F(T) \cap \Gamma \neq \emptyset$. For a given $x_1 \in H_1$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $x^* \in F(T) \cap \Gamma$, where $x^* = P_{F(T)\cap\Gamma} f(x^*)$.

On the other hand, Takahashi and Takahashi (2016) first introduced the SIP outside Hilbert spaces. Let E_1 and E_2 be two Banach spaces. Let $B_1 : E_1 \multimap E_1$ and $B_2 : E_2 \multimap E_2$ be two set-valued maximal monotone operators and $A : E_1 \to E_2$ be a bounded linear operator. They proposed the SIP in Banach spaces as follows:

$$\text{Find } x^* \in E_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.6}$$

In recent years, many authors have constructed several iterative methods for solving SIP (see, e.g., Sitthithakerngkiet et al. 2018; Takahashi and Takahashi 2016; Takahashi 2015, 2017; Takahashi and Yao 2015; Suantai et al. 2018; Jailoka and Suantai 2017; Ogbuisi and Mewomo 2017; Alofi et al. 2016).

Very recently, Alofi et al. (2016) introduced an algorithm based on Halpern's iteration for solving SIP (1.1) in a uniformly convex and smooth Banach space. They proved the following strong convergence theorem:

Theorem 1.3 *Let H be a Hilbert space and let E be a uniformly convex and smooth Banach space. Let J_E be the duality mapping on E. Let $B_1 : H \multimap H$ and $B_2 : E \multimap E^*$ be maximal monotone operators, respectively. Let $J_\lambda^{B_1}$ be the resolvent of B_1 for $\lambda > 0$ and let $J_\mu^{B_2}$ be the metric resolvent of B for $\mu > 0$. Let $A : H \to E$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Suppose that $\Gamma \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}^{B_1}(x_n - \lambda_n A^*(I - J_{\mu_n}^{B_2})Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\{\lambda_n\}$, $\{\mu_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$0 < a \leq \lambda_n \|A\|^2 \leq b < 2, \quad 0 < k \leq \mu_n, \quad 0 < c \leq \beta_n \leq d < 1,$$

for some $a, b, c, d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where $x^* = P_\Gamma u$.

However, it is observed that several iterative methods suggested require the computation of the norm of the bounded linear operator $\|A\|$, which may not be calculated easily in general. In this work, motivated by the previous works, we introduce an iterative technique with residual vectors for solving the fixed point problem of a relatively nonexpansive mapping and SIP with a way of selecting the step sizes without prior knowledge of the operator norm in the framework of p-uniformly convex and uniformly smooth Banach spaces. We prove its strong convergence of proposed algorithm to a common element of the set fixed points of a relatively nonexpansive mapping and the solutions of the SIP. As applications, we apply our result to finding the set of common fixed points of a family of mappings which is also a solution of the SIP. We also give some numerical examples and demonstrate the efficiency of the proposed algorithm. The results obtained in this paper improve and generalize many known results in the literature.

2 Preliminaries

Let E and E^* be real Banach spaces and the dual space of E, respectively. Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$ which is defined by

$$\langle A^* \bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \quad \forall x \in E_1, \quad \bar{y} \in E_2^*.$$

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}.$$

The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \left\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

Definition 2.1 A Banach space E is said to be

1. *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$;
2. *p-uniformly convex* (or to have a modulus of convexity of power type p) if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for all $\epsilon \in (0, 2]$;
3. *uniformly smooth* if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$;
4. *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \leq c_q \tau^q$ for all $\tau > 0$.

From the Definition 2.1, we observe that every p-uniformly convex space is uniformly convex and if E is q-uniformly smooth, then E is also uniformly smooth. It is known that (Agarwal et al. 2009)

$$\begin{cases} E \text{ is } p\text{-uniformly convex if and only if } E^* \text{ is } q\text{-uniformly smooth,} \\ E \text{ is } q\text{-uniformly smooth if and only if } E^* \text{ is } p\text{-uniformly convex,} \end{cases} \tag{2.1}$$

where $p \geq 2$ and $1 < q \leq 2$ are conjugate exponents, *i.e.*, p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ (see Xu and Roach 1991). For the sequence spaces ℓ_p, Lebesgue spaces L_p and Sobolev spaces W_p^m, we also know that (Agarwal et al. 2009; Hanner 1956; Xu and Roach 1991)

$$\begin{cases} \ell_p,\ L_p \text{ and } W_p^m \text{ are 2-uniformly convex and } p\text{-uniformly smooth with } 1 < p \leq 2, \\ \ell_p,\ L_p \text{ and } W_p^m \text{ are } q\text{-uniformly convex and 2-uniformly smooth with } 2 \leq q < \infty. \end{cases}$$

Definition 2.2 A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.3 The mapping $J_\varphi^E : E \multimap E^*$ associated with a gauge function φ defined by

$$J_\varphi^E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|),\ \|f\| = \varphi(\|x\|),\ \ \forall x \in E\},$$

is called the *duality mapping with gauge* φ, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*.

If $\varphi(t) = t$, then $J_\varphi^E = J_2^E = J$ is the *normalized duality mapping*. In particular, $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi^E = J_p^E$ is called the *generalized duality mapping* which is defined by

$$J_p^E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}.$$

It is well known that if E is uniformly smooth, the generalized duality mapping J_p^E is norm to norm uniformly continuous on bounded subsets of E (see Reich 1981). Furthermore, J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^* (see Reich 1992; Cioranescu 1990 for more details).

For a gauge φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s$$

is a continuous convex strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}.

We next recall the Bregman distance, which was introduced and studied in Bregman (1967).

Definition 2.4 Let E be a real smooth Banach space. The Bregman distance $D_\varphi(x, y)$ between x and y in E is defined by

$$D_\varphi(x, y) = \Phi(\|y\|) - \Phi(\|x\|) - \langle J_\varphi(x), y - x\rangle.$$

We note that the Bregman distance D_φ does not satisfy the well-known properties of a metric because D_φ is not symmetric and does not satisfy the triangle inequality. Moreover, the Bregman distance has the following important properties:

$$D_\varphi(x, y) = D_\varphi(x, z) + D_\varphi(z, y) + \langle J_\varphi^E x - J_\varphi^E z, z - y\rangle, \tag{2.2}$$

$$D_\varphi(x, y) + D_\varphi(y, x) = \langle J_\varphi^E x - J_\varphi^E y, x - y\rangle, \tag{2.3}$$

for all $x, y, z \in E$.

In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the distance $D_\varphi = D_p$ is called the *p-Lyapunov function* which was studied in Bonesky et al. (2008) and it is given by

$$D_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_p^E x, y\rangle + \frac{1}{p}\|y\|^p,$$

where p, q are conjugate exponents. For the p-uniformly convex space, the Bregman distance has the following relation (see Schöpfer et al. 2008):

$$\tau\|x - y\|^p \le D_p(x, y) \le \langle J_p^E x - J_p^E y, x - y\rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number. If $p = 2$, we get

$$D_2(x, y) := \phi(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2,$$

where ϕ is called the *Lyapunov function* which was introduced by Alber (1993, 1996).

The following Lemma can be obtained from Theorem 2.8.17 of Agarwal et al. (2009) (see also Lemma 5 of Kuo and Sahu 2013).

Lemma 2.5 *Let $p > 1$, $r > 0$ and E be a Banach space. Then the following statements are equivalent:*

(i) *E is uniformly convex;*
(ii) *There exists a strictly increasing convex function $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_r^*(0) = 0$ such that*

$$\Big\| \sum_{k=1}^{N} \alpha_k x_k \Big\|^p \leq \sum_{k=1}^{N} \alpha_k \|x_k\|^p - \alpha_i \alpha_j g_r^*(\|x_i - x_j\|),$$

for all $i, j \in \{1, 2, \ldots, N\}$, $x_k \in B_r := \{x \in E : \|x\| \leq r\}$, $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$, where $k \in \{1, 2, \ldots, N\}$.

Lemma 2.6 (Xu 1991) *Let $1 < q \leq 2$ and E be a Banach space. Then the following statements are equivalent:*

(i) *E is q-uniformly smooth;*
(ii) *there is a constant $\kappa_q > 0$ which is called the q-uniform smoothness coefficient of E such that for all $x, y \in E$*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_q^E(x)\rangle + \kappa_q \|y\|^q. \tag{2.5}$$

In what follows, we shall use the following notations: $x_n \to x$ means that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote $F(T)$ by the set of all fixed points of T, *i.e.*, $F(T) = \{x \in C : x = Tx\}$. A point $z \in C$ is called an *asymptotic fixed point* (Reich 1996) of T, if there exists a sequence $\{x_n\}$ in C which converges weakly to z and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote $\widehat{F}(T)$ by the set of asymptotic fixed points of T. A mapping $T : C \to C$ is called *Bregman relatively nonexpansive* (Butnariu et al. 2001, 2003; Censor and Reich 1996; Matsushita and Takahashi 2005), if the following conditions are satisfied:

(R1) $F(T) = \widehat{F}(T) \neq \emptyset$;
(R2) $D_p(Tx, z) \leq D_p(x, z), \quad \forall z \in F(T), \ \forall x \in C.$

Let E be a p-uniformly convex Banach space which is also uniformly smooth. Following Censor and Lent (1981) and Alber (1993), we make use of the function $V_p : E^* \times E \to \mathbb{R}^+$ which is defined by

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p \tag{2.6}$$

for all $x \in E$ and $x^* \in E^*$, where p, q are conjugate exponents. Then V_p is nonnegative and convex in the first variable. It is observed that

$$V_p(x^*, x) = D_p(J_q^{E^*}(x^*), x) \tag{2.7}$$

for all $x \in E$ and $x^* \in E^*$. In addition,

$$V_p(x^*, x) \leq V_p(x^* - y^*, x) + \langle J_q^{E^*}(x^*) - x, y^*\rangle \tag{2.8}$$

for all $x \in E$ and $x^* \in E^*$.

Lemma 2.7 (Bonesky et al. 2008) *Let $p > 1$ and E be a real p-uniformly convex and uniformly smooth Banach space. For $x \in E$ and a sequence $\{x_n\}$ in E. Then,* $\lim_{n\to\infty} D_p(x_n, x) = 0 \Longleftrightarrow \lim_{n\to\infty} \|x_n - x\| = 0$.

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$D_p(x, z) = \min_{y\in C} D_p(x, y). \tag{2.9}$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the *generalized projection* of E onto C.

Lemma 2.8 (Kuo and Sahu 2013) *Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and let $x \in E$. Then the following assertions hold:*

(i) $z = \Pi_C x$ *if and only if* $\langle J_p^E(x) - J_p^E(z), y - z\rangle \le 0$, $\forall y \in C$.
(ii) $D_p(\Pi_C x, y) + D_p(x, \Pi_C x) \le D_p(x, y)$, $\forall y \in C$.

Let $B : E \multimap E^*$ be a mapping. The effective domain of B is denoted by $D(B)$, *i.e.*, $D(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be *monotone* if

$$\langle u - v, x - y\rangle \ge 0, \quad \forall x, y \in D(B),\ u \in Bx \text{ and } v \in By. \tag{2.10}$$

A monotone operator B on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

Let E be a p-uniformly convex and uniformly smooth Banach space and let $B : E \multimap E^*$ be a maximal monotone operator. Then, for $x \in E$ and $\lambda > 0$, we define a mapping $Q_\lambda^B : E \to D(B)$ by

$$Q_\lambda^B(x) := (I + \lambda(J_p^E)^{-1}B)^{-1}(x) \text{ for all } x \in E. \tag{2.11}$$

This mapping is called the *metric resolvent* of B for $\lambda > 0$. The set of null points of B is defined by $B^{-1}(0) = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}(0)$ is closed and convex (see Takahashi 2000). We see that

$$0 \in J_p^E(Q_\lambda^B(x) - x) + \lambda B Q_\lambda^B(x). \tag{2.12}$$

Further, $F(Q_\lambda^B) = B^{-1}(0)$ for $\lambda > 0$ (see Zeidler 1984). From Kuo and Sahu (2013), we also know that

$$\langle Q_\lambda^B(x) - Q_\lambda^B(y), J_p^E(x - Q_\lambda^B(x)) - J_p^E(y - Q_\lambda^B(y))\rangle \ge 0, \tag{2.13}$$

for all $x, y \in E$ and if $B^{-1}(0) \neq \emptyset$, then

$$\langle J_p^E(x - Q_\lambda^B(x)), Q_\lambda^B(x) - z\rangle \ge 0, \tag{2.14}$$

for all $x \in E$ and $z \in B^{-1}(0)$.

In addition, we can define a single-valued mapping $R_\lambda^B : E \to D(B)$ so-called the *resolvent* of B by (Kohsaka and Takahashi 2005)

$$R_\lambda^B(x) := (J_p^E + \lambda B)^{-1} J_p^E(x) \quad \text{for all } x \in E.$$

It is known that R_λ^B is a relatively nonexpansive mapping and $F(R_\lambda^B) = B^{-1}(0)$ for $\lambda > 0$ (see Kuo and Sahu 2013).

Lemma 2.9 (Kohsaka and Takahashi 2005) *Let $B : E \multimap E^*$ be a maximal monotone operator with $B^{-1}(0) \neq \emptyset$ and let R_λ^B be a resolvent operator of B for $\lambda > 0$. Then*

$$D_p(R_\lambda^B(x), z) + D_p(R_\lambda^B(x), x) \leq D_p(x, z),$$

for all $x \in E$ and $z \in B^{-1}(0)$.

The following Theorem is proved by Kohsaka and Takahashi (see Kohsaka and Takahashi 2005, Lemma 7.2).

Lemma 2.10 (Kohsaka and Takahashi 2005) *Let $B : E \multimap E^*$ be a monotone operator. Then B is maximal if and only if for each $\lambda > 0$,*

$$R(J_p^E + \lambda B) = E^*,$$

where $R(J_p^E + \lambda B)$ is the range of $J_p^E + \lambda B$.

The following lemma was proved by Suantai et al. (2018).

Lemma 2.11 *Let E_1 and E_2 be uniformly convex and smooth Banach spaces. Let $A : E_1 \to E_2$ be a bounded linear operator with the adjoint operator A^*. Let $R_\lambda^{B_1}$ be the resolvent operator of a maximal monotone operator B_1 for $\lambda_1 > 0$ and $Q_{\lambda_2}^{B_2}$ be a metric resolvent of a maximal monotone operator B_2 for $\lambda_2 > 0$. Suppose that $\Gamma \neq \emptyset$. Let $r > 0$ and $x^* \in E_1$. Then x^* is a solution of problem (1.6) if and only if*

$$x^* = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(x^*) - rA^*J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Ax^*)).$$

Lemma 2.12 *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Suppose that $x \in E$ and $\{x_n\}$ is a sequence in E. Then the following statements are equivalent:*

(a) *$\{D_p(x_n, x)\}$ is bounded;*
(b) *$\{x_n\}$ is bounded.*

Proof For the implication $(a) \implies (b)$ was proved in Reich and Sabach (2010). For the converse implication $(b) \implies (a)$, we assume that $x \in E$ and $\{x_n\}$ are bounded. From (2.4), we observe that

$$\begin{aligned} D_p(x_n, x) &\leq \langle J_p^E x_n - J_p^E x, x_n - x \rangle \\ &\leq \|J_p^E x_n - J_p^E x\| \|x_n - x\| \\ &\leq M, \end{aligned}$$

for all $n \in \mathbb{N}$, where $M = \sup_{n \geq 1}\{\|x_n\|, \|x_n\|^{p-1}, \|x\|, \|x\|^{p-1}\}$. This implies that $\{D_p(x_n, x)\}$ is bounded. □

Lemma 2.13 (Reich 1979) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^\infty \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14 (Maingé 2008) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \ldots$ *and* $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 2.15 *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, \ldots, N)$ and $\alpha_k \in (0,1)$ with $\sum_{k=1}^N \alpha_k = 1$. Then, we have*

$$D_p\left(J_q^{E^*}\left(\sum_{k=1}^N \alpha_k J_p^E(x_k)\right), z\right) \leq \sum_{k=1}^N \alpha_k D_p(x_k, z) - \alpha_i\alpha_j g_r^*\left(\|J_p^E(x_i) - J_p^E(x_j)\|\right),$$

for all $i, j \in \{1, 2, \ldots, N\}$.

Proof Since p-uniformly convex, hence it is uniformly convex. From Lemma 2.5, we have

$$\begin{aligned}
&D_p\left(J_q^{E^*}\left(\sum_{k=1}^N \alpha_k J_p^E(x_k)\right), z\right)\\
&= V_p\left(\sum_{k=1}^N \alpha_k J_p^E(x_k), z\right)\\
&= \frac{1}{q}\Big\|\sum_{k=1}^N \alpha_k J_p^E(x_k)\Big\|^q - \left\langle \sum_{k=1}^N \alpha_k J_p^E(x_k), z\right\rangle + \frac{1}{p}\|z\|^p\\
&\leq \frac{1}{q}\sum_{k=1}^N \alpha_k \|J_p^E(x_k)\|^q - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) - \left\langle \sum_{k=1}^N \alpha_k J_p^E(x_k), z\right\rangle + \frac{1}{p}\|z\|^p\\
&= \frac{1}{q}\sum_{k=1}^N \alpha_k \|J_p^E(x_k)\|^q - \sum_{k=1}^N \alpha_k \langle J_p^E(x_k), z\rangle + \frac{1}{p}\|z\|^p - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|)\\
&= \sum_{k=1}^N \alpha_k D_p(x_k, z) - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),
\end{aligned}$$

for all $i, j \in \{1, 2, \ldots, N\}$. This completes the proof. □

3 Algorithm and strong convergence theorem

In this section, we introduce an iterative algorithm for finding a common element of the set of solutions of split inclusion problem (1.6) and the set of fixed points of a Bregman relatively nonexpansive mapping. More specifically, we assume the following assumptions:

- E_1 and E_2 are p-uniformly convex and uniformly smooth Banach spaces;
- $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ are maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively;

- $R_{\lambda_1}^{B_1}$ is the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and $Q_{\lambda_2}^{B_2}$ is the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$;
- $A : E_1 \to E_2$ is a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$;
- $T : E_1 \to E_1$ is a Bregman relatively nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$;
- The set of solution of SIP is consistent, *i.e.*, $\Gamma \neq \emptyset$;
- $\Omega := F(T) \cap \Gamma \neq \emptyset$;
- ϵ_n denotes the residual vector in E_1 such that $\lim_{n\to\infty} \epsilon_n = u \in E_1$.

Algorithm 3.1 *Choose an initial guess $u_1 \in E_1$; let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)) \\ u_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \left(\frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \right)^{\frac{1}{q-1}}, \quad n \in N, \tag{3.2}$$

for some $\epsilon > 0$, where the index set $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Note that the choice in (3.2) of the stepsize λ_n is independent of the norms $\|A\|$.

Lemma 3.2 *Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by Algorithm 3.1. Then, $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are bounded.*

Proof. By the choice of λ_n, we observe that

$$\begin{aligned} &\lambda_n^{q-1} \leq \frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \\ \iff &\kappa_q \lambda_n^{q-1} \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q \leq \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p - \epsilon\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q \\ \iff &\frac{\epsilon\kappa_q}{q}\|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q \leq \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p - \frac{\kappa_q \lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q. \end{aligned} \tag{3.3}$$

Let $z \in \Omega$. From (2.14), we observe that

$$\begin{aligned} &\langle J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n, Au_n - Az\rangle \\ &= \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p + \langle J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n, Q_{\lambda_2}^{B_2}(Au_n) - Az\rangle \\ &\geq \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p. \end{aligned} \tag{3.4}$$

Set $v_n := J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)$ for all $n \geq 1$. By (3.4) and Lemma 2.6, we have

$$\begin{aligned} &D_p(x_n, z) \leq D_p(v_n, z) \\ &= D_p\big(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n), z\big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q}\|J_q^{E_1^*}(J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n)\|^p - \langle J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, z\rangle + \frac{1}{p}\|z\|^p \\
&= \frac{1}{q}\|J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n)\|^q - \langle J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, z\rangle + \frac{1}{p}\|z\|^p \\
&\le \frac{1}{q}\|J_p^{E_1}(u_n)\|^q - \lambda_n\langle Au_n, J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\rangle + \frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q - \langle J_p^{E_1}(u_n), z\rangle \\
&\quad +\lambda_n\langle J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, Az\rangle + \frac{1}{p}\|z\|^p \\
&= \frac{1}{q}\|u_n\|^p - \langle J_p^{E_1}(u_n), z\rangle \\
&\quad +\frac{1}{p}\|z\|^p + \lambda_n\langle J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, Az-Au_n\rangle + \frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q \\
&= D_p(u_n,z) + \lambda_n\langle J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, Az-Au_n\rangle + \frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q \\
&\le D_p(u_n,z) - \lambda_n\left(\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q\right),
\end{aligned}
\tag{3.5}
$$

which implies by (3.3) that

$$D_p(x_n, z) \le D_p(u_n, z).$$

Since $\lim_{n\to\infty}\epsilon_n = u \in E_1$, which implies that $\{\epsilon_n\}$ is bounded, then from Lemma 2.12, we have $\{D_p(\epsilon_n, z)\}$ is bounded. So there exists a constant $K > 0$ such that $D_p(\epsilon_n, z) \le K$ for all $n \ge 1$. From Lemma 2.15, we have

$$
\begin{aligned}
D_p(x_{n+1}, z) &\le D_p(u_{n+1}, z) \\
&= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), z) \\
&\le \alpha_n D_p(\epsilon_n, z) + \beta_n D_p(x_n, z) + \gamma_n D_p(Tx_n, z) \\
&\quad -\beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|) \\
&\le \alpha_n D_p(\epsilon_n, z) + (1-\alpha_n) D_p(x_n, z) - \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|) \\
&\le \alpha_n K + (1-\alpha_n) D_p(x_n, z) \\
&\le \max\{K, D_p(x_n, z)\} \\
&\ \ \vdots \\
&\le \max\{K, D_p(x_1, z)\}.
\end{aligned}
\tag{3.6}
$$

By induction, we have $\{D_p(x_n, z)\}$ is bounded. Hence, $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{Au_n\}$.

Theorem 3.3 *Let $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by Algorithm 3.1. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
(C2) $0 < k \le \beta_n\gamma_n \le 1$ for some $k \in (0, 1)$.

Then $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ converge strongly to $x^* = \Pi_\Omega u$, where Π_Ω is the generalized projection from E_1 onto Ω.

Proof Let $x^* = \Pi_{F(T)\cap\Gamma}u$. From (2.7) and (3.6), we have

$$
\begin{aligned}
&D_p(x_{n+1}, x^*)\\
&\le D_p(u_{n+1}, x^*)\\
&= V_p\big(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n), x^*\big)\\
&\le V_p(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n) - \alpha_n(J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), x^*))\\
&\quad + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&= V_p(\alpha_n J_p^{E_1}(x^*) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n), x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(x^*) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), x^*)\\
&\quad + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&\le \alpha_n D_p(x^*, x^*) + \beta_n D_p(x_n, x^*) + \gamma_n D_p(Tx_n, x^*) - \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\quad + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&\le (1-\alpha_n)D_p(x_n, x^*) - \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\quad + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle. \qquad (3.7)
\end{aligned}
$$

We now divide the proof into two cases:

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_p(x_n, x^*)\}_{n=n_0}^{\infty}$ is non-increasing. So we have $\{D_p(x_n, x^*)\}_{n=1}^{\infty}$ converges and it is bounded. From (3.7), we have

$$
\begin{aligned}
0 &\le kg_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\le \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\le D_p(x_n, x^*) - D_p(x_{n+1}, x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle. \qquad (3.8)
\end{aligned}
$$

This implies that

$$\lim_{n\to\infty} g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|) = 0.$$

By the property of g_r^*, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| = 0. \qquad (3.9)$$

Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (3.10)$$

By Lemma 2.7, we also have

$$\lim_{n\to\infty} D_p(x_n, Tx_n) = 0. \qquad (3.11)$$

By the boundedness of $\{x_n\}$ and the reflexivity of E_1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x} \in E_1$. From (3.10), we obtain $\hat{x} \in \widehat{F}(T) = F(T)$. From (3.3), (3.5) and (3.6), we see that

$$\begin{aligned}\frac{\epsilon^2\kappa_q}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q &\le \lambda_n\Big(\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p-\frac{\kappa_q\lambda_n^{q-1}}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q\Big)\\ &\le D_p(u_n,\hat{x})-D_p(x_n,\hat{x})\\ &\le \alpha_{n-1}D_p(\epsilon_{n-1},\hat{x})+D_p(x_{n-1},\hat{x})-D_p(x_n,\hat{x})\to 0 \ \text{ as } \ n\to\infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|=0. \tag{3.12}$$

From (3.5) and (3.6), we have

$$\begin{aligned}\epsilon\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p &\le \lambda_n\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p\\ &\le D_p(u_n,\hat{x})-D_p(x_n,\hat{x})+\frac{\kappa_q\lambda_n^q}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q\\ &\le \alpha_{n-1}D_p(\epsilon_{n-1},\hat{x})+D_p(x_{n-1},\hat{x})-D_p(x_n,\hat{x})\\ &\quad+\frac{\kappa_q\lambda_n^q}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|Au_n-Q_{\lambda_2}^{B_2}Au_n\|=0. \tag{3.13}$$

Then, we have

$$\begin{aligned}\|J_p^{E_1}(v_n)-J_p^{E_1}(u_n)\| &\le \lambda_n\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|\\ &\le \lambda_n\|A^*\|\|J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|\\ &\le \lambda_n\|A^*\|\|Au_n-Q_{\lambda_2}^{B_2}Au_n\|^{p-1}\to 0 \ \text{ as } \ n\to\infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|J_p^{E_1}(v_n)-J_p^{E_1}(u_n)\|=0. \tag{3.14}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty}\|v_n-u_n\|=0. \tag{3.15}$$

By Lemma 2.9 and (3.6), we have

$$\begin{aligned}D_p(x_n,v_n) &= D_p(R_{\lambda_1}^{B_1}v_n,v_n)\\ &\le D_p(v_n,\hat{x})-D_p(x_n,\hat{x})\\ &\le D_p(u_n,\hat{x})-D_p(x_n,\hat{x})\\ &\le \alpha_{n-1}D_p(\epsilon_{n-1},\hat{x})+D_p(x_{n-1},\hat{x})-D_p(x_n,\hat{x})\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty}\|R_{\lambda_1}^{B_1}v_n-v_n\|=\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{3.16}$$

Since $x_{n_i}\rightharpoonup\hat{x}\in E_1$, we also have $v_{n_i}\rightharpoonup\hat{x}\in E_1$. From (3.16), we get $\hat{x}\in F(R_{\lambda_1}^{B_1})\in B_1^{-1}(0)$.

From (3.15) and (3.16), we obtain

$$\|x_n-u_n\|\le\|x_n-v_n\|+\|v_n-u_n\|\to 0 \ \text{ as } \ n\to\infty. \tag{3.17}$$

Since $x_{n_i} \rightharpoonup \hat{x} \in E_1$ and from (3.17), we also get $u_{n_i} \rightharpoonup \hat{x} \in E_1$. From (2.14), we have

$$\begin{aligned}
\|(I - Q_{\lambda_2}^{B_2})A\hat{x}\|^p &= \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}\rangle \\
&= \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), A\hat{x} - Au_{n_i}\rangle \\
&\quad + \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), Au_{n_i} - Q_{\lambda_2}^{B_2}Au_{n_i}\rangle \\
&\quad + \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), Q_{\lambda_2}^{B_2}Au_{n_i} - Q_{\lambda_2}^{B_2}A\hat{x}\rangle \\
&\le \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), A\hat{x} - Au_{n_i}\rangle \\
&\quad + \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), Au_{n_i} - Q_{\lambda_2}^{B_2}Au_{n_i}\rangle.
\end{aligned} \tag{3.18}$$

Since A is continuous, we have $Au_{n_i} \rightharpoonup A\hat{x}$ as $i \to \infty$. Then, we have

$$\|A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}\| = 0,$$

that is, $A\hat{x} = Q_{\lambda_2}^{B_2}A\hat{x}$. This shows that $A\hat{x} \in F(Q_{\lambda_2}^{B_2}) = B_2^{-1}(0)$. So $\hat{x} \in \Gamma$. Therefore, we conclude that $\hat{x} \in \Omega := F(T) \cap \Gamma$.

Now, we see that

$$\begin{aligned}
D_p(u_{n+1}, x_n) &\le D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), x_n) \\
&\le \alpha_n D_p(\epsilon_n, x_n) + \beta_n D_p(x_n, x_n) + \gamma_n D_p(Tx_n, x_n) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|u_{n+1} - x_n\| = 0. \tag{3.19}$$

So, we have

$$\|u_{n+1} - u_n\| \le \|u_{n+1} - x_n\| + \|x_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{3.20}$$

We now choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^*\rangle = \lim_{i\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^*\rangle,$$

where $x^* = \Pi_\Omega u$. From (3.17) and Lemma 2.8, we get

$$\begin{aligned}
\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_n - x^*\rangle &= \limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^*\rangle \\
&= \lim_{i\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^*\rangle \\
&= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \hat{x} - x^*\rangle \le 0.
\end{aligned}$$

From (3.20), we also have

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle \le 0. \tag{3.21}$$

By (3.7), we note that

$$\begin{aligned}
D_p(x_{n+1}, x^*) &\le (1-\alpha_n)D_p(x_n, x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle \\
&= (1-\alpha_n)D_p(x_n, x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(u), u_{n+1} - x^*\rangle \\
&\quad + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle.
\end{aligned}$$

Since $\epsilon_n \to u$ implies $J_p^{E_1}(\epsilon_n) \to J_p^{E_1}(u)$. Considering this together with (3.21), we conclude by Lemma 2.13 that $D_p(x_n, x^*) \to 0$ as $n \to \infty$. Therefore,S $x_n \to x^* \in \Omega$.

Case 2 Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then, by Lemma 2.14, we obtain

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \le \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n = D_p(x_n, x^*)$ for all $n \in \mathbb{N}$. Then, we have from (3.6) that

$$\begin{aligned}
0 &\le \lim_{n\to\infty} (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) \\
&\le \lim_{n\to\infty} (D_p(\epsilon_{\tau(n)}, x^*) + (1-\alpha_{\tau(n)})D_p(x_{\tau(n)}, x^*) - D_p(x_{\tau(n)}, x^*)) \\
&= \lim_{n\to\infty} \alpha_{\tau(n)}\big(D_p(\epsilon_{\tau(n)}, x^*) - D_p(x_{\tau(n)}, x^*)) = 0,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) = 0. \tag{3.22}$$

Following the proof line in *Case 1*, we can show that

$$\begin{aligned}
&\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0, \\
&\lim_{n\to\infty} \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2} Au_{\tau(n)}\| = 0, \\
&\lim_{n\to\infty} \|Au_{\tau(n)} - Q_{\lambda_2}^{B_2} Au_{\tau(n)}\| = 0, \\
&\lim_{n\to\infty} \|x_{\tau(n)} - v_{\tau(n)}\| = \lim_{n\to\infty} \|x_{\tau(n)} - u_{\tau(n)}\| = 0
\end{aligned}$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)+1} - u_{\tau(n)}\| = 0.$$

Furthermore, we can show that

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle \le 0.$$

From (3.7), we have

$$D_p(x_{\tau(n)+1}, x^*) \le (1-\alpha_{\tau(n)})D_p(x_{\tau(n)}, x^*) + \alpha_{\tau(n)}\langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle,$$

which implies that

$$\begin{aligned}
\alpha_{\tau(n)} D_p(x_{\tau(n)}, x^*) \le\ & D_p(x_{\tau(n)}, x^*) - D_p(x_{\tau(n)+1}, x^*) \\
&+ \alpha_{\tau(n)}\langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle.
\end{aligned}$$

Since $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$, we get

$$\begin{aligned}
D_p(x_{\tau(n)}, x^*) &\le \langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle \\
&= \langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(u), u_{\tau(n)+1} - x^*\rangle + \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle.
\end{aligned}$$

Since $\epsilon_{\tau(n)} \to u$ implies $J_p^{E_1}(\epsilon_{\tau(n)}) \to J_p^{E_1}(u)$. Hence, $\lim_{n\to\infty} D_p(x_{\tau(n)}, x^*) = 0$. From (3.22), we obtain

$$\begin{aligned} D_p(x_n, x^*) \leq D_p(x_{\tau(n)+1}, x^*) = \ & D_p(x_{\tau(n)}, x^*) + (D_p(x_{\tau(n)+1}, x^*) \\ & - D_p(x_{\tau(n)}, x^*)) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $D_p(x_n, x^*) \to 0$. That is $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

We consequently obtain the following result in Hilbert spaces:

Corollary 3.4 *Let H_1 and H_2 be Hilbert spaces. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively. Let $R_{\lambda_1}^{B_1}$ be the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and let $Q_{\lambda_2}^{B_2}$ be the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator $A^* : H_2 \to H_1$. Let $T : H_1 \to H_1$ be a relatively nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$. Assume that $\Omega := F(T) \cap \Gamma \neq \emptyset$. Choose an initial guess $u_1 \in H_1$; let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(u_n - \lambda_n A^*(I - Q_{\lambda_2}^{B_2})Au_n) \\ u_{n+1} = \alpha_n \epsilon_n + \beta_n x_n + \gamma_n T x_n, \quad \forall n \geq 1, \end{cases} \tag{3.23}$$

where $\{\epsilon_n\} \subset H_1$ is a residual vector such that $\epsilon_n \to u$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \frac{2\|(I - Q_{\lambda_2}^{B_2})Au_n\|^2}{\|A^*(I - Q_{\lambda_2}^{B_2})Au_n\|^2} - \epsilon, \quad n \in N, \tag{3.24}$$

for some $\epsilon > 0$, where the index set $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < k \leq \beta_n \gamma_n \leq 1$ *for some* $k \in (0, 1)$.

Then $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to $x^* = \Pi_\Omega u$.

4 Convergence theorems for a family of mappings

In this section, we apply our result to the common fixed point problems of a family of mappings.

4.1 A countable family of relatively nonexpansive mappings

Definition 4.1 (Aoyama et al. 2007) Let C be a subset of a real p-uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C in to E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the *AKTT-condition* if, for any bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{z \in B}\{\|J_p^E(T_{n+1}z) - J_p^E(T_n z)\|\} < \infty.$$

As in Suantai et al. (2012), we can prove the following Proposition:

Proposition 4.2 *Let C be a nonempty, closed and convex subset of a real p-uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Suppose that for any bounded subset B of C. Then there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in B \tag{4.1}$$

and

$$\lim_{n\to\infty} \sup_{z\in B} \|J_p^E(Tz) - J_p^E(T_n z)\| = 0.$$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3 *Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively. Let $R_{\lambda_1}^{B_1}$ be the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and let $Q_{\lambda_2}^{B_2}$ be the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of Bregman relatively nonexpansive mappings on E_1 such that $F(T_n) = \widehat{F}(T_n)$ for all $n \geq 1$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma \neq \emptyset$. Choose an initial guess $u_1 \in E_1$, and let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)) \\ u_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(T_n x_n)), \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\{\epsilon_n\} \subset E_1$ is a residual vector such that $\epsilon_n \to u$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \left(\frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \right)^{\frac{1}{q-1}}, \quad n \in N, \tag{4.3}$$

for some $\epsilon > 0$, where the index set $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < k \leq \beta_n \gamma_n \leq 1$ *for some* $k \in (0, 1)$.

Suppose in addition that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition and $F(T) = \widehat{F}(T)$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to $x^* = \Pi_\Omega u$, where Π_Ω is the generalized projection from E_1 onto Ω.

Proof To this end, it suffices to show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By following the method of proof in Theorem 3.3, we can show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 4.2, we see that

$$\begin{aligned}\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| + \|J_p^{E_1}(T_nx_n) - J_p^{E_1}(Tx_n)\| \\ &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| \\ &\quad + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_nx) - J_p^{E_1}(Tx)\| \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

4.2 A semigroup of relatively nonexpansive mappings

Definition 4.4 A one-parameter family $\mathcal{S} = \{T_t\}_{t\ge0}$ from E into E is said to be a *nonexpansive semigroup* if it satisfies the following conditions:

(S1) $T_0x = x$ for all $x \in E$;
(S2) $T_{s+t} = T_sT_t$ for all $s, t \ge 0$;
(S3) for each $x \in C$ the mapping $t \mapsto T_tx$ is continuous;
(S4) for each $t \ge 0$, T_t is nonexpansive, *i.e.*,

$$\|T_tx - T_ty\| \le \|x - y\|, \ \forall x, y \in E.$$

Remark 4.5 We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, *i.e.*, $F(\mathcal{S}) = \bigcap_{t\ge0} F(T_t)$.

We now give some examples of semigroup operator. The following classical examples were one of the main sources for the development of semigroup theory (see Engel and Nagel 2000):

Example 4.6 Let E be a real Banach space and let $\mathcal{L}(E)$ be the space of all bounded linear operators on E. For $A \in \mathcal{L}(E)$ and define a bounded linear operator T_t by

$$T_t := e^{tA} = \sum_{n=0}^{\infty} \frac{t^nA^n}{n!},$$

for $t \ge 0$. Then, the operator T_t is a semigroup on E.

Example 4.7 Let $E := L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Consider the initial value problem for the heat equation:

$$\begin{aligned}\tfrac{\partial u}{\partial t} &= \Delta u, \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x, 0) &= f(x), \quad \text{for } x \in \mathbb{R}^n,\end{aligned} \tag{4.4}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on E. We can solve the heat equation using Fourier transform and the solution (4.4) can be written as follows:

$$u(x, t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{\frac{-\|s-\xi\|^2}{4t}} f(\xi)d\xi,$$

where $t > 0$, $s \in \mathbb{R}^n$ and $f \in E$. Then, we can write the solution $u(x,t)$ in the form of convolution integral as follows:

$$u(x,t) = (K_t * f)(x),$$

where K_t is heat kernel given by $K_t(x) = \frac{1}{\sqrt{(4\pi t)^n}} e^{\frac{-\|x\|^2}{4t}}$. Then the solution of (4.4) can be written as follows:

$$T_t f(x) := u(x,t) = (K_t * f)(x).$$

We can check that the operator $T_t f(x)$ is a semigroup on E.

Definition 4.8 A one-parameter family $\mathcal{S} = \{T_t\}_{t\geq 0} : E \to E$ is said to be a *family of uniformly Lipschitzian mappings* if there exists a bounded measurable function $L_t : (0,\infty) \to [0,\infty)$ such that

$$\|T_t x - T_t y\| \leq L_t \|x - y\|, \ \forall x, y \in E.$$

We now first give the following definition:

Definition 4.9 A one-parameter family $\mathcal{S} = \{T_t\}_{t\geq 0} : E \to E$ is said to be a *Bregman relatively nonexpansive semigroup* if it satisfies $(S1)$, $(S2)$, $(S3)$ and the following conditions:

(a) $F(\mathcal{S}) = \widehat{F}(\mathcal{S}) \neq \emptyset$;
(b) $D_p(T_t x, z) \leq D_p(x, z)$, $\forall x \in E$, $z \in F(\mathcal{S})$ and $t \geq 0$.

Using idea in Aleyner and Censor (2005), Aleyner and Reich (2005) and Benavides et al. (2002), we define the following concept:

Definition 4.10 A continuous operator semigroup $\mathcal{S} = \{T_t\}_{t\geq 0} : E \to E$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) if for all $s \geq 0$ and any bounded subset B of E such that

$$\lim_{t\to\infty} \sup_{x\in B} \|J_p^E(T_t x) - J_p^E\big(T_s T_s x\big)\| = 0.$$

Theorem 4.11 *Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively. Let $R_{\lambda_1}^{B_1}$ be the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and let $Q_{\lambda_2}^{B_2}$ be the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$. Let $\mathcal{S} = \{T_t\}_{t\geq 0}$ be a u.a.r. Bregman relatively nonexpansive semigroup of uniformly Lipschitzian mappings on E_1 into E_1 with a bounded measurable function $L_t : (0,\infty) \to [0,\infty)$ such that $F(\mathcal{S}) := \bigcap_{h\geq 0} F(T_h) \neq \emptyset$. Assume that $\Omega := F(\mathcal{S}) \cap \Gamma \neq \emptyset$. Choose an initial guess $u_1 \in E_1$; let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)) \\ u_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(T_{t_n} x_n)), \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\{\epsilon_n\} \subset E_1$ is a residual vector such that $\epsilon_n \to u$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \left(\frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \right)^{\frac{1}{q-1}}, \quad n \in N, \tag{4.6}$$

for some $\epsilon > 0$*, where the index set* $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ *and* $\lambda_n = \lambda$ *(*λ *being any nonnegative value), otherwise. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < k \leq \beta_n\gamma_n \leq 1$ *for some* $k \in (0, 1)$*;*
(C3) $\{t_n\} \subset (0, \infty)$ *with* $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *converge strongly to* $x^* = \Pi_\Omega u$*, where* Π_Ω *is the generalized projection from* E_1 *onto* Ω.

Proof We only have to show that $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$ for all $t \geq 0$. By following the method of proof in Theorem 3.3, we can show that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - T_{t_n} x_n\| = 0. \tag{4.7}$$

Since $\{T_t\}_{t\geq 0}$ is a uniformly of Lipschitzian mappings with a bounded measurable function L_t. Then, we have

$$\begin{aligned}\|T_t T_{t_n} x_n - T_t x_n\| &\leq L_t \|T_{t_n} x_n - x_n\| \\ &\leq \sup_{t\geq 0}\{L_t\} \|T_{t_n} x_n - x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Since $J_p^{E_1}$ is uniformly norm-to-norm continuous on bounded subsets of E_1, then we also have

$$\lim_{n\to\infty} \|J_p^{E_1}(T_t T_{t_n} x_n) - J_p^{E_1}(T_t x_n)\| = 0. \tag{4.8}$$

For each $t \geq 0$, we note that

$$\begin{aligned}\|J_p^{E_1}(x_n) - J_p^{E_1}(T_t x_n)\| &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_{t_n} x_n)\| + \|J_p^{E_1}(T_{t_n} x_n) - J_p^{E_1}(T_t T_{t_n} x_n)\| \\ &\quad + \|J_p^{E_1}(T_t T_{t_n} x_n) - J_p^{E_1}(T_t x_n)\| \\ &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_{t_n} x_n)\| + \|J_p^{E_1}(T_t T_{t_n} x_n) - J_p^{E_1}(T_t x_n)\| \\ &\quad + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_{t_n} x) - J_p^{E_1}(T_t T_{t_n}) x\|.\end{aligned}$$

Since $\{T_t\}_{t\geq 0}$ is a u.a.r. Bregman relatively nonexpansive semigroup with $\lim_{n\to\infty} t_n = \infty$, then from (4.7) and (4.8), we get

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_t x_n)\| = 0$$

for all $t \geq 0$. Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, we get

$$\lim_{n\to\infty} \|x_n - T_t x_n\| = 0.$$

This completes the proof. □

5 Numerical experiments

In this section, we give some numerical examples to support our main theorem.

Table 1 Numerical results of Algorithm 5.2 with different choices of N and M

The choices of N and M	No. of iterations	cpu (time)
$N = 50$, $M = 50$	250	0.007260
$N = 100$, $M = 50$	290	0.010884
$N = 200$, $M = 200$	357	0.031999
$N = 150$, $M = 300$	347	0.024889
$N = 500$, $M = 1000$	460	0.260444

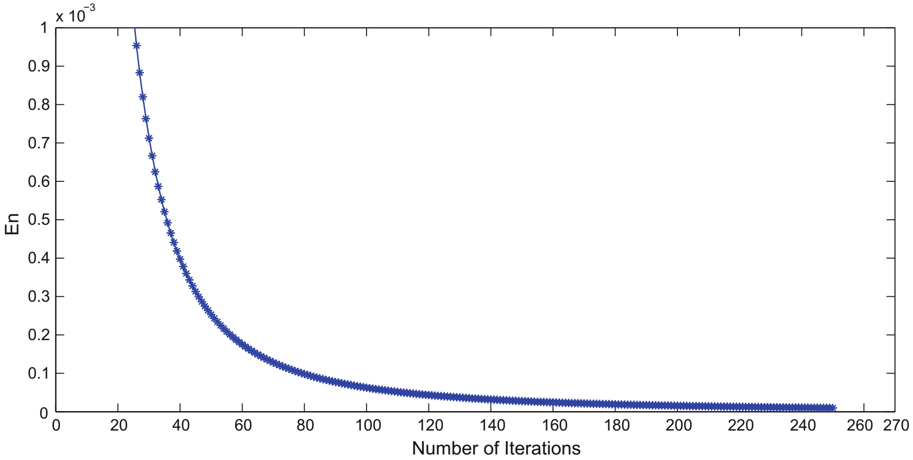

Fig. 1 The convergence behavior of E_n for $N = 50$ and $M = 50$

Example 5.1 For each $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(\mathbf{x}) := \|\mathbf{x}\|_2 \text{ and } g(\mathbf{x}) = -\sum_{i=1}^{N} \log x_i.$$

Then, we have

$$\text{prox}_{\lambda f}(\mathbf{x}) \begin{cases} \left(1 - \frac{\lambda}{\|\mathbf{x}\|_2}\right)x; & \|\mathbf{x}\|_2 \geq \lambda \\ 0; & \|\mathbf{x}\|_2 < \lambda \end{cases} \tag{5.1}$$

and

$$\text{prox}_{\lambda g}(\mathbf{x})_i = \frac{x_i + \sqrt{x_i^2 + 4\lambda}}{2}$$

for $i = 1, 2, 3, \ldots, N$. Let a mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ be defined by

$$T\mathbf{x} = (2 - x_1, 2 - x_2, 2 - x_3, \ldots, 2 - x_N).$$

We aim to solve the following SIP and the fixed point problem: find $x^* \in \Gamma \cap F(T)$, *i.e.*, find $x^* \in \text{argmin } f$ such that $Ax^* \in \text{argmin } g$ and x^* is a fixed point of T, where A is a real

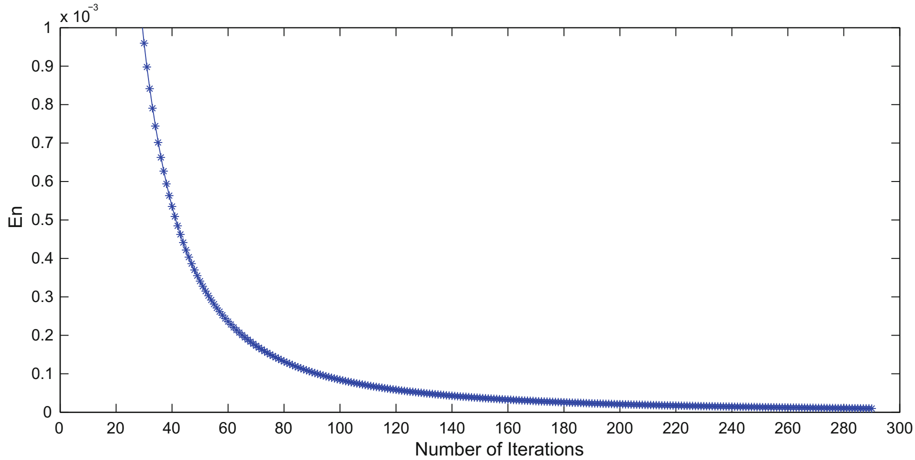

Fig. 2 The convergence behavior of E_n for $N = 100$ and $M = 50$

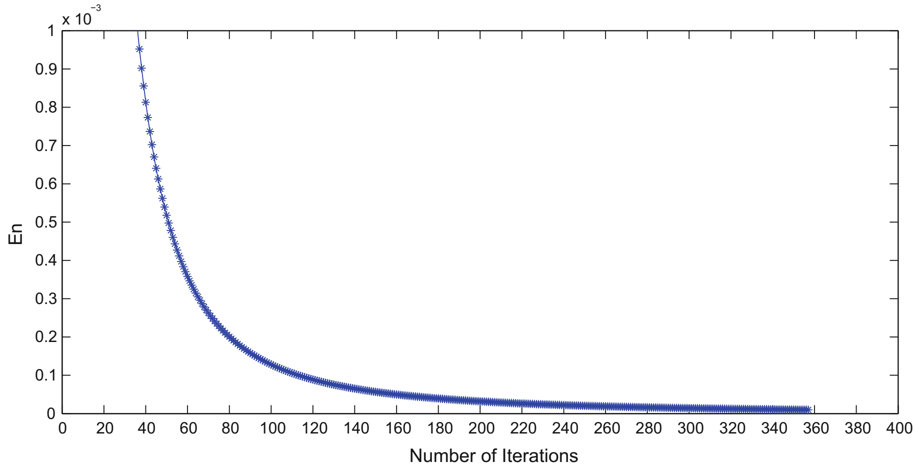

Fig. 3 The convergence behavior of E_n for $N = 200$ and $M = 200$

$N \times M$ matrix. So our iterative scheme (3.1) becomes

$$\begin{cases} \mathbf{x}_n = \mathrm{prox}_{\lambda_1}^{f}\left[\mathbf{u}_n - \lambda_n A^t(A\mathbf{u}_n - \mathrm{prox}_{\lambda_2}^{g}(A\mathbf{u}_n))\right] \\ \mathbf{u}_{n+1} = \alpha_n \epsilon_n + \beta_n \mathbf{x}_n + \gamma_n T\mathbf{x}_n, \quad \forall n \geq 1. \end{cases} \tag{5.2}$$

Let $\lambda_1 = \lambda_2 = 1$, $\alpha_n = \frac{1}{20n+1}$, $\beta_n = 0.5$, $\gamma_n = \frac{10n-0.5}{20n+1}$ and $\lambda_n = \frac{\|A\mathbf{u}_n - \mathrm{prox}_{\lambda_2}^{g}(A\mathbf{u}_n)\|^2}{\|A^t(A\mathbf{u}_n - \mathrm{prox}_{\lambda_2}^{g}(A\mathbf{u}_n))\|^2}$. The stopping criterion is defined by $E_n = \|u_{n+1} - u_n\| < 10^{-6}$. The matrix A is generated from a normal distribution with mean zero and one variance. For an initial guess $\mathbf{x}_1 \in \mathbb{R}^N$ and residual vector $\epsilon_n \in \mathbb{R}^N$ randomly, we obtain the following numerical results, given in Table 1 and Figs. 1, 2, 3, 4 and 5:

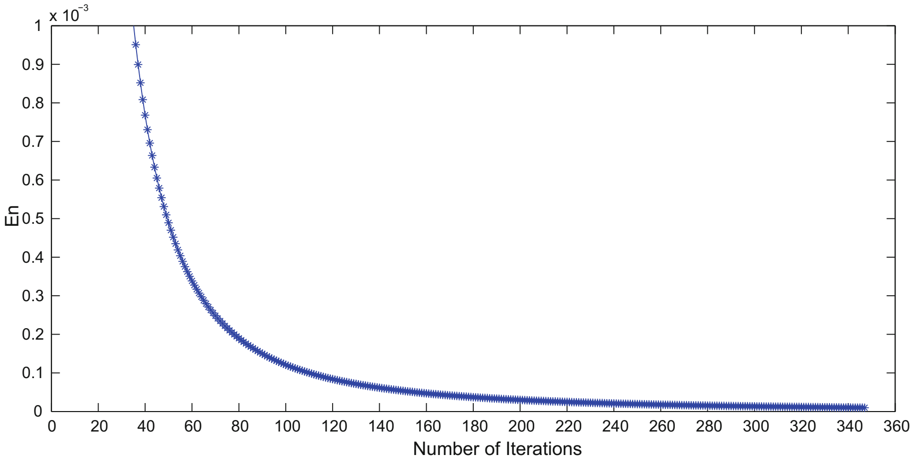

Fig. 4 The convergence behavior of E_n for $N = 150$ and $M = 300$

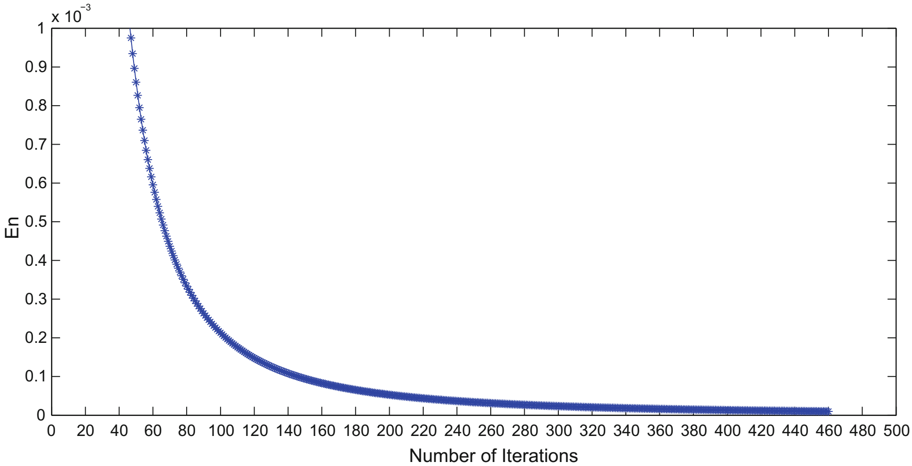

Fig. 5 The convergence behavior of E_n for $N = 500$ and $M = 1000$

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STRONG CONVERGENCE OF A GENERAL VISCOSITY EXPLICIT RULE FOR THE SUM OF TWO MONOTONE OPERATORS IN HILBERT SPACES

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and Pongsakorn Sunthrayuth[3,†]

Abstract In this paper, we study a general viscosity explicit rule for approximating the solutions of the variational inclusion problem for the sum of two monotone operators. We then prove its strong convergence under some new conditions on the parameters in the framework of Hilbert spaces. As applications, we apply our main result to the split feasibility problem and the LASSO problem. We also give some numerical examples to support our main result. The results presented in this paper extend and improve the corresponding results in the literature.

Keywords Monotone operator, Hilbert space, strong convergence, iterative method.

MSC(2010) 47H09, 47H10, 47J25, 47J05.

1. Introduction

Let H be a real Hilbert space. In this paper, we study the *variational inclusion problem* (VIP) which is the problem of finding $z \in H$ such that

$$0_H \in (A + B)z \tag{1.1}$$

where $A : H \to H$ is an operator, $B : H \multimap H$ is a set-valued operator and 0_H is a zero vector in H. The set of solutions of VIP is denoted by $(A + B)^{-1}0_H$.

It is known that the variational inclusion problem is a generalization of variational inequalities, equilibrium problem, split feasibility problem, convex minimization problem and linear inverse problem (see [23, 33, 37]). Moreover, the variational inclusion problem has many applications in applied sciences, engineering, economics and medical sciences especially image and signal processing, statistical regression and machine learning (see, *e.g.* [6, 34, 39]).

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A popular method for solving the VIP is the *forward–backward algorithm* (FBA) [3,16,21,40] which is defined by the following manner: $x_1 \in H$ and

$$x_{n+1} = J_r^B(x_n - rAx_n), \quad \forall n \geq 1, \tag{1.2}$$

where $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator and $J_r^B := (I + rB)^{-1}$ is a resolvent operator of B for $r > 0$. It was shown that the sequence $\{x_n\}\}$ generated by FBA converges weakly to a solution of VIP. This method also includes, in particular, the proximal point algorithm [5,13,18,27,32] and the gradient method [4,17].

In order to obtain the strong convergence, Lopez et al. [23] introduced the following Halpern iteration for solving the VIP: $x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n}^B(x_n - r_n(Ax_n + a_n) + b_n), \quad \forall n \geq 1, \tag{1.3}$$

where $u \in H$ is a given point, $\{a_n\}$ and $\{b_n\}$ are sequences in H, $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a solution of VIP.

Lin and Takahashi [22] proposed the following modified FBA by using the viscosity approximation method introduced by Moudafi [29]: $x_1 \in H$ and

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n) J_{r_n}^B(x_n - r_n Ax_n), \quad \forall n \geq 1, \tag{1.4}$$

where $h : H \to H$ is a contraction, $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution of VIP.

In recent years, some modifications of FBA have been investigated extensively by many researchers in the several setting (see, *e.g.*, [1,8,12,14,15,19,20,30,31,35,38]).

Takahashi et al. [37] introduced the following iteration for solving the fixed point problem of a nonexpansive mapping and the variational inclusion problem:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T(\alpha_n u + (1 - \alpha_n) J_{r_n}^B(x_n - r_n Ax_n)), \quad \forall n \geq 1, \tag{1.5}$$

where $u \in H$ is a given point, T is a nonexpansive mapping, $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator. Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common solution.

On the other hand, a typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping T on a real Hilbert space H:

$$\min_{x \in F(T)} \tfrac{1}{2}\langle Gx, x\rangle - f(x), \tag{1.6}$$

where A is a linear bounded operator and f is a potential function for γh (*i.e.*, $f'(x) = \gamma h(x)$ for $x \in H$).

Using the viscosity approximation method, Marino and Xu [26] introduced the following general iterative process for a nonexpansive mapping T on H:

$$x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)Tx_n, \quad \forall n \geq 1, \tag{1.7}$$

where h is a contraction on H and $0 < \gamma < \frac{\bar{\gamma}}{\theta}$. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a unique solution of the variational inequality

$$\langle(\gamma h - G)z, x - z\rangle \leq 0, \quad \forall x \in F(T), \tag{1.8}$$

which is also the optimality condition for the minimization problem (1.6).

Very recently, Marino et al. [25] introduced the following general viscosity explicit rule in real Hilbert spaces:

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)T(t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where T is a nonexpansive mapping and h is a contraction on H. They proved that the sequence $\{x_n\}$ generated by (1.9) strongly converges to a fixed point of T, which is also the unique solution of the variational inequality (1.8).

Motivated by the works in the literature, we aim to propose a new general viscosity explicit rule for solving variational inclusion (1.1) in the framework of Hilbert spaces. We prove its strong convergence under some suitable condition on the parameters. As applications, we apply our main result to the split feasibility problem and the LASSO problem. Some numerical experiments are also given in this paper.

2. Preliminaries and lemmas

In this section, we provide some basic definitions and lemmas which will be used in our proof.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonlinear mapping. We denote $F(T)$ by the set of fixed points of T.

• A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

• A mapping $T : C \to C$ is said to be *contractive* if there exists a constant $\theta \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \theta\|x - y\|, \quad \forall x, y \in C.$$

• A mapping $G : H \to H$ is said to be *strongly positive* if there is a constant $\bar{\gamma} > 0$ such that

$$\langle Gx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.1}$$

Let $A : H \multimap H$ be a set-valued operator. We denote the domain of an operator A by $\mathrm{dom}(A) = \{x \in H : Ax \neq \emptyset\}$. The set of all zero points of A is denoted by $A^{-1}0_H$, *i.e.*,

$$A^{-1}0_H = \{x \in H : 0_H \in Ax\}$$

where 0_H is a zero vector of H.

• An operator A is said to be *monotone* if for each $x, y \in \mathrm{dom}(A)$,

$$\langle u - v, x - y\rangle \geq 0, \quad u \in Ax \text{ and } v \in Ay.$$

• An operator A is said to be *α-inverse strongly monotone* if for each $x, y \in \mathrm{dom}(A)$, there exists $\alpha > 0$ such that

$$\langle u - v, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad u \in Ax \text{ and } v \in Ay.$$

• A monotone operator A is said to be *maximal* if the graph of A is not property contained in the graph of any other monotone operators. It is known that a monotone operator A is maximal if and only if $\mathcal{R}(I + rA) = H$ for all $r > 0$, where $\mathcal{R}(I + rA)$ is the range of $I + rA$.

In this case, we can define the *resolvent operator* of A for r by $J_r^A = (I + rA)^{-1} : H \to \mathrm{dom}(A)$. It is known that J_r^A is single-valued and nonexpansive. Moreover, $F(J_r^A) = A^{-1}0_H$ (see [36]).

Let C be a nonempty, closed and convex subset of a real Hilbert space H. The nearest point projection of H onto C is denoted by P_C with the property

$$\|x - P_C x\| \le \|x - y\|$$

for all $x \in H$ and $y \in C$. Such P_C is called the *metric projection* of H onto C. It is well known that P_C satisfies

$$\langle x - P_C x, y - P_C x\rangle \le 0$$

for all $x \in H$ and $y \in C$ (see [36]).

We next recall some facts which will be needed in the rest of this paper.

Lemma 2.1 ([36]). *Let H be a real Hilbert space. Then the following statements hold:*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$*;*
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ *for* $t \in \mathbb{R}$ *and* $x, y \in H$*.*

Lemma 2.2 ([26]). *Assume G is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|G\|^{-1}$. Then, $\|I - \rho G\| \le 1 - \rho\bar{\gamma}$.*

Let $A : H \to H$ be an α-inverse strongly monotone and $B : H \multimap H$ be a maximal monotone operator. In what follows, we shall use the following notation:

$$T_r = J_r^B(I - rA) = (I + rB)^{-1}(I - rA), \quad r > 0.$$

Lemma 2.3 ([23]). *The following statements hold:*

(i) *For* $r > 0$, $F(T_r) = (A + B)^{-1}0$.
(ii) *For* $0 < r \le s$ *and* $x \in H$, $\|x - T_r x\| \le 2\|x - T_s x\|$.

Lemma 2.4 ([23]). *Let H be a real Hilbert space. Assume that A is an α-inverse strongly monotone in H. Then, given $r > 0$, we have*

$$\begin{aligned}\|T_r x - T_r y\|^2 \le{}& \|x - y\|^2 - r(2\alpha - r)\|Ax - Ay\|^2 \\ &- \|(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y\|^2,\end{aligned}$$

for all $x, y \in B_r := \{z \in H : \|z\| \le r\}$. In particular, if $0 < r < 2\alpha$, then T_r is nonexpansive.

Lemma 2.5 ([26]). *Let H be a real Hilbert space. Let T be a nonexpansive mapping on H such that $F(T) \neq \emptyset$, G be a strongly positive linear bounded operator on H and h be a contraction on H with coefficient $\theta \in (0, 1)$ and $0 < \gamma < \bar{\gamma}/\theta$. Let $\{z_t\}$ be a net which is defined by*

$$z_t = t\gamma h(z_t) + (I - tG)Tz_t, \quad \forall t \in (0, 1).$$

Then $\{z_t\}$ converges strongly as $t \to 0^+$ to a point $z \in F(T)$, which solves the variational inequality:

$$\langle \gamma h(z) - Gz, x - z \rangle \leq 0, \ \forall x \in F(T).$$

Lemma 2.6 ([42]). *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 ([24]). *Let $\{s_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$s_{m_k} \leq s_{m_k+1} \quad \text{and} \quad s_k \leq s_{m_k+1}.$$

In fact, $m_k := \max\{j \leq k : s_j \leq s_{j+1}\}$.

3. Main results

In this section, we introduce a new general viscosity explicit rule for solving the VIP and prove the strong convergence theorem of the proposed method in real Hilbert spaces.

Theorem 3.1. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \multimap H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $G : H \to H$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and $h : H \to H$ be a contraction with coefficient $\theta \in (0, 1)$ such that $0 < \gamma < \bar{\gamma}/\theta$. Choose an initial guess $x_1 \in H$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n) J_{r_n}^B (x_n - r_n A x_n), \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G) J_{r_n}^B (I - r_n A)(t_n x_n + (1 - t_n)\bar{x}_{n+1}), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{r_n\} \subset \big(0, 2\alpha\big)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}(1 - t_n)(1 - \beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $z = P_{(A+B)^{-1}0}\gamma h(z)$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, without loss of generality, that $\alpha_n < \|G\|^{-1}$ for all $n \geq 1$. For each $n \geq 1$, we put $T_n := J^B_{r_n}(I - r_n A)$. Let $z \in (A+B)^{-1}0$. By the nonexpansivity of T_n, we have

$$\begin{aligned}
\|\bar{x}_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(T_n x_n - T_n z)\| \\
&\leq \beta_n\|x_n - z\| + (1-\beta_n)\|T_n x_n - T_n z\| \\
&\leq \beta_n\|x_n - z\| + (1-\beta_n)\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&= \|\alpha_n(\gamma h(x_n) - Gz) + (I - \alpha_n G)(T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z)\| \\
&\leq \alpha_n\|\gamma h(x_n) - Gz\| + \|I - \alpha_n G\|\|T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z\| \\
&\leq \alpha_n\gamma\|h(x_n) - h(z)\| + \alpha_n\|\gamma h(z) - Gz\| \\
&\quad + (1-\alpha_n\bar{\gamma})\|\|t_n(z_n - z) + (1-t_n)(\bar{x}_{n+1} - z)\| \\
&\leq \alpha_n\gamma\theta\|x_n - z\| + (1-\alpha_n\bar{\gamma})(t_n\|x_n - z\| + (1-t_n)\|\bar{x}_{n+1} - z\|) + \alpha_n\|\gamma h(z) - Gz\| \\
&\leq (1 - (\bar{\gamma} - \gamma\theta)\alpha_n)\|x_n - z\| + (\bar{\gamma} - \gamma\theta)\alpha_n\frac{\|\gamma h(z) - Gz\|}{\bar{\gamma} - \gamma\theta} \\
&\leq \max\left\{\|x_n - z\|, \frac{\|\gamma h(z) - Gz\|}{\bar{\gamma} - \gamma\theta}\right\}.
\end{aligned}$$

By induction, we obtain

$$\|x_n - z\| \leq \max\left\{\|x_1 - z\|, \frac{\|\gamma h(z) - Gz\|}{\bar{\gamma} - \gamma\theta}\right\}, \quad \forall n \geq 1.$$

Hence $\{x_n\}$ is bounded.

For each $n \geq 1$, we put $z_n := t_n x_n + (1-t_n)\bar{x}_{n+1}$. By Lemma 2.4, we have

$$\begin{aligned}
&\|T_n z_n - z\|^2 \\
&= \|J^B_{r_n}(I - r_n A)z_n - J^B_{r_n}(I - r_n A)z\|^2 \\
&\leq \|z_n - z\|^2 - r_n(2\alpha - r_n)\|Az_n - Az\|^2 - \|z_n - r_n Az_n - T_n z_n + r_n Az\|^2.
\end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned}
&\|z_n - z\|^2 \\
&\leq t_n\|x_n - z\|^2 + (1-t_n)\|\bar{x}_{n+1} - z\|^2 \\
&\leq t_n\|x_n - z\|^2 + (1-t_n)\Big[\beta_n\|x_n - z\|^2 + (1-\beta_n)\|T_n x_n - z\|^2\Big] \\
&\leq t_n\|x_n - z\|^2 + (1-t_n)\Big[\beta_n\|x_n - z\|^2 + (1-\beta_n)\Big(\|x_n - z\|^2 \\
&\quad - r_n(2\alpha - r_n)\|Ax_n - Az\|^2 - \|x_n - r_n Ax_n - T_n x_n + r_n Az\|^2\Big)\Big] \\
&\leq \|x_n - z\|^2 - (1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2
\end{aligned}$$

$$+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big). \tag{3.3}$$

Substituting (3.3) into (3.2), we get

$$\begin{aligned}
&\|T_nz_n - z\|^2\\
\le\ &\|x_n - z\|^2 - (1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big)\\
&-r_n(2\alpha - r_n)\|Az_n - Az\|^2 - \|z_n - r_nAz_n - T_nz_n + r_nAz\|^2.
\end{aligned}\tag{3.4}$$

From Lemma 2.1 (i) and (3.4), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2\\
=\ &\|\alpha_n(\gamma h(x_n) - Gz) + (I - \alpha_nG)(T_nz_n - z)\|^2\\
\le\ &\|(I-\alpha_nG)(T_nz_n - z)\|^2 + 2\alpha_n\gamma\langle h(x_n) - h(z), x_{n+1} - z\rangle\\
&+2\alpha_n\langle \gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &(1-\alpha_n\bar{\gamma})^2\|T_nz_n - z\|^2 + 2\alpha_n\gamma\theta\|x_n - z\|\|x_{n+1} - z\|\\
&+2\alpha_n\langle \gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &(1-\alpha_n\bar{\gamma})^2\|T_nz_n - z\|^2 + \alpha_n\gamma\theta\big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big)\\
&+2\alpha_n\langle \gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &(1-\alpha_n\bar{\gamma})^2\Big[\|x_n - z\|^2 - (1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) - r_n(2\alpha - r_n)\|Az_n - Az\|^2\\
&-\|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big] + \alpha_n\gamma\theta\big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big)\\
&+2\alpha_n\langle \gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &\big((1-\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\theta\big)\|x_n - z\|^2 + \alpha_n\gamma\theta\|x_{n+1} - z\|^2\\
&-(1-\alpha_n\bar{\gamma})^2(1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) - (1-\alpha_n\bar{\gamma})^2\Big(r_n(2\alpha - r_n)\|Az_n - Az\|^2\\
&+\|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big) + 2\alpha_n\langle \gamma h(z) - Gz, x_{n+1} - z\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - z\|^2\\
\le\ &\frac{(1-\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\theta}{1-\alpha_n\gamma\theta}\|x_n - z\|^2\\
&-\frac{(1-\alpha_n\bar{\gamma})^2(1-t_n)(1-\beta_n)}{1-\alpha_n\gamma\theta}\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) - \frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\Big(r_n(2\alpha - r_n)\|Az_n - Az\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big) + \frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle \\
&= \left[1 - \frac{2(\bar{\gamma}-\gamma\theta)\alpha_n}{1-\alpha_n\gamma\theta}\right]\|x_n - z\|^2 + \frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\|x_n - z\|^2 \\
&\quad -K_n\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2 + \|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) \\
&\quad -\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\Big(r_n(2\alpha - r_n)\|Az_n - Az\|^2 + \|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big) \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle, \qquad (3.5)
\end{aligned}
$$

where $K_n := \frac{(1-\alpha_n\bar{\gamma})^2(1-t_n)(1-\beta_n)}{1-\alpha_n\gamma\theta}$. We note that $\liminf_{n\to\infty} K_n > 0$ and $\liminf_{n\to\infty} r_n(2\alpha - r_n) > 0$. For each $n \geq 1$, we set

$$
\begin{aligned}
s_n &:= \|x_n - z\|^2, \\
\gamma_n &:= \frac{2(\bar{\gamma}-\gamma\theta)\alpha_n}{1-\alpha_n\gamma\theta}, \\
\eta_n &:= K_n\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2 + \|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) \\
&\quad +\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}(r_n(2\alpha - r_n)\|Az_n - Az\|^2 + \|z_n - r_nAz_n - T_nz_n + r_nAz\|^2), \\
\delta_n &:= \frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle + \frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\|x_n - z\|^2.
\end{aligned}
$$

Then (3.5) reduces to the the following formulae:

$$s_{n+1} \leq (1-\gamma_n)s_n - \eta_n + \delta_n, \quad \forall n \geq 1 \qquad (3.6)$$

and

$$s_{n+1} \leq (1-\gamma_n)s_n + \delta_n, \quad \forall n \geq 1. \qquad (3.7)$$

We next show that $s_n \to 0$ as $n \to \infty$ by considering two possible cases:
Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{s_n\}_{n=n_0}^{\infty}$ is non-increasing. This implies that $\{s_n\}_{n=1}^{\infty}$ is convergent. From (3.6), we have

$$0 \leq \eta_n \leq s_n - s_{n+1} + \delta_n - \gamma_n s_n.$$

Since $\lim_{n\to\infty}\gamma_n = \lim_{n\to\infty}\delta_n = 0$, which implies that $\lim_{n\to\infty}\eta_n = 0$. Then, we obtain

$$\lim_{n\to\infty}\|Az_n - Az\| = \lim_{n\to\infty}\|z_n - r_nAz_n - T_nz_n + r_nAz\| = 0$$

and

$$\lim_{n\to\infty}\|Ax_n - Az\| = \lim_{n\to\infty}\|x_n - r_nAx_n - T_nx_n + r_nAz\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|T_nz_n - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|T_nx_n - x_n\| = 0. \qquad (3.8)$$

Since $\liminf_{n\to\infty} r_n > 0$, there exists $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. Then, by Lemma 2.3 (ii), we have

$$\|T_r x_n - x_n\| \leq 2\|T_n x_n - x_n\|.$$

From (3.8), we obtain

$$\lim_{n\to\infty} \|T_r x_n - x_n\| = 0. \tag{3.9}$$

Let $z_t = t\gamma h(z_t) + (I - tG)T_r z_t,\ \forall t \in (0,1)$. Then it follows from Lemma 2.5 that $\{z_t\}$ converges strongly to a fixed point $z \in F(T_r)$. So, we obtain

$$\begin{aligned}
&\|z_t - x_n\|^2 \\
&= \|t(\gamma h(z_t) - Gx_n) + (I - tG)(T_r z_t - x_n)\|^2 \\
&\leq (1 - t\bar{\gamma})^2\|T_r z_t - x_n\|^2 + 2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&= (1 - t\bar{\gamma})^2\|T_r z_t - T_r x_n + T_r x_n - x_n\|^2 + 2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&\leq (1 - t\bar{\gamma})^2\bigg(\|T_r z_t - T_r x_n\|^2 + 2\langle T_r x_n - x_n, T_r z_t - x_n\rangle\bigg) \\
&\quad +2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&\leq (1 - t\bar{\gamma})^2\bigg(\|z_t - x_n\|^2 + 2\|T_r x_n - x_n\|\|T_r z_t - x_n\|\bigg) \\
&\quad +2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&= (1 - 2t\bar{\gamma} + (\bar{\gamma}t)^2)\|z_t - x_n\|^2 + 2t\langle \gamma h(z_t) - Gz_t, z_t - x_n\rangle \\
&\quad +2t\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t),
\end{aligned} \tag{3.10}$$

where

$$f_n(t) = 2(1 - t\bar{\gamma})^2\|T_r z_t - x_n\|\|T_r x_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Since G is strongly positive linear, we have

$$\langle Gz_t - Gx_n, z_t - x_n\rangle = \langle G(z_t - x_n), z_t - x_n\rangle \geq \bar{\gamma}\|z_t - x_n\|^2. \tag{3.12}$$

It follows (3.10) and (3.12) that

$$\begin{aligned}
&2t\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \\
&\leq (\bar{\gamma}^2 t^2 - 2t\bar{\gamma})\|z_t - x_n\|^2 + 2t\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t) \\
&\leq (\bar{\gamma}t^2 - 2t)\langle Gz_t - Gx_n, z_t - x_n\rangle + 2t\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t) \\
&= \bar{\gamma}t^2\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t),
\end{aligned} \tag{3.13}$$

which implies that

$$\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \leq \frac{\bar{\gamma}t}{2}\langle Gz_t - Gx_n, z_t - x_n\rangle + \frac{1}{2t}f_n(t). \tag{3.14}$$

Taking limit $n \to \infty$ in (3.14) and noting (3.11), we have

$$\limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \leq \frac{t}{2}M, \tag{3.15}$$

where $M > 0$ is large enough. Taking limit $t \to 0$ in (3.15), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \le 0. \tag{3.16}$$

Since

$$\begin{aligned}
&\langle \gamma h(z) - Gz, x_n - z\rangle \\
&= \langle \gamma h(z) - Gz, x_n - z\rangle - \langle \gamma h(z) - Gz, x_n - z_t\rangle + \langle \gamma h(z) - Gz, x_n - z_t\rangle \\
&\quad - \langle \gamma h(z) - Gz_t, x_n - z_t\rangle + \langle \gamma h(z) - Gz_t, x_n - z_t\rangle - \langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \\
&\quad + \langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_n - z\rangle \\
&\le \|\gamma h(z) - Gz\|\|z_t - z\| + (\|G\| + \gamma\theta)\|z_t - z\|\lim_{n\to\infty}\|x_n - z_t\| \\
&\quad + \limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle.
\end{aligned}$$

Then, from (3.16), we obtain that

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_n - z\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_n - z\rangle \\
&\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \\
&\le 0.
\end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned}
\|T_n z_n - x_n\| &\le \|T_n z_n - z_n\| + \|z_n - x_n\| \\
&\le \|T_n z_n - z_n\| + (1 - t_n)(1 - \beta_n)\|T_n x_n - x_n\| \\
&\le \|T_n z_n - z_n\| + \|T_n x_n - x_n\|.
\end{aligned}$$

This together with (3.8) implies that

$$\lim_{n\to\infty}\|T_n z_n - x_n\| = 0. \tag{3.18}$$

Further, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|x_{n+1} - T_n z_n\| + \|T_n z_n - x_n\| \\
&\le \alpha_n\|h(x_n) - T_n z_n\| + \|T_n z_n - x_n\|.
\end{aligned}$$

This together with (3.18) implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.19}$$

Combining (3.17) and (3.19), we get that

$$\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle \le 0. \tag{3.20}$$

Due to (3.7), we see that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0$. Utilizing Lemma 2.6, we can conclude that $\lim_{n\to\infty} s_n = 0$. Therefore $x_n \to z$ as $n \to \infty$.

Case 2. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that $s_{n_i} \leq s_{n_i+1}$ for all $i \in \mathbb{N}$. By Lemma 2.7, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and

$$s_{m_k} \leq s_{m_k+1} \quad \text{and} \quad s_k \leq s_{m_k+1} \tag{3.21}$$

for all $k \in \mathbb{N}$. So, we have

$$0 \leq \eta_{m_k} \leq s_{m_k} - s_{m_k+1} + \delta_{m_k} - \gamma_{m_k} s_{m_k} \to 0 \quad \text{and} \quad k \to \infty.$$

This implies that

$$\lim_{k\to\infty} \|T_{m_k} z_{m_k} - z_{m_k}\| = 0 \quad \text{as} \quad \lim_{k\to\infty} \|T_{m_k} x_{m_k} - x_{m_k}\| = 0. \tag{3.22}$$

Following the proof line in **Case 1**, we can show that

$$\lim_{k\to\infty} \|T_r x_{m_k} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty} \langle \gamma h(z) - Gz, x_{m_k} - z\rangle \leq 0.$$

Since

$$\begin{aligned}
\|T_{m_k} z_{m_k} - x_{m_k}\| &\leq \|T_{m_k} z_{m_k} - z_{m_k}\| + \|z_{m_k} - x_{m_k}\| \\
&\leq \|T_{m_k} z_{m_k} - z_{m_k}\| + (1 - s_{m_k})(1 - \beta_{m_k})\|T_{m_k} x_{m_k} - x_{m_k}\| \\
&\leq \|T_{m_k} z_{m_k} - z_{m_k}\| + \|T_{m_k} x_{m_k} - x_{m_k}\|.
\end{aligned}$$

This implies by (3.18) that

$$\lim_{k\to\infty} \|T_{m_k} z_{m_k} - x_{m_k}\| = 0.$$

Note that

$$\begin{aligned}
\|x_{m_k+1} - x_{m_k}\| &\leq \|x_{m_k+1} - T_{m_k} z_{m_k}\| + \|T_{m_k} z_{m_k} - x_{m_k}\| \\
&\leq \alpha_{m_k} \|h(x_{m_k}) - T_{m_k} z_{m_k}\| + \|T_{m_k} z_{m_k} - x_{m_k}\|.
\end{aligned}$$

Hence, we have

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0 \tag{3.23}$$

and hence

$$\limsup_{k\to\infty} \langle \gamma h(z) - Gz, x_{m_k+1} - z\rangle \leq 0. \tag{3.24}$$

From (3.6), we have

$$s_{m_k+1} \leq (1 - \gamma_{m_k}) s_{m_k} + \delta_{m_k}. \tag{3.25}$$

This implies that

$$\gamma_{m_k} s_{m_k} \leq s_{m_k} - s_{m_k+1} + \delta_{m_k}.$$

Since $s_{m_k} \leq s_{m_k+1}$ and $\alpha_{m_k} > 0$ then $\lim_{k\to\infty} s_{m_k} = 0$. By the fact that $a^2 - b^2 \leq 2a(a-b)$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned} |s_{m_k+1} - s_{m_k}| &= \|x_{m_k+1} - z\|^2 - \|x_{m_k} - z\|^2 \\ &\leq 2\|x_{m_k} - z\|(\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \\ &\leq 2\|x_{m_k} - z\|\|x_{m_k+1} - x_{m_k}\|. \end{aligned}$$

This implies by (3.23) that

$$\lim_{k\to\infty} (s_{m_k+1} - s_{m_k}) = 0.$$

So, we have

$$s_k \leq s_{m_k+1} = s_{m_k} + (s_{m_k+1} - s_{m_k}) \to 0 \quad \text{as} \quad k \to \infty,$$

which implies that $\lim_{k\to\infty} s_k = 0$ and so $x_k \to z$ as $k \to \infty$. This completes the proof. □

Next, we also study the following general viscosity explicit rule (3.1) with the error sequence.

Theorem 3.2. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \multimap H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $G : H \to H$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and $h : H \to H$ be a contraction with coefficient $\theta \in (0,1)$ such that $0 < \gamma < \bar{\gamma}/\theta$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) J_{r_n}^B (x_n - r_n A x_n), \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G) J_{r_n}^B (I - r_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) + e_n, \ \forall n \geq 1, \end{cases} \tag{3.26}$$

where $\{e_n\} \subset H$, $\{r_n\} \subset (0, 2\alpha)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0,1)$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} (1-t_n)(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$;

(C4) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $z = P_{(A+B)^{-1}0}\gamma h(z)$.

Proof. For arbitrary initial guess $y_1 \in H$, we define a sequence $\{y_n\}$ as follows:

$$\begin{cases} \bar{y}_{n+1} = \beta_n y_n + (1-\beta_n) T_n y_n, \\ y_{n+1} = \alpha_n \gamma h(y_n) + (I - \alpha_n G) T_n (t_n y_n + (1-t_n)\bar{y}_{n+1}), \quad \forall n \geq 1, \end{cases}$$

where $T_n = J_{r_n}^B (I - r_n A)$. By Theorem 3.1, we know that $\{y_n\}$ converges strongly to $z = P_{(A+B)^{-1}0}\gamma h(z)$. We next show that $x_n \to z$ as $n \to \infty$. By the nonexpansiveness of T_n, we have

$$\begin{aligned} \|\bar{x}_{n+1} - \bar{y}_{n+1}\| &\leq \beta_n \|x_n - y_n\| + (1-\beta_n)\|T_n x_n - T_n y_n\| \\ &\leq \|x_n - y_n\|. \end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1}-y_{n+1}\| \\
&= \|\alpha_n\gamma(h(x_n)-h(y_n)) + (I-\alpha_n G)(T_n(t_nx_n+(1-t_n)\bar{x}_{n+1}) \\
&\quad -T_n(t_ny_n+(1-t_n)\bar{y}_{n+1})) + e_n\| \\
&\le \alpha_n\gamma\|h(x_n)-h(y_n)\| + (1-\alpha_n\bar{\gamma})\|T_n(t_nx_n+(1-t_n)\bar{x}_{n+1}) \\
&\quad -T_n(t_ny_n+(1-t_n)\bar{y}_{n+1})\| + \|e_n\| \\
&\le \alpha_n\gamma\|h(x_n)-h(y_n)\|+(1-\alpha_n\bar{\gamma})\Big(t_n\|x_n-y_n\|+(1-t_n)\|\bar{x}_{n+1}-\bar{y}_{n+1}\|\Big)+\|e_n\| \\
&\le \alpha_n\gamma\theta\|x_n-y_n\| + (1-\alpha_n\bar{\gamma})\|x_n-y_n\| + \|e_n\| \\
&= (1-(\bar{\gamma}-\theta\gamma)\alpha_n)\|x_n-y_n\| + \|e_n\|.
\end{aligned}$$

From $(C4)$ and Lemma 2.6, we obtain $\lim_{n\to\infty}\|x_n-y_n\| = 0$. Then, we conclude that $x_n \to z$. This completes the proof. □

We remark some merits of our work as follows:

(1) The method of proof in Theorem 3.1 is very different from the proof in Theorem 3.1 of Marino et al. [25] because Algorithm (3.1) deals with the problem of finding an element of $(A+B)^{-1}0$ which involves the resolvent of maximal monotone operator.

(2) The result presented in Theorem 3.1 is proved under new assumptions on $\{\beta_n\}$ and $\{t_n\}$.

(3) The result presented in Theorem 3.1 is applicable for solving the split feasibillity problem and the LASSO problem (see, Section 4).

4. Some Applications

In this section, we utilize our main result to the split feasibility problem and the LASSO problem.

4.1. The split feasibility problem

Let C and Q be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a linear bounded operator with its adjoint T^*. The *split feasibility problem* (SFP) is to find

$$\hat{x} \in C \quad \text{such that} \quad T\hat{x} \in Q. \tag{4.1}$$

This problem was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [9] in 1994 for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction (see [7]). The SFP has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (see, *e.g.*, [10, 11, 28, 41, 45, 46]).

For solving the SFP (4.7), Byrne [7] introduce the so-called *CQ-iterative algorithm* for approximating a solution of SFP, which is defined by

$$x_{n+1} = P_C(x_n - \lambda A^*(I-P_Q)Ax_n), \quad \forall n \ge 1, \tag{4.2}$$

where $0 < \lambda < 2\alpha$ with $\alpha = 1/\|A\|^2$. Here, $\|A\|^2$ is the spectral radius of A^*A. It was shown that the sequence $\{x_n\}$ converges weakly to a solution of the SFP.

It is known that $\hat{x}$ solves the SFP (4.1) if and only if $\hat{x}$ is the solution of the following minimization problem [43]:

$$\min_{x\in C} f(x),$$

where f is the proximity function defined by $f(x) := \frac{1}{2}\|(I - P_Q)Tx\|^2$ with its gradient $\nabla f = T^*(I - P_Q)T$. Further, if $\nabla f = T^*(I - P_Q)T$ is $\|T\|^2$-Lipschitz continuous, then ∇f is $1/\|T\|^2$-inverse strongly monotone, where $\|T\|^2$ is the spectral radius of T^*T (see [6]). In fact, set $A = \nabla f$ and $B = \partial i_C$ in Theorem 3.1, where i_C is the indicator function (see [37]). So we obtain the following result.

Theorem 4.1. *Suppose that the SFP* (4.1) *is consistent. For an initial guess* $x_1 \in H_1$, *let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)P_C(x_n - r_nT^*(I-P_Q)Tx_n), \\ x_{n+1} = \alpha_n\gamma h(x_n) + (I-\alpha_n G)P_C(I - r_nT^*(I-P_Q)T)(t_nx_n + (1-t_n)\bar{x}_{n+1}), \end{cases} \tag{4.3}$$

$\forall n \geq 1$,*where* $\{r_n\} \subset (0, \frac{2}{\|T\|^2})$, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2}{\|T\|^2}$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a solution of the SFP.*

4.2. The LASSO Problem

The LASSO problem is abbreviation for the least absolute shrinkage and selection operator, which formulated as the minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Tx - b\|_2^2 \quad \text{subject to} \quad \|x\|_1 \leq \lambda, \tag{4.4}$$

where $T \in \mathbb{R}^{m\times n}$ is a given matrix, $b \in \mathbb{R}^m$ is a given vector and $\lambda \geq 0$ is a tuning parameter. The lasso was introduced by Tibshirani [39] in 1996. It has been received much attention due to the involvement of the l_1 norm which promotes sparsity, phenomenon of many practical problems arising from image and signal processing, statistics model, machine learning, and so on. It is known that an equivalent formulation of (4.4) is the following regularized minimization problem:

$$\min_{x\in\mathbb{R}^n} f(x) + g(x), \tag{4.5}$$

where $f(x) := \frac{1}{2}\|Tx - b\|_2^2$, $g(x) := \lambda\|x\|_1$ and $\lambda \geq 0$. We know that $\nabla f(x) = T^*(Tx - b)$ is $\|T^*T\|$-Lipshitz continuous. This implies that ∇f is $1/\|T^*T\|$-inverse strongly monotone. The proximal of $g(x) = \lambda\|x\|_1$ is given by

$$\text{prox}_g(x) = \text{argmin}_u \lambda\|x\|_1 + \frac{1}{2}\|u - x\|_2^2,$$

which is separable in indices. Then, for $x \in \mathbb{R}^n$,

$$\begin{aligned}\operatorname{prox}_g(x) &= \operatorname{prox}_{\lambda\|\cdot\|_1}(x)\\ &= \Big(\operatorname{prox}_{\lambda|\cdot|_1}(x_1), \operatorname{prox}_{\lambda|\cdot|_1}(x_2), ..., \operatorname{prox}_{\lambda|\cdot|_1}(x_n)\Big)\\ &= (\alpha_1, \alpha_2, ..., \alpha_n),\end{aligned}$$

where $\alpha_k = \operatorname{sgn}(x_k)\max\{|x_k| - \lambda, 0\}$ for $k = 1, 2, ..., n$.

For solving the LASSO problem, Xu [44] (see also [2]) proposed the following proximal-gradient algorithm (PGA):

$$x_{n+1} = \operatorname{prox}_{r_n g}(x_n - r_n T^*(Tx_n - b)). \tag{4.6}$$

He proved that the PGA (4.6) converges weakly to a solution of the LASSO problem (4.4).

In what follows, we present a general viscosity explicit rule for approximating solutions of the LASSO problem in infinite dimensional Hilbert spaces. Set $A = \nabla f$ and $B = \operatorname{prox}_{r_n g}$ in Theorem 3.1, we obtain the following result.

Theorem 4.2. *Suppose that the problem* (4.4) *is consistent. For an initial guess* $x_1 \in H$, *let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)\operatorname{prox}_{r_n g}(x_n - r_n T^*(Tx_n - b)),\\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)\operatorname{prox}_{r_n g}((t_n x_n + (1-t_n)\bar{x}_{n+1})\\ -r_n T^*(T(t_n x_n + (1-t_n)\bar{x}_{n+1}) - b)), \quad \forall n \geq 1, \end{cases} \tag{4.7}$$

where $\{r_n\} \subset (0, \frac{2}{\|T^*T\|^2})$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2}{\|T^*T\|^2}$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a solution of the problem* (4.4).

5. Numerical Example

We next give some numerical experiments of a general viscosity explicit rule (3.1).

Example 5.1. Let $H = \mathbb{R}^3$ with the norm $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$. Consider the mapping $G : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $G\mathbf{x} = 4\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. It is easy to see that G is a linear bounded operator on $\mathbb{R}^3$ with $\bar{\gamma} = 4$. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $h(\mathbf{x}) = 0.1\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. It is easy to see that h is a contraction on $\mathbb{R}^3$ with $\theta = 0.1$. Then, we can choose $\gamma = 10$. For any $\mathbf{x} \in \mathbb{R}^3$, let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A\mathbf{x} = 3\mathbf{x} - (1, -2, 5)^t$ and $B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $B\mathbf{x} = 2\mathbf{x}$. We see that A is a 1/3-inverse strongly monotone and B is a maximal monotone operator. Moreover, we have for $r > 0$

$$J_r^B(\mathbf{x} - rA\mathbf{x}) = (I + rB)^{-1}(\mathbf{x} - rA\mathbf{x})$$

$$= \frac{1-3r}{1+2r}\mathbf{x} + \frac{r}{1+2r}(1,-2,5)^t,$$

for all $\mathbf{x} \in \mathbb{R}^3$. Since $\alpha = 1/3$, we can choose $r_n = 0.5$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{1000n+1}$, $\beta_n = \frac{n}{2n+3}$ and $t_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Starting $\mathbf{x}_1 = (10000, -40000, 50000)^t$ and use $\|x_{n+1} - x_n\|_2 < 10^{-6}$, for stopping criterion. Then, we obtain the following numerical results.

Table 1. Numerical results of Example 5.1 for iteration process (3.1).

Time taken	No. of iterations	$\mathbf{x}_n = (x_1^n, x_2^n, x_3^n)^t$	$\|x_{n+1} - x_n\|_2$
0.110385	2	$(-2479.77097902, 9919.58191808, -12398.85489510)^t$	8.08E+04
	3	$(203.36993275, -813.08835944, 1016.84966373)^t$	1.73E+04
	4	$(-12.88336871, 51.93363061, -64.41684356)^t$	1.41E+03
	5	$(1.02739736, -3.70989923, 5.13698682)^t$	9.01E+01
	6	$(0.14598968, -0.18417851, 0.72994841)^t$	5.71E+00
	7	$(0.20354553, -0.41436724, 1.01772764)^t$	3.72E-01
	8	$(0.19966701, -0.39882655, 0.99833507)^t$	2.51E-02
	10	$(0.19993710, -0.39987152, 0.99968548)^t$	8.69E-05
	20	$(0.19997096, -0.39994192, 0.99985479)^t$	8.88E-06
	$\vdots$	$\vdots$	$\vdots$
	50	$(0.19998878, -0.39997755, 0.99994388)^t$	1.28E-06
	$\vdots$	$\vdots$	$\vdots$
	55	$(0.19998982, -0.39997964, 0.99994909)^t$	1.05E-06
	56	$(0.19999000, -0.39998001, 0.99995002)^t$	1.02E-06
	57	$(0.19999018, -0.39998037, 0.99995091)^t$	9.8E-07

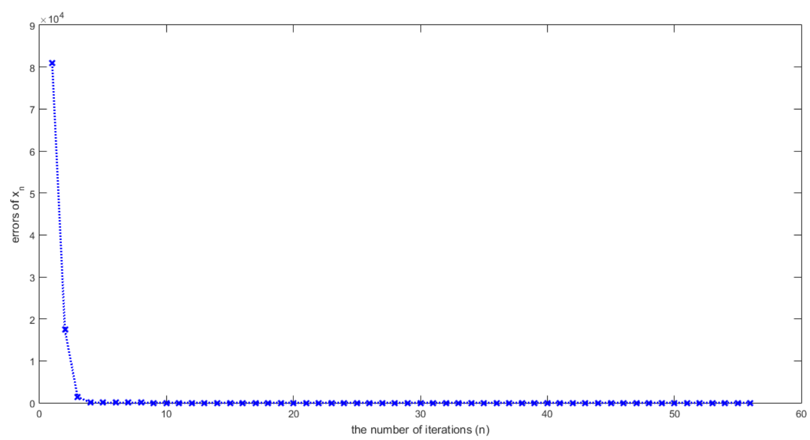

Figure 1. The error plotting of $\|x_{n+1} - x_n\|_2$ in Table 1.

6. Conclusions

In this work, we have introduced new iterative methods for solving the inclusion problem for the sum of two monotone operators in Hilbert spaces. Strong convergence was discussed under suitable conditions. Some applications to the split feasibility problem and the LASSO problem are also given. Preliminary numerical experiments are provided to support our proposed methods.

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Convergence Theorems for Generalized Viscosity Explicit Methods for Nonexpansive Mappings in Banach Spaces and Some Applications

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Abstract: In this paper, we introduce a generalized viscosity explicit method (GVEM) for nonexpansive mappings in the setting of Banach spaces and, under some new techniques and mild assumptions on the control conditions, prove some strong convergence theorems for the proposed method, which converge to a fixed point of the given mapping and a solution of the variational inequality. As applications, we apply our main results to show the existence of fixed points of strict pseudo-contractions and periodic solutions of nonlinear evolution equations and Fredholm integral equations. Finally, we give some numerical examples to illustrate the efficiency and implementation of our method.

Keywords: nonexpansive mapping; Banach space; strong convergence; viscosity iterative method; nonlinear evolution equation; Fredholm integral equation

MSC: 47H09; 47H10; 47J25; 47J05

1. Introduction

In the real world, many engineering and science problems can be reformulated as ordinary differential equations. Several numerical methods have been developed for solving ordinary differential equations (ODEs) by numerous authors. The major method in order to solve ODEs is the implicit midpoint rule, also well known as the second-order Runge–Kutta method or improved the Euler method. It is a forceful numerical method for numerically solving ODEs (in particular, stiff equations) (see [1–6]) and differential algebraic equations (see [4]). Consider the following initial value problem for the following time-dependent ordinary differential equation:

$$\begin{cases} x'(t) = f(x(t)), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. The implicit midpoint method is an implicit method, which is given by the following finite difference scheme [7]:

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right), \quad \forall n \geq 0, \end{cases} \tag{2}$$

where $h > 0$ is a step size. It is known that, if $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous and sufficiently smooth, then $\{y_n\}$ converges to the solution of Equation (1) as $h \to 0$ uniformly over $t \in [t_0, t^*]$ for any fixed $t^* > 0$. If we write the function f in the form $f = I - T$, where T is a nonlinear mapping, then equilibrium problems involving differential Equation (1) is the fixed point problem $x = Tx$. Following the procedure (2), Xu et al. [8] introduced two equivalent algorithms to approximate the fixed point of a nonexpansive mapping in a Hilbert space H as follows:

$$x_{n+1} = x_n - t_n\left[\frac{x_n + x_{n+1}}{2} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right], \quad \forall n \geq 0, \tag{3}$$

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0, \tag{4}$$

for any $x_0 \in H$, where $\{t_n\}_{n=1}^{\infty} \subset (0,1)$ is a sequence.

On the other hand, since, in 2000, Moudafi [9] introduced viscosity approximation methods for fixed point problems in Hilbert spaces, some authors have obtained some convergence theorems of viscosity approximation methods to show the existence of fixed points of some kinds of nonlinear mappings and solutions of nonlinear problems (see, for example, [10–20]). Especially, in 2015, Xu et al. [8] combined the Moudafi viscosity method [9] with the implicit midpoint method for a nonexpansive mapping in Hilbert spaces as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 1, \tag{5}$$

where f is a contraction and $\{\alpha_n\} \subset (0,1)$ is a sequence. They also proved that $\{x_n\}$ generated by (5) converges strongly to a point $x^* \in F(T)$, which is the unique solution of the following variational inequality problem:

$$\langle (f - I)x^*, z - x^* \rangle \leq 0, \quad \forall z \in F(T). \tag{6}$$

Recently, Ke and Ma [21] improved the VIMRby replacing the midpoint by any point of the interval $[x_n, x_{n+1}]$. They constructed the so-called method generalized viscosity implicit rules for a nonexpansive mapping as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(s_n x_n + (1 - s_n)x_{n+1}), \quad \forall n \geq 1. \tag{7}$$

They showed that $\{x_n\}$ defined by (7) converges strongly to $x^* \in F(T)$, which solves the variational inequality problem (6).

In fact, the computation by the implicit midpoint methods is not an easy work in practice. Therefore, we consider the explicit midpoint method proposed by the framework of the finite difference [22,23]:

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + hf(y_n), \\ y_{n+1} = y_n + hf\left(\frac{y_n + \bar{y}_{n+1}}{2}\right), \quad \forall n \geq 0. \end{cases} \tag{8}$$

It is easy to see that the explicit midpoint method calculates the state of a system at the next time from the state of the system at the current time (see [22,24]).

In 2017, Marino et al. [25] applied the sequence (7) and the explicit midpoint method (8) to established the following so-called general viscosity explicit rule for quasi-nonexpansive mappings T in Hilbert spaces:

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \ \forall n \geq 1, \end{cases} \tag{9}$$

where f is a contraction and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ are the sequences in $(0,1)$. They proved, under suitable conditions on the sequence parameters, that the generalized viscosity explicit rule (9) strongly converges to the set of $F(T)$, which is also the solution of the variational inequality problem (6).

The main objective of this paper is to introduce a generalized viscosity explicit rule (9) for nonexpansive mappings in Banach spaces. Some strong convergence theorems of the proposed algorithm are proven under new techniques and some mild assumption on the control conditions. As applications, we apply our main result to the fixed point problem of strict pseudo-contractions, a periodic solution of a nonlinear evolution equation, and a nonlinear Fredholm integral equation. Finally, some numerical examples that show the efficiency and implementation of our algorithm are presented. The results presented in the paper extend and improve the main results of Ke and Ma [21], Marino et al. [25], and previously known results in the earlier and recent literature to Banach spaces.

2. Preliminaries

Let E and E^* be a real Banach space and the dual space of E, respectively. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by:

$$J(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^2, \|\bar{x}\| = \|x\|\}, \ \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. If $E := H$ is a real Hilbert space, then $J = I$ is the identity mapping and if E is smooth, then J is single-valued, which is denoted by j.

The modulus of convexity of E is the function $\delta : (0,2] \to [0,1]$ defined by:

$$\delta(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by:

$$\rho_E(\tau) = \sup\left\{\tfrac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in E, \|x\| = \|y\| = 1\right\}.$$

The space E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Suppose that $1 < q \leq 2$, then E is said to be q-uniformly smooth if there exists $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$ for all $\tau > 0$. It is well known that, if q is uniformly smooth, then E is uniformly smooth [26]. A typical example of a uniformly convex and uniformly smooth Banach spaces is l_p, where $p > 1$. More precisely, l_p is $\min\{p,2\}$-uniformly smooth for any $p > 1$.

Recall that a mapping $T : C \to C$ is said to be: *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

A mapping $f : C \to C$ is said to be a strict contraction if there exists a constant $\alpha \in (0,1)$ satisfying:

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \ \forall x, y \in C.$$

We use Π_C to denote the collection of all contractions from C into itself. Note that each $f \in \Pi_C$ has a unique fixed point in C.

Lemma 1. ([27]) *Let E be a real Banach space. Then, for each $x, y \in E$, we have:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,$$

where $j(x+y) \in J(x+y)$.

Lemma 2. ([28]) *Given $r > 0$ and $p > 1$ are fixed real numbers in Banach space E, then the following statements are equivalent:*

(1) E is uniformly convex.

(2) There is a strictly-increasing, continuous, and convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and:

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - t(1-t)\phi(\|x-y\|)$$

for all $x, y \in B_r[0] := \{x \in E : \|x\| \leq r\}$.

The following lemma can be found in [29–31].

Lemma 3. *Let C be a closed, convex subset and nonempty uniformly smooth in E. Let T be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then the sequence $\{z_t\}$ defined by $z_t = tf(z_t) + (1-t)Tz_t$ for all $t \in (0,1)$ converges strongly to a point $x^* \in F(T)$, which solves the variational inequality problem:*

$$\langle f(x^*) - x^*, j(z - x^*)\rangle \leq 0, \ \ \forall f \in \Pi_C, \ z \in F(T).$$

Lemma 4. ([32]) *Let C be a closed, convex subset and nonempty in E, which has the uniformly Gâteaux differentiable norm and T is a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{z_t\}$ strongly converges to $x^* \in F(T)$, where $\{z_t\}$ is defined by $z_t = tf(z_t) + (1-t)Tz_t$ for all $t \in (0,1)$. Suppose that $\{x_n\}$ is bounded in C such that:*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Then, we have:

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, j(x_n - x^*)\rangle \leq 0.$$

Lemma 5. ([33,34]) *Assume that $\{a_n\}$ is a positive real sequence such that:*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \ \ \forall n \geq 0,$$

where $\{\gamma_n\} \in (0,1)$ and $\{\delta_n\} \in \mathbb{R}$ such that:

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

In order to prove our main result with new techniques, we needed the following Maingé lemma [35]:

Lemma 6. *Given $\{a_n\}$ are real sequences so that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ that $m_k \to \infty$, and the following properties are satisfied for all numbers $k \in \mathbb{N}$:*

$$a_k \le a_{m_k+1}, \quad a_{m_k} \le a_{m_k+1}.$$

where $m_k := \max\{j \le k : a_j \le a_{j+1}\}$.

3. The Main Results

First, we prove a lemma for our main results.

Lemma 7. *Let C be a nonempty closed and convex subset of a real Banach space E. Let T be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$ with coefficient $\alpha \in (0,1)$. For any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad \forall n \ge 1, \end{cases} \tag{10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ are the sequences in $(0,1)$. Then, $\{x_n\}$ is bounded.

Proof. For each $n \ge 1$, we put $z_n := s_n x_n + (1-s_n)\bar{x}_{n+1}$. Let $z \in F(T)$; we have:

$$\begin{aligned} \|z_n - z\| &= \|s_n(x_n - z) + (1-s_n)(\bar{x}_{n+1} - z)\| \\ &\le s_n\|x_n - z\| + (1-s_n)\|\bar{x}_{n+1} - z\| \\ &\le s_n\|x_n - z\| + (1-s_n)\big(\beta_n\|x_n - z\| + (1-\beta_n)\|Tx_n - z\|\big) \\ &\le s_n\|x_n - z\| + (1-s_n)\beta_n\|x_n - z\| + (1-s_n)(1-\beta_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned}$$

It follows that:

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - f(z)) + \alpha_n(f(z) - z) + (1-\alpha_n)(Tz_n - z)\| \\ &\le \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|Tz_n - z\| \\ &\le \alpha_n\alpha\|x_n - z\| + (1-\alpha_n)\|z_n - z\| + \alpha_n\|f(z) - z\| \\ &= (1-(1-\alpha)\alpha_n)\|x_n - z\| + (1-\alpha)\alpha_n\frac{\|f(z) - z\|}{1-\alpha} \\ &\le \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1-\alpha}\right\}. \end{aligned}$$

By induction, we have:

$$\|x_n - z\| \le \max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1-\alpha}\right\}, \quad \forall n \ge 1.$$

Hence, $\{x_n\}$ is bounded. This completes the proof. □

Theorem 1. *Let C be closed, convex subset and nonempty uniformly convex and uniformly smooth in E. Let T be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$ with coefficient $\alpha \in (0,1)$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} \beta_n(1-\beta_n)(1-s_n)$.

Then, $\{x_n\}$ generated by (10) converges strongly to a point $x^* \in F(T)$, which solves the variational inequality problem:

$$\langle f(x^*) - x^*, j(z - x^*)\rangle \le 0, \quad \forall z \in F(T). \tag{11}$$

Proof. Let $x^* \in F(T)$. Since $z_n := s_n x_n + (1-s_n)\bar{x}_{n+1}$, by the convexity of $\|\cdot\|^2$ and Lemma 2, we have:

$$\begin{aligned}
\|Tz_n - x^*\|^2 &\leq \|z_n - x^*\|^2 \\
&= \|s_n(x_n - x^*) + (1-s_n)(\bar{x}_{n+1} - x^*)\|^2 \\
&\leq s_n\|x_n - x^*\|^2 + (1-s_n)\|\bar{x}_{n+1} - x^*\|^2 \\
&= s_n\|x_n - x^*\|^2 + (1-s_n)\|\beta_n(x_n - x^*) + (1-\beta_n)(Tx_n - x^*)\|^2 \\
&\leq s_n\|x_n - x^*\|^2 + (1-s_n)\Big[\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|Tx_n - x^*\|^2 \\
&\quad -\beta_n(1-\beta_n)\phi(\|x_n - Tx_n\|)\Big] \\
&\leq \|x_n - x^*\|^2 - \beta_n(1-\beta_n)(1-s_n)\phi(\|x_n - Tx_n\|). \qquad (12)
\end{aligned}$$

It follows from Lemma 1 and (12) that:

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
= \;& \|\alpha_n(f(x_n) - f(x^*)) + \alpha_n(f(x^*) - x^*) + (1-\alpha_n)(Tz_n - x^*)\|^2 \\
\leq \;& \|\alpha_n(f(x_n) - f(x^*)) + (1-\alpha_n)(Tz_n - x^*)\|^2 + 2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
\leq \;& \alpha_n\|f(x_n) - f(x^*)\|^2 + (1-\alpha_n)\|Tz_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
\leq \;& \alpha_n\|f(x_n) - f(x^*)\|^2 + (1-\alpha_n)\Big[\|x_n - x^*\|^2 - \beta_n(1-\beta_n)(1-s_n)\phi(\|x_n - Tx_n\|)\Big] \\
& +2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
\leq \;& (1-(1-\alpha^2)\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)(1-s_n)\phi(\|x_n - Tx_n\|) \\
& +2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle. \qquad (13)
\end{aligned}$$

Now, we show that $\{x_n\}$ converges strongly to x^* as $n \to \infty$ by considering two possible cases:

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=n_0}^{\infty}$ is non-increasing. This implies that $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is convergent. From (13), it follows that:

$$(1-\alpha_n)\beta_n(1-\beta_n)(1-s_n)\phi(\|x_n - Tx_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M,$$

where $M = \sup_{n\geq 1}\{2\|f(x^*) - x^*\|\|x_{n+1} - x^*\|, (1-\alpha^2)\|x_n - x^*\|^2\} < \infty$. From the conditions $(C1)$and $(C2)$, we have:

$$\phi(\|x_n - Tx_n\|) \to 0 \text{ as } n \to \infty,$$

which implies by the property of ϕ that:

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty. \qquad (14)$$

Let:

$$z_t = f(z_t) + (1-t)Tz_t, \;\; \forall t \in (0,1).$$

By Lemma 3, $\{z_t\}$ converges strongly to x^*, which solves the variational inequality problem:

$$\langle f(x^*) - x^*, j(z - x^*)\rangle \leq 0, \;\; \forall z \in F(T).$$

By (14) and Lemma 4, it follows that:

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j(x_n - x^*)\rangle \leq 0. \qquad (15)$$

Since:

$$\begin{aligned}
\|Tz_n - x_n\| &\leq \|Tz_n - Tx_n\| + \|Tx_n - x_n\| \\
&\leq \|z_n - x_n\| + \|Tx_n - x_n\| \\
&= (1 - s_n)(1 - \beta_n)\|Tx_n - x_n\| + \|Tx_n - x_n\| \\
&\leq 2\|x_n - Tx_n\|,
\end{aligned}$$

it follows from (14) that:

$$\lim_{n\to\infty} \|Tz_n - x_n\| = 0. \tag{16}$$

Moreover, we note that:

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|\alpha_n(f(x_n) - x_n) + (1 - \alpha_n)(Tz_n - x_n)\| \\
&\leq \alpha_n\|f(x_n) - x_n\| + (1 - \alpha_n)\|Tz_n - x_n\|.
\end{aligned}$$

It follows from (16) that:

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{17}$$

Furthermore, we have:

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \leq 0. \tag{18}$$

From (13), we note that:

$$\|x_{n+1} - x^*\|^2 \leq (1 - (1 - \alpha^2)\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle. \tag{19}$$

Applying Lemma 5, (18) and (19), we can conclude that $x_n \to x^*$ as $n \to \infty$.

Case 2. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that:

$$\|x_{n_i} - x^*\| \leq \|x_{n_{i+1}} - x^*\|, \ \forall i \geq 1.$$

By Lemma 6, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and:

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\|,$$

$$\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \ \forall k \geq 1. \tag{20}$$

Again, from (13), we have:

$$\begin{aligned}
&(1 - \alpha_{m_k})\beta_{m_k}(1 - \beta_{m_k})(1 - s_{m_k})\phi(\|x_{m_k} - Tx_{m_k}\|) \\
\leq\ & \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \alpha_{m_k}M \\
\leq\ & \alpha_{m_k}M,
\end{aligned}$$

which implies by the property of ϕ that:

$$\|x_{m_k} - Tx_{m_k}\| \to 0 \ \text{ as } \ k \to \infty. \tag{21}$$

Following the proof lines in Case 1, we can show that:

$$\limsup_{k\to\infty}\langle f(x^*)-x^*, j(x_{m_k}-x^*)\rangle \le 0. \tag{22}$$

Consequently, we have:

$$\begin{aligned}\|Tz_{m_k}-x_{m_k}\| &\le \|Tz_{m_k}-Tx_{m_k}\|+\|Tx_{m_k}-x_{m_k}\|\\ &\le \|z_{m_k}-x_{m_k}\|+\|Tx_{m_k}-x_{m_k}\|\\ &= (1-s_{m_k})(1-\beta_{m_k})\|Tx_{m_k}-x_{m_k}\|+\|Tx_{m_k}-x_{m_k}\|\\ &\le 2\|x_{m_k}-Tx_{m_k}\|\to 0 \text{ as } k\to\infty.\end{aligned}$$

It follows that:

$$\begin{aligned}\|x_{m_k+1}-x_{m_k}\| &\le \|\alpha_{m_k}(f(x_{m_k})-x_{m_k})+(1-\alpha_{m_k})(Tz_{m_k}-x_{m_k})\|\\ &\le \alpha_{m_k}\|f(x_{m_k})-x_{m_k}\|+(1-\alpha_{m_k})\|Tz_{m_k}-x_{m_k}\|\to 0 \text{ as } k\to\infty.\end{aligned}$$

Therefore, we have:

$$\limsup_{k\to\infty}\langle f(x^*)-x^*, j(x_{m_k+1}-x^*)\rangle \le 0. \tag{23}$$

This together with (19) implies that:

$$\begin{aligned}&\|x_{m_k+1}-x^*\|^2\\ \le\ &(1-(1-\alpha^2)\alpha_{m_k})\|x_{m_k}-x^*\|^2+2\alpha_{m_k}\langle f(x^*)-x^*, j(x_{m_k+1}-x^*)\rangle.\end{aligned} \tag{24}$$

We see that:

$$\begin{aligned}&(1-\alpha^2)\alpha_{m_k}\|x_{m_k}-x^*\|^2\\ \le\ &\|x_{m_k}-x^*\|^2-\|x_{m_k+1}-x^*\|^2+2\alpha_{m_k}\langle f(x^*)-x^*, j(x_{m_k+1}-x^*)\rangle\\ \le\ &2\alpha_{m_k}\langle f(x^*)-x^*, j(x_{m_k+1}-x^*)\rangle.\end{aligned} \tag{25}$$

Since $\alpha_{m_k}>0$, we have $\lim_{k\to\infty}\|x_{m_k}-x^*\|=0$. Therefore, we have:

$$\begin{aligned}\|x_k-x^*\| &\le \|x_{m_k+1}-x^*\|\\ &= \|x_{m_k}-x^*\|+\|x_{m_k+1}-x^*\|-\|x_{m_k}-x^*\|\\ &\le \|x_{m_k}-x^*\|+\|x_{m_k+1}-x_{m_k}\|\to 0 \text{ as } k\to\infty,\end{aligned}$$

which implies that $x_k\to x^*$ as $k\to\infty$. This completes the proof. □

Corollary 1. *Let C be a nonempty closed and convex subset of a Hilbert space H. Let $T: C\to C$ be a nonexpansive self-mapping such that $F(T)\neq\emptyset$ and $f\in\Pi_C$ with coefficient $\alpha\in(0,1)$. For any $x_1\in C$, let $\{x_n\}$ be a sequence generated by:*

$$\begin{cases}\bar{x}_{n+1}=\beta_n x_n+(1-\beta_n)Tx_n,\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T(s_n x_n+(1-s_n)\bar{x}_{n+1}),\ \ \forall n\ge 1,\end{cases} \tag{26}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ belonging in $(0,1)$ satisfy (C1) and (C2) in Theorem 1. Then, the sequence $\{x_n\}$ converges strongly to $x^*\in F(T)$, which solves the variational inequality problem:

$$\langle f(x^*)-x^*, z-x^*\rangle \le 0,\ \ \forall z\in F(T). \tag{27}$$

Now, we give some remarks on our results as follows:

(1) We get the idea that the results of Ke and Ma [21] and Marino et al. [25] in Hilbert spaces extend to Banach spaces.
(2) The proof methods of our result are very different from the ones of Ke and Ma [21]. Further, we remove the following conditions:
$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad 0 < \epsilon \le s_n \le s_{n+1} < 1$$
in Theorem 3.1 of [21].
(3) We give new control conditions and techniques to prove our results.
(4) The proof methods of our results are simpler than those of the results given by some authors (see, for example, [8,21,36,37]).
(5) Our results are applicable for the family of nonexpansive mappings, for example W_n-mapping, a countable family of nonexpansive mappings, and nonexpansive semigroups.

Open Problem

Is it possible to obtain the convergence results of the sequence (10) in the setting of more general spaces, such as reflexive, strictly-convex, and smooth Banach spaces, which admit the duality mapping j_φ without the weak continuity assumption, where φ is a gauge function?

4. Convergence Theorems for a Strict Pseudo-Contraction Mapping

Let C be a closed, convex subset, and nonempty in E. A self-mapping T is called λ-strictly pseudo-contractive if there exists $\lambda > 0$ such that:
$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2, \ \forall x, y \in C \tag{28}$$
for some $j(x-y) \in J(x-y)$. It is easy to check that (28) is equivalent to the following inequality:
$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \lambda\|(I-T)x - (I-T)y\|^2, \ \forall x, y \in C.$$

Lemma 8. ([38]) *Let C be closed, convex subset and nonempty two-uniformly smooth in E. Let T be a λ-strict pseudo-contractive mapping. For all $x \in C$, we define $T_\theta x := (1-\theta)x + \theta Tx$. Then, for any $\theta \in \big(0, \frac{\lambda}{K^2}\big]$, where $K > 0$ is the two-uniformly smooth constant, T_θ is a nonexpansive mapping such that $F(T_\theta) = F(T)$.*

Using Theorem 1 and Lemma 8, we get the result as follows:

Theorem 2. *Let C be a closed, convex subset and nonempty uniformly convex and two-uniformly smooth in E. Let T be a λ-strict pseudo-contractions with $F(T) \neq \emptyset$ and $f \in \Pi_C$ with coefficient $\alpha \in (0,1)$. Define a mapping $T_\theta x := (1-\theta)x + \theta Tx$ for all $x \in C$, where $\theta \in \big(0, \frac{\lambda}{K^2}\big]$. For any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by:*
$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)T_\theta x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\theta(s_n x_n + (1-s_n)\bar{x}_{n+1}), \ \forall n \ge 1, \end{cases} \tag{29}$$
where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ belonging in $(0,1)$ satisfy (C1) and (C2) of Theorem 1. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$, which solves the variational inequality problem:
$$\langle f(x^*) - x^*, j(z - x^*)\rangle \le 0, \ \forall z \in F(T). \tag{30}$$

5. Some Applications

In this section, we give some applications of Theorem 1 in the framework of Hilbert spaces.

5.1. Periodic Solution of a Nonlinear Evolution Equation

Let H be a (possibly complex) Hilbert space. Consider the following time-dependent nonlinear evolution equation in H:

$$\frac{du}{dt} + A(t)u = g(t,u), \ \ \forall t > 0, \tag{31}$$

where $A(t)$ is a family of closed, linear operators and $g : \mathbb{R} \times H \to H$. Recall that u is a mild solution of Equation (31) with initial value $u(0) = v$ if, for any $t > 0$,

$$u(t) = U(t,0)v + \int_0^t U(t,s)g(s,u(s))ds,$$

where $\{U(t,s)\}_{t \geq s \geq 0}$ is the evolution system in the case of a homogeneous linear system:

$$\frac{du}{dt} + Au(t) = 0.$$

The following useful result on the existence of periodic solutions of the problem (31) can be found in [39].

Theorem 3. *Assume that $A(t)$ and $g(t,u)$ are periodic in t of period $\xi > 0$ and satisfy:*

(1) $Re\langle g(t,u) - g(t,v), u - v\rangle \leq 0 \ \ \forall t > 0$ and $u, v \in H$;
(2) $u \in D(A(t))$ and $Re\langle D(A(t))u, u\rangle \geq 0 \ \ \forall t > 0$;
(3) There exists a mild solution u of the Equation (31) on $\mathbb{R}^+$ for each initial value $v \in H$;
(4) There exists some $R > 0$ such that $Re\langle g(t,u), u\rangle < 0$ for all $u \in H$ with $\|u\| = R$ and $t \in [0, \xi]$.

Then, there exists $v \in H$ with $\|v\| \leq R$ such that solution of Equation (31) with initial $u(0) = v$ is of period ξ.

If we define a mapping $T : H \to H$ by:

$$Tv = u(\xi), \ \ \forall v \in H, \tag{32}$$

where u is the solution of the Equation (31) satisfying initial $u(0) = v$, then T is a nonexpansive of closed ball $B := \{v \in H : v \leq R\}$ into itself with $F(T) \neq \emptyset$. Moreover, each fixed point of T corresponding to solution u of Equation (31) is the periodic solution of Equation (31) with the initial $u(0) = v$ (see [39]). For the other case, finding a periodic solution of (31) is equivalent to finding a fixed point set (see [40], Section 10).

Theorem 4. *Let H be a Hilbert space. Let $A(t)$ and $g(t,u)$ be periodic in t of period $\xi > 0$, which satisfy the conditions (i)–(iv) of Theorem 3. Let $T : H \to H$ be a mapping defined by (32) and $f \in \Pi_H$ with coefficient $\alpha \in (0,1)$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ are the sequences in $(0,1)$ that satisfy the conditions $(C1)$ and $(C2)$ of Theorem 1. For any $x_1 \in H$, let $\{x_n\}$ be a sequence generated by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \ \ \forall n \geq 1. \end{cases} \tag{33}$$

Then, the sequence $\{x_n\}$ converges strongly to fixed element v of T, and then, the corresponding solution of the Equation (31) with initial $u(0) = \xi$ is a periodic solution of Equation (31).

5.2. Nonlinear Fredholm Integral Equation

Consider the following nonlinear Fredholm integral equation:

$$x(t) = g(t) + \int_0^1 F(t, s, x(s))ds, \ \ \forall t \in [0, 1], \tag{34}$$

where g is a continuous function on $[0, 1]$.

$F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In this case, if we assume that F satisfies the Lipschitz continuity condition, i.e.,

$$|F(t, s, x) - F(t, s, y)| \leq |x - y|, \ \ \forall t, s \in [0, 1], \ x, y \in \mathbb{R}, \tag{35}$$

then Equation (34) has at least one solution in $L_2[0, 1]$ (see [41], Theorem 3.3). Define a mapping $T : L_2[0, 1] \to L_2[0, 1]$ by:

$$(Tx)(t) = g(t) + \int_0^1 F(t, s, x(s))ds, \ \ \forall t \in [0, 1]. \tag{36}$$

Then, for any $x, y \in L_2[0, 1]$, we have:

$$\begin{aligned}
\|Tx - Ty\|^2 &= \int_0^1 |(Tx)(t) - (Ty)(t)|^2 dt \\
&= \int_0^1 \left| \int_0^1 \big(F(t, s, x(s)) - F(t, s, y(s))\big)ds \right|^2 dt \\
&\leq \int_0^1 \left| \int_0^1 |x(s) - y(s)|ds \right|^2 dt \\
&\leq \int_0^1 |x(s) - y(s)|^2 ds \\
&\leq \|x - y\|^2,
\end{aligned}$$

which implies that T is a nonexpansive mapping on $L_2[0, 1]$. Thus, we see that finding a solution of Equation (34) is equivalent to finding a fixed point of T in $L_2[0, 1]$.

Theorem 5. *Let $F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a mapping satisfying the Lipschitz continuity condition and g be a continuous function on $[0, 1]$. Let $T : L_2[0, 1] \to L_2[0, 1]$ be a mapping defined by (36) and $f \in \Pi_{L_2[0,1]}$ with coefficient $\alpha \in (0, 1)$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ are the sequences in $(0, 1)$ that satisfy the conditions (C1) and (C2) of Theorem 1. For any $x_1(t) \in L_2[0, 1]$, let $\{x_n\}$ be a sequence generated by:*

$$\begin{cases} \bar{x}_{n+1}(t) = \beta_n x_n(t) + (1 - \beta_n)Tx_n(t), \\ x_{n+1}(t) = \alpha_n f(x_n(t)) + (1 - \alpha_n)T(s_n x_n(t) + (1 - s_n)\bar{x}_{n+1}(t)), \ \ \forall n \geq 1, \end{cases} \tag{37}$$

where $t \in [0, 1]$. Then, the sequence $\{x_n(t)\}$ converges strongly in $L_2[0, 1]$ to the solution of the integral Equation (34).

Remark 1. *Our result can be applied to show the existence of solutions of some nonlinear problems, that is (general system) variational inequality problems, constrained convex minimization problems, hierarchical minimization problems, and split feasibility problems (see [21,36,42]).*

6. Numerical Examples

In this section, we present a numerical example of the sequence (10) in the ℓ_3 space, which is uniformly convex with uniformly smooth setting of a Banach space, but not a Hilbert space. Further, using MATLAB, numerical results of the sequence (7) are obtained. In particular, we perform the comparison speed of the convergence to show that the sequence (10) is faster than the sequence (7).

Example 1. Let $C = E = \ell_3$ and $\mathbf{x} = (x_1, x_2, x_3, \cdots) \in \ell_3$, where $x_i \in \mathbb{R}$ for $i = 1, 2, 3, \cdots$ and $\|\cdot\|_{\ell_3} : \ell_3 \to \mathbb{R}^+$ be the norm defined by:

$$\|\mathbf{x}\|_{\ell_3} = \Big(\sum_{i=1}^{\infty} |x_i|^3\Big)^{1/3}.$$

Let $T : \ell_3 \to \ell_3$ be a nonexpansive mapping and $f : \ell_3 \to \ell_3$ be a contraction defined by:

$$T\mathbf{x} = \Big(\frac{x_1 + x_2}{10}, \frac{x_2 + x_3}{2}, \frac{x_3 - x_1}{5}, 0, 0, 0, \cdots\Big), \quad f(\mathbf{x}) = \frac{1}{4}(x_1, x_2, x_3, \cdots),$$

respectively. Let $\alpha_n = \frac{1}{10n+1}$, $\beta_n = \frac{1}{20n+1} + 0.9$ and $s_n = \frac{1}{30n+1} + 0.8$. It is easy to see that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ satisfy the conditions $(C1)$ and $(C2)$ of Theorem 1. Moreover, we have $F(T) = \{(0, 0, 0, 0, 0, 0, \cdots)\}$. Therefore, our sequence (10) has the following form:

$$\begin{cases} \bar{\mathbf{x}}_{n+1} = \big(\frac{1}{20n+1} + 0.9\big)\mathbf{x}_n + \big(0.1 - \frac{1}{20n+1}\big)T\mathbf{x}_n, \\ \mathbf{x}_{n+1} = \frac{1}{10n+1} f(\mathbf{x}_n) + \frac{10n}{10n+1} T\Big(\big(\frac{1}{30n+1} + 0.8\big)\mathbf{x}_n + \big(0.2 - \frac{1}{30n+1}\big)\bar{\mathbf{x}}_{n+1}\Big), \quad \forall n \geq 1 \end{cases} \tag{38}$$

and the sequence (7) has the following form:

$$\mathbf{x}_{n+1} = \frac{1}{10n+1} f(\mathbf{x}_n) + \frac{10n}{10n+1} T\Big(\big(\frac{1}{30n+1} + 0.8\big)\mathbf{x}_n + \big(0.2 - \frac{1}{30n+1}\big)\mathbf{x}_{n+1}\Big), \quad \forall n \geq 1. \tag{39}$$

Let $\mathbf{x_1} = (1, -2, 3, 0, 0, 0, \cdots)$ be an initial point. Then, we obtain the following numerical results:

Remark 2. *We see that from Tables 1 and 2 and Figure 1, the sequence (38) converges to a fixed point of T faster than the sequence (39).*

Table 1. Numerical results of the sequence (38).

Number of Iterates	$\mathbf{x}_n = (x_1^n, x_2^n, x_3^n, x_4^n, x_5^n, x_6^n, \ldots)$	$\|\mathbf{x}_n - F(T)\|_3$
1	(1.0000000, −2.0000000, 3.0000000, 0, 0, 0, ...)	2.7144176
2	(0.2327849, −0.4480184, 0.7292703, 0, 0, 0, ...)	0.6771823
3	(0.0526107, −0.0931765, 0.1750471, 0, 0, 0, ...)	0.1675113
4	(0.0118323, −0.0180499, 0.0418631, 0, 0, 0, ...)	0.0410437
5	(0.0026730, −0.0031216, 0.0099981, 0, 0, 0, ...)	0.0099602
6	(0.0006105, −0.0004172, 0.0023865, 0, 0, 0, ...)	0.0023955
7	(0.0001418, -9.9203×10^{-6}, 0.0005694, 0, 0, 0, ...)	5.7233×10^{-4}
8	(3.3678×10^{-5}, 2.2004×10^{-5}, 0.0001358, 0, 0, 0, ...)	1.3688×10^{-4}
9	(8.2042×10^{-6}, 1.1855×10^{-5}, 3.2365×10^{-5}, 0, 0, 0, ...)	3.3056×10^{-5}
10	(2.0542×10^{-6}, 4.6141×10^{-6}, 7.7051×10^{-6}, 0, 0, 0, ...)	8.2639×10^{-6}
$\vdots$	$\vdots$	$\vdots$
15	(2.8321×10^{-9}, 1.3627×10^{-8}, 5.6702×10^{-9}, 0, 0, 0, ...)	1.3986×10^{-8}

Table 2. Numerical results of the sequence (39).

Number of Iterates	$x_n = (x_1^n, x_2^n, x_3^n, , x_4^n, x_5^n, x_6^n, \ldots)$	$\|x_n - F(T)\|_3$
1	(1.0000000, −2.0000000, 3.0000000, 0, 0, 0, ...)	2.7144176
2	(−0.0470355, 0.3985926, 0.3853261, 0, 0, 0, ...)	0.4938371
3	(0.0337876, 0.3481921, 0.0732708, 0, 0, 0, ...)	0.3493757
4	(0.0344241, 0.1863456, 0.0570267, 0, 0, 0, ...)	0.1867381
5	(0.0196135, 0.0844269, −0.0054563, 0, 0, 0, ...)	0.0847707
6	(0.0091690, 0.0345417, −0.0045213, 0, 0, 0, ...)	0.0347302
7	(0.0038295, 0.0130688, −0.0024315, 0, 0, 0, ...)	0.0131499
8	(0.0014729, 0.0046081, −0.0011026, 0, 0, 0, ...)	0.0046371
9	$(5.2742 \times 10^{-4}, 0.0015100, -4.5084 \times 10^{-4}, 0, 0, 0, \ldots)$	0.0015180
10	$(1.7573 \times 10^{-4}, 4.5282 \times 10^{-4}, -1.7034 \times 10^{-4}, 0, 0, 0, \ldots)$	4.5360×10^{-4}
⋮	⋮	⋮
15	$(-1.7307 \times 10^{-7}, -1.6353 \times 10^{-6}, -3.0799 \times 10^{-7}, 0, 0, 0, \ldots)$	8.1981×10^{-7}

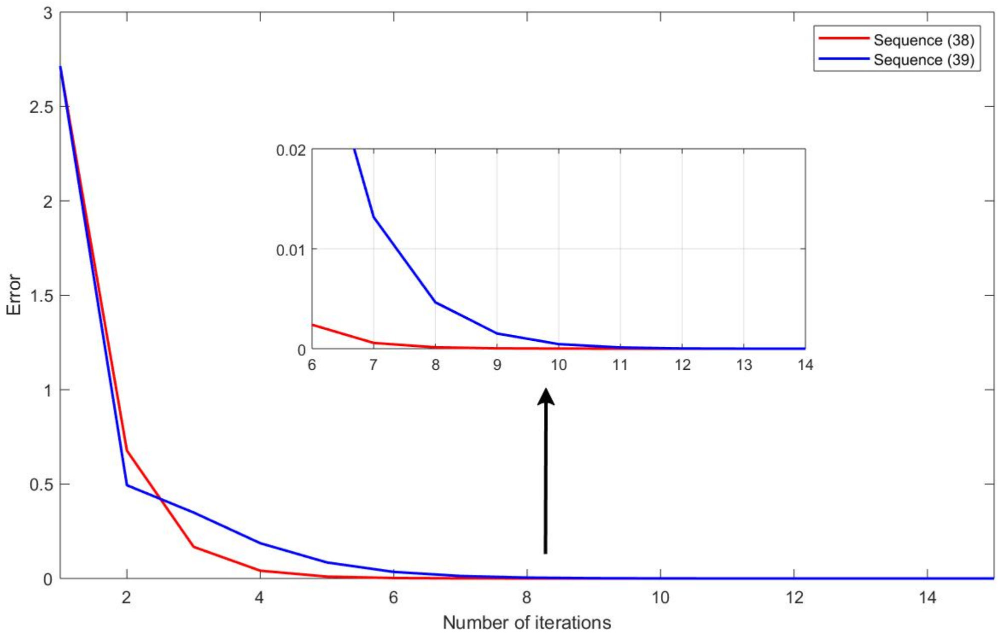

Figure 1. The convergence behavior of error values for the sequences (38) and (39).

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ORIGINAL PAPER

Iterative methods with perturbations for the sum of two accretive operators in q-uniformly smooth Banach spaces

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Abstract In this work, we introduce implicit and explicit iteration processes with perturbations for solving the fixed point problem of nonexpansive mappings and the quasi-variational inclusion problem. We then prove its strong convergence under some suitable conditions. In the last section of the paper, some applications are given also. The results obtained in this paper extend and improve some known others presented in the literature.

Keywords Variational inclusion · Banach space · Strong convergence · Iterative method · m-Accretive operator

Mathematics Subject Classification 47H09 · 47H10 · 47H17 · 47J25 · 49J40

1 Introduction

Let C be a nonempty, closed and convex of a real Banach space X. Let $S : C \longrightarrow C$ be a mapping. We use $F(S)$ to denote the set of all fixed points of S, i.e., $F(S) = \{x \in C : x = Sx\}$. Recall that a mapping $S : C \longrightarrow C$ is said to be L-Lipschitzian if there exists $L > 0$ such

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that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

A mapping $S : C \longrightarrow C$ is said to be nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A popular way to solve the fixed point problem for nonexpansive mappings is to employ iterative methods which now have received vast investigations. This is because of its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing.

Let C be a nonempty closed convex subset of a real Banach space X. Let $A : C \to X$ be a single-valued nonlinear mapping and let $B : X \to 2^X$ be a multi-valued mapping. The so called quasi-variational inclusion problem is to find a point $x \in X$ such that

$$0 \in (A + B)x. \tag{1.1}$$

We denote the solution set of (1.1) by $(A+B)^{-1}0$. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see, for instance [1–3]. The problem (1.1) includes many optimization problems as special cases.

Takahashi et al. [4] proved the following theorem for maximal monotone operators with nonlinear operator in Hilbert spaces:

Theorem T Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)S(\alpha_n x + (1 - \alpha_n)J_{\lambda_n}(x_n - \lambda_n Ax_n)), \quad \forall n \geq 1, \tag{1.2}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

(i) $0 < a \leq \lambda_n \leq b < 2\alpha$;
(ii) $0 < c \leq \beta_n \leq d < 1$;
(iii) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to a point of $F(S) \cap (A + B)^{-1}0$.

Manaka–Takahashi [5] introduced the following iteration process in Hilbert spaces $H : x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SJ_{\lambda_n}(x_n - \lambda_n Ax_n), \ \ \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\}$ is a positive sequence, S is a nonexpansive mapping on C, $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and S is a nonexpansive mapping on C. They showed that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a point in $F(S) \cap (A + B)^{-1}0$ under some mild conditions.

Recently, Lopez et al. [6] considered the following iteration process in the framework of Banach spaces: $u, x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{\lambda_n}(x_n - \lambda_n(Ax_n + a_n)) + b_n), \quad \forall n \geq 1, \tag{1.4}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in X. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution of $(A+B)^{-1}0$.

We note that, in applications, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. This leads us, in this paper, to introduce implicit and explicit iterative schemes with perturbations for solving the fixed point problem for nonexpansive mappings and the quasi-variational inclusion problem. We then prove its strong convergence under some suitable conditions. Finally, we provide some applications to the main result. The obtained results improve and extend some known results appeared in the literature.

2 Preliminaries

In this section, we collect some definitions and lemmas which will be used in the sequel. In what follows, we shall use the following notations: $x_n \to x$ mean that $\{x_n\}$ converges strongly to x; $x_n \rightharpoonup x$ mean that $\{x_n\}$ converges weakly to x.

A Banach space X is said to be strictly convex, if whenever x and y are not collinear, then: $\|x+y\| < \|x\| + \|y\|$. Let $S(X) = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. The *modulus of convexity* of X is the function $\delta : (0,2] \longrightarrow [0,1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S(X), \|x-y\| \geq \epsilon\right\}.$$

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

The *modulus of smoothness* of X is the function $\rho : \mathbb{R}^+ := [0,\infty) \longrightarrow \mathbb{R}^+$ defined by

$$\rho(\tau) = \sup\left\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S(X)\right\}.$$

A Banach space X is said to be *uniformly smooth* if $\frac{\rho(t)}{t} \longrightarrow 0$ as $t \longrightarrow 0$. Suppose that $q > 1$, a Banach space X is said to be q-*uniformly smooth* if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. If X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth.

Let X^* be a dual space of a Banach space X. Let $q > 1$ be a real number. The *generalized duality mapping* $J_q : X \longrightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the *normalized duality mapping* and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. If X is a real Hilbert space, then $J_q = I$, where I is the identity mapping. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q. The generalized duality mapping j_q is said to be *weakly sequentially continuous generalized duality mapping* if for each $\{x_n\}$ in X with $x_n \rightharpoonup x$, we have $j_q(x_n) \rightharpoonup^* j_q(x)$.

The following facts are well known (see [7,8]):

(1) Each uniformly convex Banach space (uniformly smooth Banach space) is reflexive and strictly convex.
(2) If a Banach space X admits a weakly sequentially continuous generalized duality mapping, then X satisfies Opials condition, and X is smooth smooth.

(3) All Hilbert spaces, L_p (or l_p) spaces and the Sobolev spaces W_m^p with $p \geq 2$ are 2-uniformly smooth, while L_p (or l_p) spaces and the Sobolev spaces W_m^p with $1 < p \leq 2$ are p-uniformly smooth.
(4) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. More precisely, L_p is $\min\{p, 2\}$ -uniformly smooth for each $p > 1$.

Let $A : X \longrightarrow 2^X$ be a set-valued mapping. We denote the domain and range of an operator $A : X \longrightarrow 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. Let $q > 1$. A set-valued mapping $A : D(A) \subset X \longrightarrow 2^X$ is said to be *accretive* of order q if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0, \quad u \in Ax \text{ and } v \in Ay.$$

An accretive operator A is said to be *m-accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone (see [8]).

Let A be an m-accretive operator on X, we use $A^{-1}0$ to denote the set of all zeros of A, i.e., $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. For an accretive operator A, we can define a single valued operator $J_\lambda^A : R(I + \lambda A) \longrightarrow D(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$, which is called the resolvent of A for λ. It is well known that J_λ^A is a nonexpansive mapping with $F(J_\lambda^A) = A^{-1}0$.

Let $\alpha > 0$ and $q > 1$. A mapping $A : C \longrightarrow X$ is said to be *α-inverse strongly accretive* (α-isa) of order q if for each $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^q.$$

It is obvious that A is also $1/\alpha$-Lipschitz continuous. If $X := H$ is a real Hilbert space, then $A : C \longrightarrow H$ is called α-inverse strongly monotone (α-ism).

Lemma 2.1 [6] *Let C be a subset of a real q-uniformly smooth Banach space X and $A : C \longrightarrow X$ be an α-isa of order q. Then the following inequality holds:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(\alpha q - \kappa_q\lambda^{q-1})\|Ax - Ay\|^q.$$

for all $x, y \in X$. In particular, if $0 < \lambda \leq \left(\frac{\alpha a}{\kappa_q}\right)^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Using the concept of sub-differentials, we have the following inequality:

Lemma 2.2 [9] *Let $q > 1$ and X be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in X$, we have*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle, \tag{2.1}$$

where $j_q(x + y) \in J_q(x + y)$.

Lemma 2.3 [10] *Let $1 < q \leq 2$ and X be a Banach space. Then the following are equivalent.*

(i) *X is q-uniformly smooth.*
(ii) *There is a constant $\kappa_q > 0$ which is called the q-*uniform smoothness coefficient of X *such that for all $x, y \in X$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + \kappa_q\|y\|^q.$$

In particular, if X is a real 2-uniformly smooth Banach space, then there exists a constant $K > 0$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2.$$

Lemma 2.4 [10] *Let $p > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then the following are equivalent.*

(i) *X is uniformly convex.*
(ii) *There is a strictly increasing, continuous and convex function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \|x\|^p - p\langle x, j_p(y)\rangle + (p-1)\|y\|^p, \quad \forall x, y \in B_r.$$

Lemma 2.5 [11] *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X and $S : C \longrightarrow C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Sx_n \longrightarrow 0$ implies $x = Sx$.*

Following the proof line as in Lemma 2.7 of [12], we obtain the following result.

Lemma 2.6 *Let C be a nonempty, closed and convex subset of a real smooth Banach space X and let $j_q : X \longrightarrow X^*$ be a generalized duality mapping. Assume that the mapping $F : C \longrightarrow X$ is accretive and weakly continuous along segments, that is, $F(x+ty) \rightharpoonup F(x)$ as $t \longrightarrow 0$. Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, j_q(x - x^*)\rangle \geq 0, \quad x \in C$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, j_q(x - x^*)\rangle \geq 0, \quad x \in C.$$

Proposition 2.7 [13] *Let $q > 1$. Then the following inequality holds:*

$$a^q - b^q \leq qa^{q-1}(a - b),$$

for arbitrary positive real numbers a, b.

Lemma 2.8 [14] *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\longrightarrow\infty} \|l_n - x_n\| = 0$.*

Lemma 2.9 [15] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) *$\sum_{n=0}^{\infty} \gamma_n = \infty$;*
(ii) *$\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < \infty$.*

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.10 (The Resolvent Identity [16]) *Let X be a real Banach space. Let A be an m-accretive operator. For $\lambda, \mu > 0$ and $x \in X$, then*

$$J_\lambda^A x = J_\mu^A\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^A x\right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ and $J_\mu^A = (I + \mu A)^{-1}$.

From the Resolvent Identity, we also have the following result.

Lemma 2.11 *For each $r, s > 0$ then*

$$\|J_r^A x - J_s^A x\| \leq \left|1 - \frac{s}{r}\right| \|J_r^A x - x\| \text{ for all } x \in X.$$

Proposition 2.12 *Let X be a real q-uniformly smooth Banach space. Let A be an m-accretive operator on X and let J_λ^A be the resolvent operator associated with A and λ. Then J_λ^A is firmly nonexpansive,* i.e.*,*

$$\|J_\lambda^A x - J_\lambda^A y\|^q \leq \langle x - y, j_q(J_\lambda^A x - J_\lambda^A y)\rangle, \ \ \forall x, y \in X.$$

Proof For each $x, y \in X$ and $\lambda > 0$, we set $u = J_\lambda^A x$ and $v = J_\lambda^A y$. By definition of the accretive operator, we have $x - u \in \lambda Au$ and $y - v \in \lambda Av$. Since A is m-accretive, we also have

$$\begin{aligned} 0 &\leq \langle x - u - (y - v), j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \langle u - v, j_q(u - v)\rangle \\ &= \langle x - y, j_q(u - v)\rangle - \|u - v\|^q, \end{aligned}$$

which implies that

$$\|u - v\|^q \leq \langle x - y, j_q(u - v)\rangle,$$

i.e.,

$$\|J_\lambda^A x - J_\lambda^A y\|^q \leq \langle x - y, j_q(J_\lambda^A x - J_\lambda^A y)\rangle, \ \ \forall x, y \in X.$$

This completes the proof. □

3 Main results

In this section, we prove a strong convergence theorem which is generated by an implicit iteration process.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real uniformly convex and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let $A : C \longrightarrow X$ be an α-isa of order q and let $B : D(B) \longrightarrow 2^X$ be an m-accretive operator such that $D(B) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let λ be a real positive constant such that $0 < \lambda < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ and let $\{u_t\} \subset X$ be a perturbation with $\lim_{t \longrightarrow 0^+} u_t = u' \in X$. For each $0 < t < 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}}$, let $\{x_t\}$ be a net defined by*

$$x_t = SJ_\lambda^B(tu_t + (1 - t)x_t - \lambda Ax_t), \tag{3.1}$$

where $J_\lambda^B = (I + \lambda B)^{-1}$. Then the net $\{x_t\}$ converges strongly as $t \longrightarrow 0^+$ to a point $x^* \in \Omega$, which solves uniquely the following variational inequality:

$$\langle u' - x^*, j_q(z - x^*)\rangle \leq 0, \ \ \forall z \in \Omega. \tag{3.2}$$

Proof We first show that the net $\{x_t\}$ is well defined. For each $t \in (0, 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}})$, we define a mapping $S_t : C \longrightarrow C$ by

$$S_t x := SJ_\lambda^B(tu_t + (1 - t)x - \lambda Ax), \ \ \forall x \in C.$$

Since S, J_λ^B and $I - \frac{\lambda}{1-t}A$ (see Lemma 2.1) are nonexpansive. For each $x, y \in C$, we have

$$\begin{aligned}
\|S_t x - S_t y\| &= \|SJ_\lambda^B(tu_t + (1-t)x - \lambda Ax) - SJ_\lambda^B(tu_t + (1-t)y - \lambda Ay)\| \\
&\leq \|(tu_t + (1-t)x - \lambda Ax) - (tu_t + (1-t)y - \lambda Ay)\| \\
&= (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x - \left(I - \frac{\lambda}{1-t}A\right)y\right\| \\
&\leq (1-t)\|x - y\|,
\end{aligned}$$

which implies that S_t is a contraction. Hence, S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point Eq. (3.1). Therefore, $\{x_t\}$ is well defined.

Take any $p \in \Omega$. It is observed that

$$\begin{aligned}
p &= Sp = SJ_\lambda^B(p - \lambda Ap) \\
&= SJ_\lambda^B\left(tp + (1-t)\left(p - \frac{\lambda}{1-t}Ap\right)\right), \quad \forall t \in \left(0, 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}}\right).
\end{aligned}$$

Set $x_t = Sy_t$, where $y_t = J_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t)$. Since S, J_λ^B and $I - \frac{\lambda}{1-t}A$ (see Lemma 2.1) are nonexpansive, we have

$$\begin{aligned}
\|y_t - p\| &= \left\|J_\lambda^B\left(tu_t + (1-t)\left(I - \frac{\lambda}{1-t}A\right)x_t\right) - J_\lambda^B\left(tp + (1-t)\left(I - \frac{\lambda}{1-t}A\right)p\right)\right\| \\
&\leq \left\|t(u_t - p) + (1-t)\left[\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right]\right\| \\
&\leq t\|u_t - p\| + (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right\| \\
&\leq t\|u_t - p\| + (1-t)\|x_t - p\|. \qquad (3.3)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_t - p\| &= \|Sy_t - Sp\| \\
&\leq \|y_t - p\| \\
&\leq t\|u_t - p\| + (1-t)\|x_t - p\|,
\end{aligned}$$

which implies that

$$\|x_t - p\| \leq \|u_t - p\|.$$

Since $\lim_{t \longrightarrow 0^+} u_t = u'$, then there exists a constant $K_1 > 0$ such that $K_1 = \sup_{t>0}\{\|u_t\|\}$. Hence, $\{x_t\}$ is bounded, so are $\{y_t\}$, $\{Sx_t\}$ and $\{Ax_t\}$.

Next, we show that $\lim_{t\longrightarrow 0^+}\|x_t - Sx_t\| = 0$. Since $\|x_t - p\| \le \|y_t - p\|$. By using the convexity of $\|\cdot\|^q$ for all $q > 1$ and Lemma 2.3, we derive

$$
\begin{aligned}
\|x_t - p\|^q &\le \|y_t - p\|^q \\
&\le \left\|(1-t)\left[\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(p - \frac{\lambda}{1-t}Ap\right)\right] + t(u_t - p)\right\|^q \\
&\le (1-t)\left\|\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(p - \frac{\lambda}{1-t}Ap\right)\right\|^q + t\|u_t - p\|^q \\
&= (1-t)\left\|(x_t - p) - \frac{\lambda}{1-t}(Ax_t - Ap)\right\|^q + t\|u_t - p\|^q \\
&\le (1-t)\left[\|x_t - p\|^q - \frac{q\lambda}{1-t}\langle Ax_t - Ap, j_q(x_t - p)\rangle \right.\\
&\quad \left. + \frac{\kappa_q\lambda^q}{(1-t)^q}\|Ax_t - Ap\|^q\right] + t\|u_t - p\|^q \\
&\le (1-t)\left[\|x_t - p\|^q - \frac{\alpha q\lambda}{1-t}\|Ax_t - Ap\|^q \right.\\
&\quad \left. + \frac{\kappa_q\lambda^q}{(1-t)^q}\|Ax_t - Ap\|^q\right] + t\|u_t - p\|^q \\
&= (1-t)\left[\|x_t - p\|^q - \frac{\lambda}{1-t}\left(\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t - Ap\|^q + t\|u_t - p\|^q\right. \\
&\le \|x_t - p\|^q - \lambda\left(\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t - Ap\|^q + t\|u_t - p\|^q,
\end{aligned}
$$

which implies that

$$
\lambda\left(\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}}\right)\|Ax_t - Ap\|^q \le t\|u_t - p\|^q. \tag{3.4}
$$

Since $t \in \left(0, 1 - \lambda\left(\frac{\kappa_q}{\alpha q}\right)^{\frac{1}{q-1}}\right)$, we have $\alpha q - \frac{\kappa_q\lambda^{q-1}}{(1-t)^{q-1}} > 0$. Also, it follows from (3.4) that

$$
\lim_{t\longrightarrow 0^+}\|Ax_t - Ap\| = 0.
$$

By Proposition 2.12 and Lemma 2.4, we have

$$
\begin{aligned}
\|y_t - p\|^q &= \|J_\lambda^B(tu_t + (1-t)x_t - \lambda Ax_t) - J_\lambda^B(p - \lambda Ap)\|^q \\
&\le \langle tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap), j_q(y_t - p)\rangle \\
&\le \frac{1}{q}[\|tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap)\|^q \\
&\quad + (q-1)\|y_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|)],
\end{aligned}
$$

which implies that

$$\begin{aligned}
\|y_t - p\|^q &\le \|tu_t + (1-t)x_t - \lambda Ax_t - (p - \lambda Ap)\|^q - g(\|tu_t \\
&\quad + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\
&= \left\|(1-t)\left[\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right]\right. \\
&\quad \left. + t(u_t - p)\right\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\
&\le (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)x_t - \left(I - \frac{\lambda}{1-t}A\right)p\right\|^q \\
&\quad + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\
&\le (1-t)\|x_t - p\|^q + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\
&\le \|x_t - p\|^q + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \\
&\le \|y_t - p\|^q + t\|u_t - p\|^q - g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|).
\end{aligned}$$

Hence, we have

$$g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) \le t\|u_t - p\|^q,$$

and so

$$\lim_{t \longrightarrow 0^+} g(\|tu_t + (1-t)x_t - \lambda(Ax_t - Ap) - y_t\|) = 0.$$

By the property of g, we have

$$\lim_{t \longrightarrow 0^+} \|x_t - y_t\| = 0. \tag{3.5}$$

Also, we obtain

$$\lim_{t \longrightarrow 0^+} \|y_t - Sy_t\| = \lim_{t \longrightarrow 0^+} \|y_t - x_t\| = 0.$$

Moreover, we observe that

$$\begin{aligned}
\|x_t - Sx_t\| &\le \|x_t - y_t\| + \|y_t - Sy_t\| + \|Sy_t - Sx_t\| \\
&\le 2\|x_t - y_t\| + \|y_t - Sy_t\| \longrightarrow 0 \ \text{ as } \ t \longrightarrow 0^+.
\end{aligned} \tag{3.6}$$

For any $z \in \Omega$, we note that

$$\begin{aligned}
\|x_t - z\|^q &\le \left\|(1-t)\left[\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right] + t(u_t - z)\right\|^q \\
&\le (1-t)^q\left\|\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right\|^q + qt\langle u_t - z, j_q(x_t - z)\rangle \\
&\le (1-t)\|x_t - z\|^q + qt\langle u' - z, j_q(x_t - z)\rangle + qt\langle u_t - u', j_q(x_t - z)\rangle,
\end{aligned}$$

which implies that

$$\|x_t - z\|^q \le q\langle u' - z, j_q(x_t - z)\rangle + q\langle u_t - u', j_q(x_t - z)\rangle. \tag{3.7}$$

Next, we show that the net $\{x_t\}$ is relatively norm-compact. Assume that $\{t_n\} \subset (0, 1)$ is a sequence such that $t_n \longrightarrow 0^+$ as $n \longrightarrow \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$, $\lambda_n := \lambda_{t_n}$ and $u_n := u_{t_n}$. From (3.7), we have

$$\|x_n - z\|^q \le q\langle u' - z, j_q(x_n - z)\rangle + q\langle u_n - u', j_q(x_n - z)\rangle. \tag{3.8}$$

By the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$. In addition, by (3.6), we also have $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. It follows from Lemma 2.5 that $x^* \in F(S)$. Furthermore, we show that $x^* \in (A+B)^{-1}0$. Let $v \in Bu$. Since

$$y_n = J^B_{\lambda_n}(t_n u_n + (1-t_n)x_n - \lambda_n A x_n).$$

It is observed that

$$\begin{aligned} &t_n u_n + (1-t_n)x_n - \lambda_n A x_n \in (I + \lambda_n B)y_n \\ &\Longleftrightarrow \frac{1}{\lambda_n}\big(t_n u_n + (1-t_n)x_n - \lambda_n A x_n - y_n\big) \in B y_n. \end{aligned}$$

Since B is accretive, we have for all $(u, v) \in B$,

$$\begin{aligned} &\left\langle \frac{1}{\lambda_n}\big(t_n u_n + (1-t_n)x_n - \lambda_n A x_n - y_n\big) - v,\, j_q(y_n - u)\right\rangle \geq 0 \\ &\Longleftrightarrow \langle t_n u_n + (1-t_n)x_n - \lambda_n A x_n - y_n - \lambda_n v,\, j_q(y_n - u)\rangle \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \langle Ax_n + v, j_q(y_n - u)\rangle &\leq \frac{1}{\lambda_n}\langle x_n - y_n, j_q(y_n - u)\rangle + \frac{t_n}{\lambda_n}\langle u_n - x_n, j_q(y_n - u)\rangle \\ &\leq \frac{1}{\lambda_n}\|x_n - y_n\|\|y_n - u\|^{q-1} + \frac{t_n}{\lambda_n}\|u_n - x_n\|\|y_n - u\|^{q-1} \\ &\leq (\|x_n - y_n\| + t_n)K_2, \end{aligned} \tag{3.9}$$

where $K_2 > 0$ is a constant such that $K_2 = \sup_{n\geq 1}\left\{\frac{1}{\lambda_n}\big(\|y_n - u\|^{q-1}, \|u_n - x_n\|\|y_n - u\|^{q-1}\big)\right\}$.

Since a Banach space X has a weakly sequentially continuous generalized duality mapping and from (3.5), we get $\langle Ax^* + v, j_q(x^* - u)\rangle \leq 0$, or $\langle -Ax^* - v, j_q(x^* - u)\rangle \geq 0$. Since B is m-accretive, we have $-Ax^* \in Bx^*$. This shows that $x^* \in (A+B)^{-1}0$. Thus $x^* \in \Omega := F(S) \cap (A+B)^{-1}0$.

Now, replacing z in (3.8) with x^*, we have

$$\|x_n - x^*\|^q \leq \langle u' - x^*, j_q(x_n - x^*)\rangle + \langle u_n - u', j_q(x_n - x^*)\rangle. \tag{3.10}$$

Since $x_n \rightharpoonup x^*$, we get $x_n \longrightarrow x^*$. This proves the relatively norm compactness of the net $\{x_t\}$ as $t \longrightarrow 0^+$.

Now, returning to (3.8) and taking the limit as $n \longrightarrow \infty$, we have

$$\|x^* - z\|^q \leq \langle u' - z, \quad j_q(x^* - z)\rangle.$$

In particular, x^* solves the variational inequality

$$\langle u' - z, j_q(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega, \tag{3.11}$$

which is equivalent to the dual variational inequality (see Lemma 2.6):

$$\langle u' - x^*, j_q(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega. \tag{3.12}$$

Hence, $x^* \in \Omega$ is a solution of variational inequality (3.2). Furthermore, we show that the solution of (3.2) is singleton. Assume that $\hat{x}, x^* \in \Omega$ are solutions of (3.2). Then, we have

$$\langle u' - \hat{x}, j_q(x^* - \hat{x})\rangle \leq 0$$

and

$$\langle u' - x^*, j_q(\hat{x} - x^*)\rangle \leq 0.$$

Adding up above two inequalities, we have

$$\|x^* - \hat{x}\|^q \leq 0,$$

which implies that $\hat{x} = x^*$ and the uniqueness is proved.

In summary, we have shown that each cluster point of $\{x_t\}$ equal to x^* as $t \longrightarrow 0^+$. Therefore, we can conclude that the net $\{x_t\}$ converges strongly to x^*. This completes the proof. □

Next, we prove a strong convergence theorem which is generated by an explicit iteration process.

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a real uniformly convex and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let $A : C \longrightarrow X$ be an α-isa of order q and let $B : D(B) \longrightarrow 2^X$ be an m-accretive operator such that $D(B) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^B_{\lambda_n}(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \quad \forall n \geq 1, \end{cases} \tag{3.13}$$

where $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ and $\{u_n\} \subset X$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in X$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a' \leq \beta_n \leq b' < 1$*;*
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (3.13) converges strongly to a point $x^* \in \Omega$, which solves uniquely the variational inequality (3.2).

Proof We first show that $\{x_n\}$ is bounded. Since $\lim_{n\longrightarrow\infty} u_n = u' \in X$, which implies that $\{u_n\}$ is bounded. Take any $p \in \Omega$, then there exists a constant $M_1 > 0$ such that $M_1 = \sup_{n\geq 1}\{\|u_n - p\|\}$. It is observed that

$$p = Sp = J^B_{\lambda_n}(p - \lambda_n Ap) = J^B_{\lambda_n}\left(\alpha_n p + (1-\alpha_n)\left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right).$$

Since S, $J_{\lambda_n}^B$ and $I - \frac{\lambda_n}{1-\alpha_n}A$ are nonexpansive (see Lemma 2.1), we have

$$\begin{aligned}
\|y_n - p\| &= \Big\| J_{\lambda_n}^B\Big(\alpha_n u_n + (1-\alpha_n)\Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)x_n\Big) \\
&\quad - J_{\lambda_n}^B\Big(\alpha_n p + (1-\alpha_n)\Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)p\Big)\Big\| \\
&\le \Big\|\alpha_n(u_n - p) + (1-\alpha_n)\Big[\Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)x_n - \Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)p\Big]\Big\| \\
&\le \alpha_n\|u_n - p\| + (1-\alpha_n)\Big\|\Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)x_n - \Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)p\Big\| \\
&\le \alpha_n\|u_n - p\| + (1-\alpha_n)\|x_n - p\|.
\end{aligned} \tag{3.14}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(Sy_n - p)\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|Sy_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\Big[\alpha_n\|u_n - p\| + (1-\alpha_n)\|x_n - p\|\Big] \\
&= \big(1 - (1-\beta_n)\alpha_n\big)\|x_n - p\| + (1-\beta_n)\alpha_n\|u_n - p\| \\
&\le \max\{\|x_n - p\|, M_1\}.
\end{aligned}$$

By the mathematical induction, we have

$$\|x_n - p\| \le \max\{\|x_1 - p\|, M_1\}, \quad \forall n \ge 1.$$

Thus, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{Ax_n\}$ and $\{Sx_n\}$.

Next, we show that $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = 0$. Set $y_n = J_{\lambda_n}^B z_n$, where $z_n = \alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n$. Then, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|J_{\lambda_{n+1}}^B z_{n+1} - J_{\lambda_n}^B z_n\| \le \|J_{\lambda_{n+1}}^B z_{n+1} - J_{\lambda_{n+1}}^B z_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&\le \|z_{n+1} - z_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&= \|\alpha_{n+1}u_{n+1} + (1-\alpha_{n+1})x_{n+1} \\
&\quad - \lambda_{n+1}Ax_{n+1} - (\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n)\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&= \Big\|\alpha_{n+1}(u_{n+1} - u_n) + (\alpha_{n+1} - \alpha_n)(u_n - x_n) \\
&\quad + (1-\alpha_{n+1})\Big[\Big(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\Big)x_{n+1} - \Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)x_n\Big] \\
&\quad + (\lambda_n - \lambda_{n+1})Ax_n\Big\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \le \alpha_{n+1}\big(\|u_{n+1}\| + \|u_n\|\big) \\
&\quad + |\alpha_{n+1} - \alpha_n|\big(\|u_n\| + \|x_n\|\big) + (1-\alpha_{n+1})\Big\|\Big(I - \frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\Big)x_{n+1} \\
&\quad - \Big(I - \frac{\lambda_n}{1-\alpha_n}A\Big)x_n\Big\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\| \\
&\le (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}\big(\|u_{n+1}\| + \|u_n\|\big) + |\alpha_{n+1} \\
&\quad - \alpha_n|\big(\|u_n\| + \|x_n\|\big) + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \|J_{\lambda_{n+1}}^B z_n - J_{\lambda_n}^B z_n\|.
\end{aligned}$$

By Lemma 2.11, we have

$$\|J^B_{\lambda_{n+1}} z_n - J^B_{\lambda_n} z_n\| \leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J^B_{\lambda_{n+1}} z_n - z_n\|.$$

It follows that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \alpha_{n+1}\big(\|u_{n+1}\| + \|u_n\|\big) + |\alpha_{n+1} \\
&\quad - \alpha_n|\big(\|u_n\| + \|x_n\|\big) + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|J^B_{\lambda_{n+1}} z_n - z_n\| \\
&\leq (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \Bigg(\alpha_{n+1} + |\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n| \\
&\quad + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\Bigg) M_2,
\end{aligned}$$

where $M_2 = \sup_{n\geq 1}\big\{\|u_{n+1}\| + \|u_n\|, \|u_n\| + \|x_n\|, \|Ax_n\|, \|J^B_{\lambda_{n+1}} z_n - z_n\|\big\}$. Then, we have

$$\begin{aligned}
\|Sy_{n+1} - Sy_n\| \leq \|y_{n+1} - y_n\| &\leq (1-\alpha_{n+1})\|x_{n+1} - x_n\| \\
&\quad + \left(\alpha_{n+1} + |\alpha_{n+1} - \alpha_n| + |\lambda_{n+1} - \lambda_n| + \frac{|\lambda_{n+1} - \lambda_n|}{a'}\right) M_2.
\end{aligned}$$

From (C1) and (C3), we have

$$\limsup_{n\longrightarrow\infty} \big(\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

By Lemma 2.8, we get

$$\lim_{n\longrightarrow\infty} \|Sy_n - x_n\| = 0. \tag{3.15}$$

Consequently,

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty} (1-\beta_n)\|Sy_n - x_n\| = 0. \tag{3.16}$$

Next, we show that $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. By the convexity of $\|\cdot\|^q$ for all $q > 1$ and Lemma 2.3, we have

$$\begin{aligned}
\|y_n - p\|^q &= \left\|(1-\alpha_n)\left[\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right] + \alpha_n(u_n - p)\right\|^q \\
&\leq (1-\alpha_n)\left\|\left(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n\right) - \left(p - \frac{\lambda_n}{1-\alpha_n}Ap\right)\right\|^q + \alpha_n\|u_n - p\|^q \\
&= (1-\alpha_n)\left\|(x_n - p) - \frac{\lambda_n}{1-\alpha_n}(Ax_n - Ap)\right\|^q + \alpha_n\|u_n - p\|^q \\
&\leq (1-\alpha_n)\Bigg[\|x_n - p\|^q - \frac{q\lambda_n}{1-\alpha_n}\langle Ax_n - Ap, j_q(x_n - p)\rangle \\
&\quad + \frac{\kappa_q\lambda_n^q}{(1-\alpha_n)^q}\|Ax_n - Ap\|^q\Bigg] + \alpha_n\|u_n - p\|^q \leq (1-\alpha_n)\Bigg[\|x_n - p\|^q \\
&\quad - \frac{\alpha q\lambda_n}{1-\alpha_n}\|Ax_n - Ap\|^q + \frac{\kappa_q\lambda_n^q}{(1-\alpha_n)^q}\|Ax_n - Ap\|^q\Bigg] \\
&\quad + \alpha_n\|u_n - p\|^q = (1-\alpha_n)\Big[\|x_n - p\|^q
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_n}{1-\alpha_n}\left(\alpha q - \frac{\kappa_q \lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q\Bigg] + \alpha_n\|u_n - p\|^q \\
&\le \|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q \lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\|u_n - p\|^q.
\end{aligned} \tag{3.17}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^q &\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|Sy_n - p\|^q \\
&\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|y_n - p\|^q \le \beta_n\|x_n - p\|^q + (1-\beta_n) \\
&\quad \times\left[\|x_n - p\|^q - \lambda_n\left(\alpha q - \frac{\kappa_q \lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q + \alpha_n\|u_n - p\|^q\right] \\
&= \|x_n - p\|^q - \lambda_n(1-\beta_n)\left(\alpha q - \frac{\kappa_q \lambda_n^{q-1}}{(1-\alpha_n)^{q-1}}\right)\|Ax_n - Ap\|^q \\
&\quad + \alpha_n(1-\beta_n)\|u_n - p\|^q,
\end{aligned}
$$

which implies from (C2), (C3) and Proposition 2.7 that

$$
\begin{aligned}
&c'(1-b')\big(\alpha q - \kappa_q (d')^{q-1}\big)\|Ax_n - Ap\|^q \\
&\quad \le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n(1-\beta_n)\|u_n - p\|^q \\
&\quad \le q\|x_n - p\|^{q-1}\big(\|x_n - p\| - \|x_{n+1} - p\|\big) + \alpha_n(1-\beta_n)\|u_n - p\|^q \\
&\quad \le q\|x_n - p\|^{q-1}\|x_{n+1} - x_n\| + \alpha_n(1-\beta_n)\|u_n - p\|^q.
\end{aligned}
$$

Moreover, from (C1), (C3) and (3.16), we have

$$
\lim_{n\longrightarrow\infty}\|Ax_n - Ap\| = 0. \tag{3.18}
$$

By Proposition 2.12 and Lemma 2.4, we have

$$
\begin{aligned}
\|y_n - p\|^q &= \|J^B_{\lambda_n}(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n) - J^B_{\lambda_n}(p - \lambda_n Ap)\|^q \\
&\le \langle \alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap), j_q(y_n - p)\rangle \\
&\le \frac{1}{q}\Big[\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^q + (q-1)\|y_n - p\|^q \\
&\quad - g(\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|)\Big],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|y_n - p\|^q &\le \|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^q - g(\|\alpha_n u_n \\
&\quad + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|) \\
&\le \alpha_n\|u_n - p\|^q + \|x_n - p\|^q \\
&\quad - g(\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^q &\le \beta_n\|x_n - p\|^q + (1-\beta_n)\|y_n - p\|^q \le \beta_n\|x_n - p\|^q \\
&\quad + (1-\beta_n)\Big[\alpha_n\|u_n - p\|^q + \|x_n - p\|^q - g(\|\alpha_n u_n
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|)\Big] \\
&= \|x_n - p\|^q + \alpha_n(1-\beta_n)\|u_n - p\|^q - (1-\beta_n)g(\|\alpha_n u_n \\
&\quad + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|),
\end{aligned}
$$

which implies by (C2) and Proposition 2.7 that

$$
\begin{aligned}
&(1-b')g(\|\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n(Ax_n - Ap) - y_n\|) \\
&\quad \le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_n(1-\beta_n)\|u_n - p\|^q \\
&\quad \le q\|x_n - p\|^{q-1}\big(\|x_n - p\| - \|x_{n+1} - p\|\big) + \alpha_n(1-\beta_n)\|u_n - p\|^q \\
&\quad \le q\|x_n - p\|^{q-1}\|x_{n+1} - x_n\| + \alpha_n(1-\beta_n)\|u_n - p\|^q.
\end{aligned}
$$

Then, from (C1), (C2) and (3.16), we have

$$
\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{3.19}
$$

Consequently,

$$
\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\
&\le \|x_n - Sy_n\| + \|y_n - x_n\| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.
\end{aligned} \tag{3.20}
$$

Next, we show that

$$
\limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(y_n - x^*)\rangle \le 0,
$$

where x^* is the same as in Theorem 3.1. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(x_n - x^*)\rangle = \lim_{i\longrightarrow\infty} \langle u' - x^*, j_q(x_{n_i} - x^*)\rangle.
$$

By the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in C$. From (3.19) and (3.20), we also have $y_n - Sy_n \longrightarrow 0$. Then from Lemma 2.5, we have $z \in F(S)$. Furthermore, by the similar method in the proof of Theorem 3.1, we can show that $z \in \Omega$. Since a Banach space X has a weakly sequentially continuous generalized duality mapping. Then, we have

$$
\begin{aligned}
\limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(y_n - x^*)\rangle &= \limsup_{n\longrightarrow\infty} \langle u' - x^*, j_q(x_n - x^*)\rangle \\
&= \langle u' - x^*, j_q(z - x^*)\rangle \le 0.
\end{aligned} \tag{3.21}
$$

Finally, we show that $x_n \longrightarrow x^*$. From (3.14) and Lemma 2.2, we have

$$
\begin{aligned}
\|y_n - x^*\|^q &= \left\|(1-\alpha_n)\left[\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*\right] + \alpha_n(u_n - x^*)\right\|^q \\
&\le (1-\alpha_n)^q\left\|\left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x_n - \left(I - \frac{\lambda_n}{1-\alpha_n}A\right)x^*\right\|^q \\
&\quad + q\alpha_n\langle u_n - x^*, j_q(y_n - x^*)\rangle \le (1-\alpha_n)^q\|x_n - x^*\|^q \\
&\quad + q\alpha_n\langle u_n - x^*, j_q(y_n - x^*)\rangle.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^q &\le \beta_n \|x_n - x^*\|^q + (1-\beta_n)\|Sy_n - x^*\|^q \\
&\le \beta_n \|x_n - x^*\|^q + (1-\beta_n)\|y_n - x^*\|^q \\
&\le \beta_n \|x_n - x^*\|^q + (1-\beta_n)\big[(1-\alpha_n)^q \|x_n - x^*\|^q \\
&\quad + q\alpha_n \langle u_n - x^*, j_q(y_n - x^*)\rangle\big] \\
&\le \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - x^*\|^q + q\alpha_n(1-\beta_n)\langle u_n - u', j_q(y_n - x^*)\rangle \\
&\quad + q\alpha_n(1-\beta_n)\langle u_n - x^*, j_q(y_n - x^*)\rangle \\
&\le \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - x^*\|^q \\
&\quad + q\alpha_n(1-\beta_n)\|u_n - u'\|\|y_n - x^*\|^{q-1} \\
&\quad + q\alpha_n(1-\beta_n)\langle u_n - x^*, j_q(y_n - x^*)\rangle. \qquad (3.22)
\end{aligned}
$$

Then (3.22) reduces to

$$\|x_{n+1} - x^*\|^q \le (1-\gamma_n)\|x_n - x^*\|^q + \gamma_n \delta_n,$$

where $\gamma_n := \alpha_n(1-\beta_n)$ and $\delta_n := q\|u_n - u'\|\|y_n - x^*\|^{q-1} + q\langle u' - x^*, j_q(y_n - x^*)\rangle$. It is easily seen that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\longrightarrow\infty} \delta_n \le 0$. We can therefore apply Lemma 2.9 to conclude that $x_n \longrightarrow x^*$. This completes the proof. □

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping j. Let $A : C \longrightarrow X$ be an α-isa of order 2 and let $B : D(B) \longrightarrow 2^X$ be an m-accretive operator such that $D(B) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A+B)^{-1}0 \ne \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$
\begin{cases}
y_n = J^B_{\lambda_n}(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n), \\
x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \quad \forall n \ge 1,
\end{cases}
\tag{3.23}
$$

where $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ and $\{u_n\} \subset X$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in X$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \le \beta_n \le b' < 1$;
(C3) $0 < c' \le \lambda_n < \frac{\lambda_n}{1-\alpha_n} \le d' < \frac{\alpha}{K^2}$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (3.23) converges strongly to a point $x^* \in \Omega$, which solves uniquely the following variational inequality:

$$\langle u' - x^*, j(z - x^*)\rangle \le 0, \quad \forall z \in \Omega.$$

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a real Hilbert H. Let $A : C \longrightarrow H$ be an α-ism and let $B : D(B) \longrightarrow 2^H$ be a maximal monotone operator such that $D(B) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\Omega := F(S) \cap (A+B)^{-1}0 \ne \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$
\begin{cases}
y_n = J^B_{\lambda_n}(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n), \\
x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \quad \forall n \ge 1,
\end{cases}
\tag{3.24}
$$

where $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ *and* $\{u_n\} \subset H$ *is a perturbation for the* n*-step iteration with* $\lim_{n\longrightarrow\infty} u_n = u' \in H$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a' \leq \beta_n \leq b' < 1$*;*
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2\alpha$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by* (3.24) *converges strongly to a point* $x^* \in \Omega$*, which solves uniquely the following variational inequality:*

$$\langle u' - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega.$$

4 Applications

In this section, we give some applications of Theorem 3.2 in the framework of Hilbert spaces. Throughout this section, let C be a nonempty, closed and convex subset of a real Hilbert space H.

4.1 Application to variational inequality problems

Let $A : C \longrightarrow H$ be a nonlinear monotone operator. The *variational inequality problem* is to find $z \in C$ such that

$$\langle Az, y - z \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of problem (4.1) is denoted by $VI(C, A)$. In the context of the variational inequality problem, it well known that

$$z \in VI(C, A) \Longleftrightarrow z = P_C(z - \lambda Az), \quad \forall \lambda > 0,$$

where P_C is the metric projection from H onto C.

Let $g : H \longrightarrow (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then the *subdifferential* ∂g of g is defined as follows:

$$\partial g(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y \rangle, \ \forall z \in H\}, \quad \forall x \in H.$$

It is known that ∂g is maximal monotone (see [17]). Let i_C be the indicator function of C defined by

$$i_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & x \notin C. \end{cases} \tag{4.2}$$

Since i_C is a proper lower semi-continuous convex function on H, then subdifferential ∂i_C of i_C is a maximal monotone operator. So, we can define the resolvent $J^{\partial i_C}_{\lambda}$ of ∂i_C for $\lambda > 0$ by

$$J^{\partial i_C}_{\lambda} x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$.

Lemma 4.1 [18] *Let* ∂i_C *be the subdifferential of* i_C*, where* i_C *defined as in* (4.2) *and let* $J^{\partial i_C}_{\lambda}$ *be the resolvent of* ∂i_C *for* $\lambda > 0$. *Then, we have*

$$y = J^{\partial i_C}_{\lambda} x \Longleftrightarrow y = P_C x, \quad \forall x \in H, y \in C.$$

Further, we have $(A + \partial i_C)^{-1} 0 = VI(C, A)$.

Theorem 4.2 *Let $A : C \longrightarrow H$ be an α-ism. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C(\alpha_n u_n + (1-\alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S y_n, \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{u_n\} \subset H$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in H$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a' \leq \beta_n \leq b' < 1$*;*
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2\alpha$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (4.3) converges strongly to a point $x^* \in F(S) \cap VI(C, A)$.

4.2 Application to equilibrium problems

Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of all real numbers. The *equilibrium problem* is to find $z \in C$ such that

$$G(z, y) \geq 0, \tag{4.4}$$

for all $y \in C$. The set of solutions of problem (4.6) is denoted by $EP(G)$. For solving the equilibrium problem, let us assume that a bifunction $G : C \times C \longrightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $G(x, x) = 0$ for all $x \in C$;
(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)y, y) \leq G(x, y)$;
(A4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semi-continuous.

Lemma 4.3 [19] *Let $G : C \times C \longrightarrow \mathbb{R}$ satisfying the conditions $(A1)-(A4)$. Let $\lambda > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 4.4 [20] *Assume that $G : C \times C \longrightarrow \mathbb{R}$ satisfies the conditions $(A1)-(A4)$. For $\lambda > 0$ and $x \in H$, define a mapping $T_\lambda : H \longrightarrow C$ as follows:*

$$T_\lambda(x) = \Big\{z \in C : G(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\Big\}, \ \forall x \in H.$$

Then, the following hold:

(1) *T_λ is single-valued;*
(2) *T_λ is firmly nonexpansive, i.e., for each $x, y \in H$,*

$$\|T_\lambda x - T_\lambda y\|^2 \leq \langle T_\lambda x - T_\lambda y, x - y\rangle;$$

(3) *$F(T_\lambda) = EP(G)$;*
(4) *$EP(G)$ is closed and convex.*

We call such T_λ the resolvent of G for $\lambda > 0$.

Lemma 4.5 [18] *Let $G : C \times C \longrightarrow \mathbb{R}$ satisfies the conditions $(A1)-(A4)$. Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & x \in C; \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(G) = A_G^{-1}0$ and A_G is a maximal monotone operator with $D(A_G) \subset C$. Further, for any $x \in H$ and $\lambda > 0$, the resolvent T_λ of G coincides with the resolvent of A_G, that is,

$$T_\lambda x = (I + \lambda A_G)^{-1} x.$$

Theorem 4.6 *Let $A : C \longrightarrow H$ be an α-ism. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction which satisfies the conditions $(A1) - -(A4)$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $F(S) \cap EP(G) \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = T_{\lambda_n}(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n A x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S y_n, \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\{u_n\} \subset H$ is a perturbation for the n-step iteration with $\lim_{n\longrightarrow\infty} u_n = u' \in H$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < a' \leq \beta_n \leq b' < 1$*;*
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2\alpha$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ defined by (4.5) converges strongly to a point $x^* \in F(S) \cap EP(G)$.

4.3 Application to convex minimization problems

Let $f : H \longrightarrow \mathbb{R}$ be a convex smooth function and $g : H \longrightarrow \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. The *convex minimization problem* is to find $z \in C$ such that

$$f(z) + g(z) \leq f(x) + g(x), \tag{4.6}$$

for all $x \in C$. The set of solutions of problem (4.6) is denoted by $CMP(f, g)$. By Fermat's rule, it is known that the problem (4.6) is equivalent to the problem of finding $z \in C$ such that

$$0 \in \nabla f(z) + \partial g(z),$$

where ∇f is a gradient of f and ∂g is a subdifferential of g. In fact, we can set $A = \nabla f$ and $B = \partial g$ in Theorem 3.2. It is also known ∇f is $(1/L)$-Lipschitz continuous, then it is also L-ism (see [21]). Further, ∂g is maximal monotone (see [17]). So we obtain the following result.

Theorem 4.7 *Let $f : H \longrightarrow \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇f and $G : H \longrightarrow \mathbb{R}$ be a convex and lower semi-continuous function such that $D(\partial G) \subset C$. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $F(S) \cap CMP(f, g) \neq \emptyset$. Let $\{\lambda_n\}$ be a real positive sequence and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. For an initial guess $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J_{\lambda_n}(\alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n \nabla f(x_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S y_n, \quad \forall n \geq 1, \end{cases} \tag{4.7}$$

where $\{u_n\} \subset H$ *is a perturbation for the n-step iteration with* $\lim_{n\longrightarrow\infty} u_n = u' \in H$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \leq \beta_n \leq b' < 1$;
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < 2L$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.7) *converges strongly to a point* $x^* \in F(S) \cap CMP(f, g)$.

4.4 Application to linear inverse problems

Let T be a bounded linear operator on H and $b \in H$. The unconstrained linear problem is to find $x \in H$ such that

$$Tx = b. \tag{4.8}$$

The set of solutions of problem (4.8) is denoted by $\Gamma = \{x \in H : x = T^{-1}b\}$. For each $x \in H$, we define $f : H \longrightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{2}\|Tx - b\|^2.$$

It is well known that $\nabla f = T^t(Tx - b)$ and ∇f is K-Lipschitz continuous with K the largest eigenvalue of T^tT [22]. So we obtain immediately the following result.

Theorem 4.8 *Let* $T : H \longrightarrow H$ *be a bonded linear operator and* $b \in H$ *with* K *the largest eigenvalue of* T^tT. *Let* $S : H \longrightarrow H$ *be a nonexpansive mapping such that* $F(S) \cap \Gamma \neq \emptyset$. *Let* $\{\lambda_n\}$ *be a real positive sequence and let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0, 1)$. *For an initial guess* $x_1 \in H$, *define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)x_n - \lambda_n T^t(Tx_n - b), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Sy_n, \quad \forall n \geq 1, \end{cases} \tag{4.9}$$

where $\{u_n\} \subset H$ *is a perturbation for the n-step iteration with* $\lim_{n\longrightarrow\infty} u_n = u' \in H$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a' \leq \beta_n \leq b' < 1$;
(C3) $0 < c' \leq \lambda_n < \frac{\lambda_n}{1-\alpha_n} \leq d' < \frac{2}{K}$ *and* $\lim_{n\longrightarrow\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.9) *converges strongly to a point* $x^* \in F(S) \cap \Gamma$.

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On Solving the Split Feasibility Problem and the Fixed Point Problem in Banach Spaces

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Abstract : In this paper, we introduce an iterative method for solving the split feasibility problem and the fixed point problem of countable family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Then, we prove strong convergence theorem of the sequence generated by our iterative scheme with a new way of selecting the step-size which does not require the computation on the norm of the bounded linear operator. The obtained result of this paper complements many recent results in this direction.

Keywords : Split feasibility problem; Banach space; Strong convergence; Iterative method; Fixed point
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1 Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also

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uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be its adjoint of A. The *split feasibility problem* (SFP) is to find an element

$$\hat{x} \in C \quad \text{such that} \quad A\hat{x} \in Q. \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by Γ, *i.e.*, $\Gamma := \{x \in C : Ax \in Q\}$. It is well known that if Γ is nonempty then Γ is a closed and convex subset of E_1. The SFP was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [1] in 1994 for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction (see [2]). The SFP has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (see, *e.g.*, [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

For solving the SFP in Banach spaces, Schöpfer et al. [14] first introduced the following algorithm for solving the SFP: $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J_{E_1}^* \left[J_{E_1}(x_n) - \lambda_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))\right], \; n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ is a positive sequence, Π_C denotes the generalized projection on E, P_Q is the metric projection on E_2, J_{E_1} is the duality mapping on E_1 and $J_{E_1}^*$ is the duality mapping on E_1^*. It was proved that the sequence $\{x_n\}$ converges weakly to a solution of SFP, under some mild conditions, in p-uniformly convex and uniformly smooth Banach spaces.

Recently, Shehu et al. [15] introduced an iterative scheme for solving the SFP and the fixed point problem of Bregman strongly nonexpansive mapping T in the framework of p-uniformly convex real Banach spaces which are also uniformly smooth as follows: Let $u \in C$, $u_1 \in E_1$ and

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)Tx_n\big)\right], \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and the step-size λ_n is chosen by $0 < t \leq \lambda_n \leq k < \big(\frac{q}{\kappa_q \|A\|^q}\big)^{\frac{1}{q-1}}$.

They proved that the sequence $\{x_n\}$ and $\{u_n\}$ defined by (1.3) converge strongly to a point in $F(T) \cap \Gamma$ under some mild conditions. However, it is observed that iterative method (1.3) involves step-size that depend on the operator norm $\|A\|$ (matrix in the finite-dimensional space), which may not be calculated easily in general. It makes the implementation of the iteration process inefficient when the computation of the operator norm $\|A\|$ is not explicit (see [16, 17]).

Motivated by the previous works, we introduce an iterative method for solving the split feasibility problem and the fixed point problem of countable family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Then, we prove strong convergence theorem of the sequence generated by our iterative scheme with a new way of selecting the step-size which does not require the computation on the norm of the bounded linear operator. Our result complements the results of Byrne [2], Schöpfer et al. [14], Wang [18], Shehu et al. [15], Shehu et al. [19] and many other recent results in the literature.

2 Preliminaries

Let E and E^* be real Banach spaces and the dual space of E, respectively. Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$ which is defined by

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \quad \forall x \in E_1, \quad \bar{y} \in E_2^*.$$

Let $S(E) := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S(E), \|x-y\| \geq \epsilon\right\}.$$

The space E is said to be *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. Let $p > 1$. Then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0,2]$. Observe that every p-uniformly convex space is uniformly convex. The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\left\{\tfrac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S(E)\right\}.$$

The space E is said to be *uniformly smooth* if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Suppose that $q > 1$, a Banach space E is said to be *q-uniformly smooth* if there exists a $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth. Moreover, we note that a Banach space E is p-uniformly convex if and only if E is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ (see [20]).

Let $p > 1$ be a real number. The *generalized duality mapping* $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\| = \|x\|^{p-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. In particular, $J_p^E = J_2^E$ is called the *normalized duality mapping*.

In this case, we assume that E is a p-uniformly convex and uniformly smooth, which implies that its dual space, E^* is q-uniformly smooth and uniformly convex. It is known that the generalized duality mapping J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^*.

Moreover, if E is uniformly smooth then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E. (see [21, 22] for more details).

Definition 2.1. ([23]) *Let* $f : E \to \mathbb{R}$ *be a convex and Gâteaux differentiable function. The function* $D_f : E \times E \to [0, +\infty)$ *defined by*

$$D_f(x, y) := f(y) - f(x) - \langle f'(x), y - x\rangle,$$

is called the Bregman distance with respect to f.

We remark that the Bregman distance D_f is not satisfy the well-known properties of a metric because D_f is not symmetric and does not satisfy the triangle inequality.

It is well known that the duality mapping J_p^E is the sub-differential of the functional $f_p(\cdot) = \frac{1}{p}\|\cdot\|^p$ for $p > 1$ (see [24]). Then, we have the Bregman distance with respect to f_p that

$$D_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_p^E x, y\rangle + \frac{1}{p}\|y\|^p. \tag{2.1}$$

If $p = 2$, we get

$$D_2(x, y) := \phi(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2,$$

where ϕ is called the *Lyapunov function* which was introduced by Alber [25, 26]. Moreover, the Bregman distance has the following properties:

$$D_p(x, y) = D_p(x, z) + D_p(z, y) + \langle J_p^E x - J_p^E z, z - y\rangle, \tag{2.2}$$

$$D_p(x, y) + D_p(y, x) = \langle J_p^E x - J_p^E y, x - y\rangle, \tag{2.3}$$

for all $x, y, z \in E$. For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [14]):

$$\tau\|x - y\|^p \le D_p(x, y) \le \langle J_p^E x - J_p^E y, x - y\rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number. In what follows, we shall use the following notations:

- $x_n \to x$ mean that $\{x_n\}$ converges strongly to x;
- $x_n \rightharpoonup x$ mean that $\{x_n\}$ converges weakly to x.

Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote $F(T)$ by the set of all fixed points of T, *i.e.*, $F(T) = \{x \in C : x = Tx\}$. A point $z \in C$ called an *asymptotic fixed point* of T, if there exists a sequence $\{x_n\}$ in C which $x_n \rightharpoonup z$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ by the set of asymptotic fixed points of T.

Definition 2.2. ([27, 28]) *A mapping* $T : C \to C$ *is called* Bregman relatively nonexpansive*, if the following conditions are satisfied:*

(R1) $F(T) = \widehat{F}(T) \neq \emptyset$;

(R2) $D_p(Tx, z) \leq D_p(x, z), \quad \forall z \in F(T), \ \forall x \in C$.

Clearly, in a Hilbert space H, Bregman relatively nonexpansive mappings and quasi-nonexpansive mappings are equivalent, for $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$, *i.e.*,

$$\phi(Tx, z) \leq \phi(x, z) \Longleftrightarrow \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C \ \text{ and } \ z \in F(T).$$

Definition 2.3. ([29]) *Let C be a subset of a real p-uniformly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the* AKTT-condition *if, for any bounded subset B of C,*

$$\sum_{n=1}^{\infty} \sup_{z \in B} \{\|J_p^E(T_{n+1}z) - J_p^E(T_n z)\|\} < \infty.$$

As in [30], we can prove the following fact.

Proposition 2.1. *Let C be a nonempty, closed and convex subset of a real p-uniformly convex Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Suppose that for any bounded subset B of C. Then there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n \to \infty} T_n x, \quad \forall x \in B, \tag{2.5}$$

and

$$\lim_{n \to \infty} \sup_{z \in B} \|J_p^E(Tz) - J_p^E(T_n z)\| = 0.$$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (2.5) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Recall that the metric projection from E onto C, denote by $P_C x$, satisfying the property

$$\|x - P_C x\| \leq \inf_{y \in C} \|x - y\|, \quad \forall x \in E.$$

It is well known that $P_C x$ is the unique minimizer of the norm distance. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle J_p^E(x - P_C x), y - P_C x \rangle \leq 0, \quad \forall y \in C. \tag{2.6}$$

Similarly, one can define the Bregman projection from E onto C, denote by Π_C, satisfying the property

$$D_p\big(x, \Pi_C(x)\big) = \inf_{y \in C} D_p(x, y), \quad \forall x \in E. \tag{2.7}$$

Lemma 2.2. ([19]) *Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and let $x \in E$. Then the following assertions hold:*

(i) $z = \Pi_C x$ *if and only if* $\langle J_p^E(x) - J_p^E(z), y - z\rangle \leq 0$, $\forall y \in C$.

(ii) $D_p(\Pi_C x, y) + D_p(x, \Pi_C x) \leq D_p(x, y)$, $\forall y \in C$.

Lemma 2.3. [31] *Let $1 < q \leq 2$ and E be a Banach space. Then the following are equivalent.*

(i) E is q-uniformly smooth;

(ii) There is a constant $\kappa_q > 0$ such that for all $x, y \in E$

$$\|x - y\|^q \leq \|x\|^q - q\langle j_q(x), y\rangle + \kappa_q \|y\|^q. \tag{2.8}$$

Remark 2.4. *The constant κ_q satisfying* (2.8) *is called the* q-uniform smoothness coefficient of E.

The following Lemma can be obtained from Theorem 2.8.17 of [21] (see also Lemma 5 of [32]).

Lemma 2.5. *Let $p > 1$, $r > 0$ and E be a Banach space. Then the following statements are equivalent:*

(i) E is uniformly convex;

(ii) There exists a strictly increasing convex function $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_r^*(0) = 0$ such that

$$\Big\|\sum_{k=1}^{N} \alpha_k x_k\Big\|^p \leq \sum_{k=1}^{N} \alpha_k \|x_k\|^p - \alpha_i \alpha_j g_r^*(\|x_i - x_j\|),$$

for all $i, j \in \{1, 2, ..., N\}$, $x_k \in B_r := \{x \in E : \|x\| \leq r\}$, $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$, where $k \in \{1, 2, ..., N\}$.

Lemma 2.6. ([19]) *Let E be a real p-uniformly convex and uniformly smooth Banach space. Thus, for all $z \in E$, we have*

$$D_p\bigg(J_q^{E^*}\bigg(\sum_{i=1}^{N} t_i J_p^E(x_i)\bigg), z\bigg) \leq \sum_{i=1}^{N} t_i D_p(x_i, z),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

The following lemmas can be found in [15, 19].

Lemma 2.7. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \quad \forall x \in E, \quad x^* \in E^*.$$

Then the following assertions hold:

(i) V_p *is nonnegative and convex in the first variable;*

(ii) $D_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \quad \forall x \in E, \quad x^* \in E^*$.

(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x \rangle \leq V_p(x^* + y^*, x), \quad \forall x \in E, \quad x^*, y^* \in E^*$.

Following the proof line as in Proposition 2.5 of [33], we obtain the following result:

Lemma 2.8. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $x \in E$ and $\{x_n\}$ is a sequence in E. If $\{D_p(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.9. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*

(a) $\lim_{n\to\infty} D_p(x_n, y_n) = 0$;

(b) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. For the implication $(a) \Longrightarrow (b)$. Suppose that $\lim_{n\to\infty} D_p(x_n, y_n) = 0$. From (2.4), we have

$$0 \leq \tau \|x_n - y_n\|^p \leq D_p(x_n, y_n),$$

where $\tau > 0$ is a fixed number. It follows that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

For the converse implication $(b) \Longrightarrow (a)$, we assume that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. From (2.4), we observe that

$$\begin{aligned} 0 \leq D_p(x_n, y_n) &\leq \langle J_p^E x_n - J_p^E y_n, x_n - y_n \rangle \\ &\leq \|J_p^E x_n - J_p^E y_n\| \|x_n - y_n\| \\ &\leq \|x_n - y_n\| M, \end{aligned}$$

where $M = \sup_{n\geq 1}\{\|x_n\|^{p-1}, \|y_n\|^{p-1}\}$. It follows that $\lim_{n\to\infty} D_p(x_n, y_n) = 0$. This completes the proof. □

Lemma 2.10. ([34]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. ([35]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \ldots$ *and* $\tau(n) \to \infty$*;*

(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 2.12. *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$. Then, we have*

$$D_p\bigg(J_q^{E^*}\bigg(\sum_{k=1}^N \alpha_k J_p^E(x_k)\bigg), z\bigg) \leq \sum_{k=1}^N \alpha_k D_p(x_k, z) - \alpha_i\alpha_j g_r^*\big(\|J_p^E(x_i) - J_p^E(x_j)\|\big),$$

for all $i, j \in \{1, 2, ..., N\}$.

Proof. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$. Since p-uniformly convex, hence it is uniformly convex. From Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
& D_p\bigg(J_q^{E^*}\bigg(\sum_{k=1}^N \alpha_k J_p^E(x_k)\bigg), z\bigg) \\
= \;& V_p\bigg(\sum_{k=1}^N \alpha_k J_p^E(x_k), z\bigg) \\
= \;& \frac{1}{q}\Big\|\sum_{k=1}^N \alpha_k J_p^E(x_k)\Big\|^q - \Big\langle \sum_{k=1}^N \alpha_k J_p^E(x_k), z\Big\rangle + \frac{1}{p}\|z\|^p \\
\leq \;& \frac{1}{q}\sum_{k=1}^N \alpha_k \|J_p^E(x_k)\|^q - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) \\
& -\Big\langle \sum_{k=1}^N \alpha_k J_p^E(x_k), z\Big\rangle + \frac{1}{p}\|z\|^p \\
= \;& \frac{1}{q}\sum_{k=1}^N \alpha_k \|J_p^E(x_k)\|^q - \sum_{k=1}^N \alpha_k \langle J_p^E(x_k), z\rangle + \frac{1}{p}\|z\|^p \\
& -\alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) \\
= \;& \sum_{k=1}^N \alpha_k D_p(x_k, z) - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),
\end{aligned}$$

for all $i, j \in \{1, 2, ..., N\}$. This completes the proof. □

3 Main Results

Theorem 3.1. *Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces and let C and Q be nonempty, closed and convex subsets* *of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be its adjoint of A. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of Bregman relatively nonexpansive mappings of C into E_1 such that $F(T_n) = \widehat{F}(T_n)$ for all $n \geq 1$. Suppose that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma \neq \emptyset$. For given $u \in E_1$, let $\{u_n\}$ be a sequence generated by $u_1 \in C$ and*

$$\begin{cases} x_n = \Pi_C J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\big) \\ u_{n+1} = \Pi_C J_q^{E_1^*}\big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\big(\beta_n J_p^{E_1}(x_n) + (1-\beta_n) J_p^{E_1}(T_n x_n)\big)\big], \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that the step-size $\{\lambda_n\}$ is a bounded sequence chosen in such a way that for small enough $\epsilon > 0$,

$$0 < \epsilon < \lambda_n < \left(\frac{q\|(I-P_Q)Au_n\|^p}{\kappa_q\|A^* J_p^{E_2}(I-P_Q)Au_n\|^q} - \epsilon\right)^{\frac{1}{q-1}}, \quad n \in N, \tag{3.2}$$

where the index set $N := \{n \in \mathbb{N} : (I - P_Q)Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \leq \beta_n \leq b < 1$.

Suppose in addition that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to an element $x^* = \Pi_\Omega u$, where Π_Ω is the Bregman projection from C onto Ω.

Proof. By the choice of λ_n, we observe that

$$\begin{aligned} & \lambda_n^{q-1} < \frac{q\|(I-P_Q)Au_n\|^p}{\kappa_q\|A^* J_p^{E_2}(I-P_Q)Au_n\|^q} - \epsilon \\ \iff & \quad \kappa_q \lambda_n^{q-1}\|A^* J_p^{E_2}(I-P_Q)Au_n\|^q \\ & < q\|(I-P_Q)Au_n\|^p - \epsilon\kappa_q\|A^* J_p^{E_2}(I-P_Q)Au_n\|^q \\ \iff & \frac{\epsilon\kappa_q}{q}\|A^* J_p^{E_2}(I-P_Q)Au_n\|^q \\ < & \quad \|(I-P_Q)Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I-P_Q)Au_n\|^q. \end{aligned} \tag{3.3}$$

For each $n \geq 1$, we put $x_n = \Pi_C v_n$, where

$$v_n := J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\big).$$

Let $z \in \Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma$. From (2.6), we observe that

$$\begin{aligned} & \langle J_p^{E_2}(Au_n - P_Q(Au_n)), Au_n - Az\rangle \\ = \;& \langle J_p^{E_2}(Au_n - P_Q(Au_n)), Au_n - P_Q(Au_n)\rangle \\ & +\langle J_p^{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Az\rangle \\ = \;& \|Au_n - P_Q(Au_n)\|^p \\ & +\langle J_p^{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Az\rangle \\ \geq \;& \|Au_n - P_Q(Au_n)\|^p. \end{aligned} \tag{3.4}$$

Then from Lemma 2.3 and (3.4), we have

$$\begin{aligned} & D_p(x_n, z) \\ \leq \;& D_p\big(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n), z\big) \\ = \;& \frac{1}{q}\|J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n)\|^p \\ & -\langle J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n, z\rangle + \frac{1}{p}\|z\|^p \\ = \;& \frac{1}{q}\|J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ & -\langle J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - P_Q)Au_n, z\rangle + \frac{1}{p}\|z\|^p \\ \leq \;& \frac{1}{q}\|J_p^{E_1}(u_n)\|^q - \lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Au_n\rangle \\ & +\frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q - \langle J_p^{E_1}(u_n), z\rangle \\ & +\lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Az\rangle + \frac{1}{p}\|z\|^p \\ = \;& \frac{1}{q}\|u_n\|^p - \langle J_p^{E_1}(u_n), z\rangle + \frac{1}{p}\|z\|^p + \lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Az - Au_n\rangle \\ & +\frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ = \;& D_p(u_n, z) + \lambda_n\langle J_p^{E_2}(I - P_Q)Au_n, Az - Au_n\rangle \\ & +\frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ \leq \;& D_p(u_n, z) - \lambda_n\left(\|(I - P_Q)Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q\right), \end{aligned} \tag{3.5}$$

which implies that

$$D_p(x_n, z) \leq D_p(u_n, z).$$

Now, we put

$$y_n := J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1 - \beta_n)J_p^{E_1}(T_n x_n))$$

for all $n \geq 1$. From Lemma 2.12, we have

$$
\begin{aligned}
& D_p(y_n, z) \\
= \;& D_p(J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n)J_p^{E_1}(T_n x_n)), z) \\
\leq \;& \beta_n D_p(x_n, v) + (1-\beta_n)D_p(T_n x_n, z) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \\
\leq \;& D_p(x_n, z) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \qquad (3.6) \\
\leq \;& D_p(x_n, z) \qquad (3.7)
\end{aligned}
$$

It follows from (3.7) that

$$
\begin{aligned}
D_p(x_{n+1}, z) &\leq D_p(u_{n+1}, z) \\
&\leq D_p\big(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)), z\big) \\
&\leq \alpha_n D_p(u, z) + (1-\alpha_n)D_p(y_n, z) \\
&\leq \alpha_n D_p(u, z) + (1-\alpha_n)D_p(x_n, z) \\
&\leq \max\{D_p(u, z), D_p(x_n, z)\} \\
&\;\;\vdots \\
&\leq \max\{D_p(u, z), D_p(x_1, z)\}. \qquad (3.8)
\end{aligned}
$$

Hence, $\{D_p(x_n, z)\}$ is bounded, which implies by Lemma 2.8 that $\{x_n\}$ is bounded.

Put $u_{n+1} = \Pi_C z_n$, where $z_n := J_q^{E_1^*}\big[\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n)\big]$ for all $n \geq 1$. From Lemma 2.7 and (3.6), we have

$$
\begin{aligned}
& D_p(x_{n+1}, z) \\
\leq \;& D_p(u_{n+1}, z) \\
\leq \;& D_p(z_n, z) \\
= \;& V_p\big(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n), z\big) \\
\leq \;& V_p(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)J_p^{E_1}(y_n) - \alpha_n(J_p^{E_1}(u) - J_p^{E_1}(z), z)) \\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \\
= \;& V_p(\alpha_n J_p^{E_1}(z) + (1-\alpha_n)J_p^{E_1}(y_n), z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \\
\leq \;& \alpha_n V_p(J_p^{E_1}(z), z) + (1-\alpha_n)V_p(J_p^{E_1}(y_n), z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - v\rangle \\
= \;& \alpha_n D_p(z, z) + (1-\alpha_n)D_p(y_n, z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \\
\leq \;& (1-\alpha_n)[D_p(x_n, z) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|)] \\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(v), z_n - z\rangle \\
\leq \;& (1-\alpha_n)D_p(x_n, z) - \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|)] \\
& +\alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \qquad (3.9) \\
\leq \;& (1-\alpha_n)D_p(x_n, z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle. \qquad (3.10)
\end{aligned}
$$

Next, we will divide the proof into two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_p(x_n, z)\}_{n=n_0}^{\infty}$ is non-increasing. By the boundedness of $\{D_p(x_n, z)\}_{n=1}^{\infty}$, we have $\{D_p(x_n, z)\}_{n=1}^{\infty}$ is convergent. Furthermore, we have

$$D_p(x_n, z) - D_p(x_{n+1}, z) \to 0 \quad \text{as} \quad n \to \infty.$$

Then, from (3.9), we have

$$\begin{aligned} 0 &\leq a(1-b)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \\ &\leq \beta_n(1-\beta_n)g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\|) \\ &\leq D_p(x_n, z) - D_p(x_{n+1}, z) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(z), z_n - z\rangle \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies by the property of g_r^* that

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0. \tag{3.11}$$

Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, then

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.12}$$

From Lemma 2.9, we also have

$$\lim_{n\to\infty} D_p(T_n x_n, x_n) = 0. \tag{3.13}$$

Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 2.1, we observe that

$$\begin{aligned} & \|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| \\ \leq\ & \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \|J_p^{E_1}(T_n x_n) - J_p^{E_1}(Tx_n)\| \\ \leq\ & \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_n x) - J_p^{E_1}(Tx)\| \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

By the reflexivity of a Banach space E and the boundedness of $\{x_n\}$, without loss of generality, we may assume that $x_{n_i} \rightharpoonup v \in C$ as $i \to \infty$. Then, we get $v \in \widehat{F}(T_n) = F(T_n)$ for all $n \geq 1$, *i.e.*, $v \in \bigcap_{n=1}^{\infty} F(T_n)$. Further, we show that $v \in \Gamma$. From (3.3) and (3.5), we have

$$\begin{aligned} & \frac{\epsilon^2\kappa_q}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q \\ <\ & \lambda_n\left(\|(I - P_Q)Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I - P_Q)Au_n\|^q\right) \\ \leq\ & D_p(u_n, v) - D_p(x_n, v) \\ \leq\ & \alpha_{n-1}D_p(u, v) + D_p(x_{n-1}, v) - D_p(x_n, v), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{3.14}$$

Since $v_n := J_q^{E_1^*}\big(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(Au_n - P_Q(Au_n))\big)$ for all $n \geq 1$, it follows that

$$\begin{aligned} 0 \leq \|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| &\leq \lambda_n \|A^*\| \|J_p^{E_2}(Au_n - P_Q(Au_n))\| \\ &\leq \left(\frac{q}{\kappa_q \|A\|^q}\right)^{\frac{1}{q-1}} \|A^*\| \|Au_n - P_Q(Au_n)\|^{p-1}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| = 0, \tag{3.15}$$

and hence

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.16}$$

By Lemma 2.2 (ii) and (3.6), we have

$$\begin{aligned} D_p(v_n, x_n) &= D_p(v_n, \Pi_C v_n) \leq D_p(v_n, v) - D_p(x_n, xv) \\ &\leq D_p(u_n, v) - D_p(x_n, v) \\ &\leq \alpha_{n-1} D_p(u, v) + D_p(x_{n-1}, v) - D_p(x_n, v) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

By Lemma 2.9, we get

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.17}$$

Then from (3.16) and (3.17), we have

$$\|x_n - u_n\| \leq \|v_n - u_n\| + \|v_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.18}$$

Since $x_{n_i} \rightharpoonup v \in C$ and from (3.18), we also get $u_{n_i} \rightharpoonup v \in C$. From (2.6), we have

$$\begin{aligned} &\|(I - P_Q)Av\|^p \\ = \;& \langle J_p^{E_2}(Av - P_Q(Av)), Av - P_Q(Av)\rangle \\ = \;& \langle J_p^{E_2}(Av - P_Q(Av)), Av - Au_{n_i}\rangle + \langle J_p^{E_2}(Av - P_Q(Av)), Au_{n_i} - P_Q(Au_{n_i})\rangle \\ &+ \langle J_p^{E_2}(Av - P_Q(Av)), P_Q(Au_{n_i}) - P_Q(Av)\rangle \\ \leq \;& \langle J_p^{E_2}(Av - P_Q(Av)), Av - Au_{n_i}\rangle + \langle J_p^{E_2}(Av - P_Q(Av)), Au_{n_i} - P_Q(Au_{n_i})\rangle. \end{aligned}$$

Since A is continuous, we have $Au_{n_i} \rightharpoonup Av$ as $i \to \infty$. From (3.14), we obtain

$$\|(I - P_Q)Av\| = 0,$$

i.e., $Av = P_Q(Av)$, this shows that $Av \in Q$. Thus $v \in \Omega := F(T) \cap \Gamma$. From Lemma 2.6 and (3.13), we have

$$\begin{aligned} D_p(y_n, x_n) &= D_p(J_q^{E_1^*}(\beta_n J_p^{E_1}(x_n) + (1-\beta_n) J_p^{E_1}(T_n x_n)), x_n) \\ &\leq \beta_n D_p(x_n, x_n) + (1-\beta_n) D_p(T_n x_n, x_n) \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

It follows that

$$\begin{aligned} D_p(z_n, x_n) &= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(u) + (1-\alpha_n) J_p^{E_1}(y_n)), x_n) \\ &\leq \alpha_n D_p(u, x_n) + (1-\alpha_n) D_p(y_n, x_n) \to 0 \ \text{ as } \ n \to \infty, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

Next, we show that

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_n - x^* \rangle \leq 0,$$

where $x^* = \Pi_\Omega u$. From (3.19), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), z_n - x^* \rangle &= \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^* \rangle \\ &= \lim_{i\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^* \rangle. \end{aligned}$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. It follows from Lemma 2.2 that

$$\begin{aligned} \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^* \rangle &= \lim_{i\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^* \rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), v - x^* \rangle \leq 0. \end{aligned} \tag{3.20}$$

Applying Lemma 2.10 to (3.10) and (3.20), we can conclude that $D_p(x_n, x^*) \to 0$ as $n \to \infty$. Therefore, $x_n \to x^*$ as $n \to \infty$.
Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then, by Lemma 2.11, we obtain

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \ \text{ and } \ \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n := D_p(x_n, x^*)$ for all $n \in \mathbb{N}$. Then, we have from (3.8) that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) \\ &\leq \lim_{n\to\infty} (D_p(u, x^*) + (1-\alpha_{\tau(n)}) D_p(x_{\tau(n)}, x^*) - D_p(x_{\tau(n)}, x^*)) \\ &= \lim_{n\to\infty} \alpha_{\tau(n)} \big(D_p(u, x^*) - D_p(x_{\tau(n)}, x^*) \big) = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty}(D_p(x_{\tau(n)+1},x^*)-D_p(x_{\tau(n)},x^*))=0. \tag{3.21}$$

Following the proof line in **Case 1**, we can show that

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0,$$

$$\lim_{n\to\infty}\|Au_{\tau(n)}-P_Q(Au_{\tau(n)})\|=0.$$

Further, we can show that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle\leq 0.$$

From (3.10), we have

$$\begin{aligned} D_p(x_{\tau(n)+1},x^*) &\leq (1-\alpha_{\tau(n)})D_p(x_{\tau(n)},x^*) \\ &\quad +\alpha_{\tau(n)}\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \alpha_{\tau(n)}D_p(x_{\tau(n)},x^*) &\leq D_p(x_{\tau(n)},x^*)-D_p(x_{\tau(n)+1},x^*) \\ &\quad +\alpha_{\tau(n)}\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle. \end{aligned}$$

Since $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)}>0$, we get

$$D_p(x_{\tau(n)},x^*) \leq \langle J_p^{E_1}(u)-J_p^{E_1}(x^*),z_{\tau(n)}-x^*\rangle.$$

Hence, $\lim_{n\to\infty}D_p(x_{\tau(n)},x^*)=0$. From (3.21), we have

$$\begin{aligned} D_p(x_n,x^*) &\leq D_p(x_{\tau(n)+1},x^*)=D_p(x_{\tau(n)},x^*)+(D_p(x_{\tau(n)+1},x^*)-D_p(x_{\tau(n)},x^*)) \\ &\to 0 \text{ as } n\to\infty, \end{aligned}$$

which implies that $D_p(x_n,x^*)\to 0$. Therefore $x_n\to x^*$ as $n\to\infty$. Thus from above two cases, we conclude that $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^*=\Pi_\Omega u$. This completes the proof. □

We consequently obtain the following result in Hilbert spaces.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A:E_1\to E_2$ be a bounded linear operator and $A^*:E_2^*\to E_1^*$ be its adjoint of A. Let $\{T_n\}_{n=1}^\infty$ be a countable family of quasi-nonexpansive mappings of C into E_1 such that $F(T_n)=\widehat{F}(T_n)$ for all $n\geq 1$. Suppose that $\Omega:=\bigcap_{n=1}^\infty F(T_n)\cap\Gamma\neq\emptyset$. For given $u\in E_1$, let $\{u_n\}$ be a sequence generated by $u_1\in C$ and*

$$\begin{cases} x_n=P_C(u_n-\lambda_nA^*(I-P_Q)Au_n) \\ u_{n+1}=P_C(\alpha_nu+(1-\alpha_n)(\beta_nx_n+(1-\beta_n)T_nx_n)), \quad \forall n\geq 1, \end{cases} \tag{3.22}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Suppose that the step-size $\{\lambda_n\}$ is a bounded sequence chosen in such a way that for small enough $\epsilon > 0$,

$$0 < \epsilon < \lambda_n < \frac{2\|(I-P_Q)Au_n\|^2}{\|A^*(I-P_Q)Au_n\|^2} - \epsilon, \quad n \in N, \tag{3.23}$$

where the index set $N := \{n \in \mathbb{N} : (I-P_Q)Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < a \leq \beta_n \leq b < 1$.

Suppose in addition that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to an element $x^* = P_\Omega u$, where P_Ω is the metric projection from C onto Ω.

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Convergence analysis of generalized viscosity implicit rules for a nonexpansive semigroup with gauge functions

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Abstract

In this paper, we introduce an iterative algorithm for finding the set of common fixed points of nonexpansive semigroups by the generalized viscosity implicit rule in certain Banach spaces which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function. We prove strong convergence theorems of proposed algorithm under appropriate conditions. As applications, we apply main result to solving the fixed point problems of countable family of nonexpansive mappings and the problems of zeros of accretive operators. Furthermore, we give some numerical examples for supporting our main results.

Keywords: Nonexpansive semigroup, Banach spaces, strong convergence, fixed point problem, iterative method.

2010 MSC: 47H09, 47H10, 47J05, 47J25.

1. Introduction

In this paper, we assume that E is a real Banach space with dual space E^* and C is a nonempty subset of E. Let $T : C \to C$ be a mapping. We denote the set of all fixed points of T by $F(T) = \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is called *nonexpansive* if for each $x, y \in C$ such that

$$\|Tx - Ty\| \leqslant \|x - y\|.$$

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A mapping $f : C \to C$ is called a *contraction*, if there exists a constant $\rho \in (0, 1)$ and for each $x, y \in C$

$$\|f(x) - f(y)\| \leqslant \rho\|x - y\|.$$

The viscosity approximation method has been successfully applied to various problems from calculus of variations as minimal surface problems and plasticity theory and phase transition. Various applications can be obtained in optimal control theory, singular perturbations, game theory, and partial differential equations (see [4] and references therein). In recent years, viscosity approximation method for approximating the set of (common) fixed points of nonlinear mappings have been investigated extensively by many authors in Hilbert and Banach spaces (see [10, 11, 13, 19, 20, 23–25, 30] and the references therein).

Very recently, the implicit midpoint rule (IMR) has become a powerful numerical method for numerically solving ordinary differential equations (in particular, the stiff equations) (see [5, 6, 14, 21, 22, 28]) and differential algebraic equations (see [32]).

Xu et al. [31] combined the Moudafi's viscosity method [19] (see also [30]) with IMR for nonexpansive mappings T and proposed the following *viscosity implicit midpoint rule* (VIMR) in Hilbert spaces H as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geqslant 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a real control condition in $(0, 1)$. They also proved that VIMR converges strongly to a point $x^* \in F(T)$ which also solves the variational inequality

$$\langle (f - I)x^*, z - x^*\rangle \leqslant 0, \quad \forall z \in F(T), \tag{1.2}$$

where I is the identity on H.

Later, Ke and Ma [17] improved the VIMR (1.1) by replacing the midpoint by any point of interval $[x_n, x_{n+1}]$. They introduced the so-call *generalized viscosity implicit midpoint rules* to approximating the fixed point of nonexpansive mapping T in Hilbert spaces H. They obtained the following result.

Theorem 1.1 (Theorem KM)**.** *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let f be a contraction on C with coefficient $\rho \in (0, 1)$. Let $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}), \quad \forall n \geqslant 1, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{s_n\}$ are sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \gamma_n = 1$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C3) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(C4) $0 < \kappa \leqslant s_n \leqslant s_{n+1} < 1$ *for all* $n \geqslant 1$.

Then $\{x_n\}$ converges strongly to a point $x^* \in F(T)$, which also solves (1.2).

The above results naturally bring us to the following questions.

Question 1: Can we obtain strong convergence result of Theorem 1.1 to higher spaces other than Hilbert spaces? Such as a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function.

Question 2: Can we remove the control condition (C1) in Theorem 1.1?

Question 3: Can we weaken the control conditions (C2) and (C3) in Theorem 1.1?

Question 4: Can we extend the generalized viscosity implicit midpoint rules (1.3) to finding the set of common fixed points of a family of mappings? Such as one-parameter semigroups of nonexpansive mappings.

The main objective in this paper is to give an affirmative answer to above questions, we introduce an iterative algorithm for finding the set of common fixed points of nonexpansive semigroups by the generalized viscosity implicit rule in a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function. Then, we prove strong convergence theorems of proposed algorithm with different approach on control conditions. As applications, we apply main results to solving the fixed point problems of family of nonexpansive mappings and the problems of zeros of accretive operators. Furthermore, we also give some numerical examples for support our main results.

2. Preliminaries

The continuous and strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ is said to be *gauge function* if $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|),\ \|f^*\| = \varphi(\|x\|),\ \ \forall x \in E\},$$

where $\langle \cdot,\cdot \rangle$ denotes the generalized duality paring. In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J is referred to as the normalized duality mapping. In this case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called *generalized duality mapping*. It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ for each $x \neq 0$, and $J_q(x) = \|x\|^{q-2}J(x)$, $q > 1$ (see [9]).

Remark 2.1. For the gauge function φ, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(t) = \int_0^t \varphi(\tau)d\tau$ is continuous, convex, and strictly increasing function on $[0,\infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.

Remark 2.2. It is observe that if $k \in [0,1]$ then $\varphi(ky) \leqslant \varphi(y)$. Then, we have

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(ky)dy \leqslant k\int_0^t \varphi(y)dy = k\Phi(t).$$

Remark 2.3. If a Banach space E has a uniformly Gâteaux differentiable, then $J\varphi$ is single-valued and also denoted by j_φ.

Lemma 2.4 ([18]). *Let E be a Banach space. Then for each $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leqslant \Phi(\|x\|) + \langle y, j_\varphi(x+y)\rangle,\ \ j_\varphi(x+y) \in J_\varphi(x+y).$$

Definition 2.5. A one-parameter family $\mathcal{S} = \{T_t\}_{t\geqslant 0} : C \to C$ is said to be a *nonexpansive semigroup* if it satisfies the following conditions:

(S1) $T_0x = x$ for $x \in C$;

(S2) $T_{s+t} = T_sT_t$ for $s, t \geqslant 0$;

(S3) $\lim_{t\to 0^+} T(t)x = x$ for $x \in C$;

(S4) for each $t \geqslant 0$, T_t is nonexpansive, i.e.,

$$\|T_tx - T_ty\| \leqslant \|x - y\|,\ \forall x, y \in C.$$

Remark 2.6. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t\geqslant 0} F(T_t)$.

Now, we give some examples of semigroup operator. The following classical examples were the main sources for the development of semigroup theory (see [15]).

Example 2.7. Let E be a real Banach space and let $\mathcal{L}(E)$ be the space of all bounded linear operators on E. For $A \in \mathcal{L}(E)$, consider the initial value problem for a linear autonomous differential equation on $[0, \infty)$:

$$u'(t) = Au(t), \ u(0) = x. \tag{2.1}$$

Notice that the solution of problem (2.1) is given by

$$u(t) := T_t x \text{ for all } t \geqslant 0.$$

Then, we can show that the operator $T_t x$ is a semigroup on E.

Example 2.8. Let $E := L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$. Consider the initial value problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u, \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x, 0) &= f(x), \quad \text{for } x \in \mathbb{R}^n, \end{aligned} \tag{2.2}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on E. By using Fourier transform, we can write the solution $u(x, t)$ in the form of convolution integral as:

$$u(x, t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{\frac{-\|x-\xi\|^2}{4t}} f(\xi) d\xi = (K_t * f)(x),$$

where $t > 0$, $f \in E$, and K_t is the heat kernel given by $K_t(x) = \frac{1}{\sqrt{(4\pi t)^n}} e^{\frac{-\|x\|^2}{4t}}$. Then the solution of problem (2.2) can be written as:

$$T_t f(x) := u(x, t) = (K_t * f)(x).$$

We can show that the operator $T_t f(x)$ is a semigroup on E.

Definition 2.9 ([1, 2, 8]). A continuous operator semigroup $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) if for all $s \geqslant 0$ and any bounded subset B of C,

$$\lim_{t \to \infty} \sup_{x \in B} \|T_t x - T_s T_t x\| = 0.$$

Example 2.10. Let C be a closed convex subset of a uniformly convex Banach space E. Let $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ be a nonexpansive semigroup. Let $\{\sigma_t\}_{t > 0}$ defined by $\sigma_t x = \frac{1}{t} \int_0^t T_s x ds$. Then, for each $h > 0$ and any bounded subset B of C, we have

$$\|\sigma_t x - \sigma_h \sigma_t x\| = \Big\|\sigma_t x - \frac{1}{h} \int_0^h T_s \sigma_t x ds\Big\| = \Big\|\frac{1}{h} \int_0^h (\sigma_t x - T_s \sigma_t x) ds\Big\| \leqslant \frac{1}{h} \int_0^h \|\sigma_t x - T_s \sigma_t x\| ds.$$

From Lemma 2.7 of [12], we have

$$\lim_{t \to \infty} \sup_{x \in B} \|\sigma_t x - \sigma_h \sigma_t x\| \leqslant \frac{1}{h} \int_0^h \lim_{t \to \infty} \sup_{x \in B} \|\sigma_t x - T_s \sigma_t x\| ds = 0,$$

i.e., $\{\sigma_t\}_{t > 0}$ is u.a.r..

Theorem 2.11 ([13]). *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ be a u.a.r. nonexpansive semigroup such that $F(\mathcal{S}) := \bigcap_{t \geqslant 0} F(T_t) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. Suppose that $\{t_n\}$ is a real divergent sequence and $\{\alpha_n\}$ is a real sequence in $(0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$. Then, the sequence $\{x_n\}$ defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) T_{t_n} x_n, \quad \forall n \geqslant 1,$$

converges strongly to a point $p \in F(\mathcal{S})$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z - p) \rangle \leqslant 0, \quad \forall z \in F(\mathcal{S}).$$

Lemma 2.12 ([26]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geqslant 0$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then, $\lim_{n\to\infty} \|l_n - x_n\| = 0$.*

Lemma 2.13 ([29]). *Assume that $\{a_n\}$ is a nonnegative real sequence such that*

$$a_{n+1} \leqslant (1-\theta_n)a_n + \theta_n \sigma_n,$$

where $\{\theta_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \theta_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\theta_n \sigma_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\mathcal{S} = \{T_t\}_{t\geqslant 0} : C \to C$ be a u.a.r. nonexpansive semigroup such that $F(\mathcal{S}) := \bigcap_{t\geqslant 0} F(T_t) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0,1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{t_n}\big(s_n x_n + (1-s_n)x_{n+1}\big), \ \ \forall n \geqslant 1, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $\{t_n\} \subset (0,\infty)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
(C3) $t_{n+1} = h + t_n$ *for all* $h \geqslant 0$ *and* $\lim_{n\to\infty} t_n = \infty$;
(C4) $0 < \kappa \leqslant s_n \leqslant s_{n+1} < 1$ *for all* $n \geqslant 1$.

Then, $\{x_n\}$ defined by (3.1) converges strongly to a point $p \in F(\mathcal{S})$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z-p)\rangle \leqslant 0, \ \ \forall z \in F(\mathcal{S}). \tag{3.2}$$

Proof. First, we will show that $\{x_n\}$ generated by (3.1) is well defined. For each $x, u \in C$, define the mapping $S_n : C \to C$ by

$$S_n x := \alpha_n f(x) + \beta_n x + \gamma_n T_{t_n}\big(s_n u + (1-s_n)x\big), \ \ \forall n \geqslant 1.$$

For each $x, y \in C$, we have

$$\begin{aligned}\|S_n x - S_n y\| &= \|\alpha_n(f(x) - f(y)) + \beta_n(x-y) + \gamma_n\big[T_{t_n}\big(s_n u + (1-s_n)x\big) - T_{t_n}\big(s_n u + (1-s_n)y\big)\big]\| \\ &\leqslant \alpha_n\|f(x) - f(y)\| + \beta_n\|x-y\| + \gamma_n\|T_{t_n}\big(s_n u + (1-s_n)x\big) - T_{t_n}\big(s_n u + (1-s_n)y\big)\| \\ &\leqslant \alpha_n \rho\|x-y\| + \beta_n\|x-y\| + \gamma_n(1-s_n)\|x-y\| \\ &= (1-(1-\rho)\alpha_n - \gamma_n\kappa)\|x-y\| \leqslant (1-(1-\rho)\alpha_n)\|x-y\|,\end{aligned}$$

this mean that S_n is a contraction. So S_n has a unique fixed point. Therefore, the sequence $\{x_n\}$ defined by (3.1) is well-defined.

Next, we show that $\{x_n\}$ is bounded. For each $p \in F(\mathcal{S})$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(T_{t_n}\big(s_n x_n + (1-s_n)x_{n+1}\big) - p)\| \\ &\leqslant \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|T_{t_n}\big(s_n x_n + (1-s_n)x_{n+1}\big) - p\|\end{aligned}$$

$$
\begin{aligned}
&\leqslant \alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|s_n(x_n-p)+(1-s_n)(x_{n+1}-p)\|\\
&\leqslant \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+\beta_n\|x_n-p\|+r_n(s_n\|x_n-p\|+(1-s_n)\|x_{n+1}-p\|)\\
&\leqslant (\alpha_n\rho+\beta_n+\gamma_n s_n)\|x_n-p\|+\alpha_n\|f(p)-p\|+\gamma_n(1-s_n)\|x_{n+1}-p\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-p\| &\leqslant \frac{\alpha_n\rho+\beta_n+\gamma_n s_n}{1-\gamma_n(1-s_n)}\|x_n-p\|+\frac{\alpha_n}{1-\gamma_n(1-s_n)}\|f(p)-p\|\\
&=\left(1-\frac{(1-\rho)\alpha_n}{1-\gamma_n(1-s_n)}\right)\|x_n-p\|+\frac{(1-\rho)\alpha_n}{1-\gamma_n(1-s_n)}\frac{\|f(p)-p\|}{1-\rho}\\
&\leqslant \max\left\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\}.
\end{aligned}
$$

By induction, we have

$$
\|x_n-p\|\leqslant \max\left\{\|x_1-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\},\quad \forall n\geqslant 1.
$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{f(x_n)\}$ and $\{T_{t_n}(s_nx_n+(1-s_n)x_{n+1})\}$ are bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Let $z_n=s_nx_n+(1-s_n)x_{n+1}$ for all $n\geqslant 1$. Then, we have

$$
\begin{aligned}
\|z_{n+1}-z_n\| &= \|s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2}-(s_nx_n+(1-s_n)x_{n+1})\|\\
&\leqslant (1-s_{n+1})\|x_{n+2}-x_{n+1}\|+s_n\|x_{n+1}-x_n\|.
\end{aligned}
$$

Let $y_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}$ for all $n\geqslant 1$. Then, we drive that

$$
\begin{aligned}
y_{n+1}-y_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}T_{t_{n+1}}z_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+\gamma_nT_{t_n}z_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}f(x_n)+T_{t_{n+1}}z_{n+1}-T_{t_n}z_n+\frac{\alpha_n}{1-\beta_n}T_{t_n}z_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}T_{t_{n+1}}z_{n+1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|y_{n+1}-y_n\| &\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-T_{t_{n+1}}z_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|T_{t_n}z_n-f(x_n)\|+\|T_{t_{n+1}}z_{n+1}-T_{t_{n+1}}z_n\|\\
&\quad+\|T_{t_{n+1}}z_n-T_{t_n}z_n\|\\
&\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-T_{t_{n+1}}z_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|T_{t_n}z_n-f(x_n)\|\\
&\quad+(1-s_{n+1})\|x_{n+2}-x_{n+1}\|+s_n\|x_{n+1}-x_n\|+\|T_{t_{n+1}}z_n-T_{t_n}z_n\|.
\end{aligned} \tag{3.3}
$$

Now, we estimate $\|x_{n+2}-x_{n+1}\|$. Observe that

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}f(x_{n+1})+\beta_{n+1}x_{n+1}+\gamma_{n+1}T_{t_{n+1}}z_{n+1}-(\alpha_nf(x_n)+\beta_nx_n+\gamma_nT_{t_n}z_n)\|\\
&=\|\alpha_{n+1}(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)(f(x_n)-T_{t_n}z_n)+\beta_{n+1}(x_{n+1}-x_n)\\
&\quad+(\beta_{n+1}-\beta_n)(x_n-T_{t_n}z_n)+\gamma_{n+1}(T_{t_{n+1}}z_{n+1}-T_{t_n}z_n)\|\\
&\leqslant \alpha_{n+}\|f(x_{n+1})-f(x_n)\|+|\alpha_{n+1}-\alpha_n|\|f(x_n)-T_{t_n}z_n\|+\beta_{n+1}\|x_{n+1}-x_n\|\\
&\quad+|\beta_{n+1}-\beta_n|\|x_n-T_{t_n}z_n\|+\gamma_{n+1}\|T_{t_{n+1}}z_{n+1}-T_{t_{n+1}}z_n\|+\gamma_{n+1}\|T_{t_{n+1}}z_n-T_{t_n}z_n\|
\end{aligned}
$$

$$\begin{aligned}
&\leqslant (\alpha_{n+1}\rho + \beta_{n+1})\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - T_{t_n}z_n\| \\
&\quad + |\beta_{n+1} - \beta_n|\|x_n - T_{t_n}z_n\| \\
&\quad + \gamma_{n+1}((1 - s_{n+1})\|x_{n+2} - x_{n+1}\| + s_n\|x_{n+1} - x_n\|) + \gamma_{n+1}\|T_{t_{n+1}}z_n - T_{t_n}z_n\| \\
&\leqslant (\alpha_{n+1}\rho + \beta_{n+1} + \gamma_{n+1}s_n)\|x_{n+1} - x_n\| + \big(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|\big)M_1 \\
&\quad + \gamma_{n+1}(1 - s_{n+1})\|x_{n+2} - x_{n+1}\| + \gamma_{n+1}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|,
\end{aligned}$$

where $M_1 = \sup_{n\geqslant 1}\{\|f(x_n)\| + \|T_{t_n}z_n\|, \|x_n\| + \|T_{t_n}z_n\|\}$. It follows that

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\leqslant \frac{\alpha_{n+1}\rho + \beta_{n+1} + \gamma_{n+1}s_n}{1 - \gamma_{n+1}(1 - s_{n+1})}\|x_{n+1} - x_n\| + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1 \\
&\quad + \frac{\gamma_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\| \\
&= \left(1 - \frac{(1-\rho)\alpha_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)\|x_{n+1} - x_n\| + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1 \\
&\quad + \frac{\gamma_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|.
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), we get that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leqslant \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - T_{t_{n+1}}z_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|T_{t_n}z_n - f(x_n)\| \\
&\quad + (1 - s_{n+1})\Bigg\{\left(1 - \frac{(1-\rho)\alpha_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)\|x_{n+1} - x_n\| \\
&\quad + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1 \\
&\quad + \frac{\gamma_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|\Bigg\} \\
&\quad + s_n\|x_{n+1} - x_n\| + \|T_{t_{n+1}}z_n - T_{t_n}z_n\| \\
&\leqslant \left(1 - \frac{(1-\rho)\alpha_{n+1}(1 - s_{n+1})}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)\|x_{n+1} - x_n\| + \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - T_{t_{n+1}}z_{n+1}\| \\
&\quad + \frac{\alpha_n}{1 - \beta_n}\|T_{t_n}z_n - f(x_n)\| + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1 \\
&\quad + \frac{1}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|.
\end{aligned} \tag{3.5}$$

Since $t_{n+1} = h + t_n$ for all $h \geqslant 0$, we have

$$\lim_{n\to\infty}\|T_{t_{n+1}}z_n - T_{t_n}z_n\| = \lim_{n\to\infty}\|T_hT_{t_n}z_n - T_{t_n}z_n\| \leqslant \lim_{n\to\infty}\sup_{x\in\{z_n\}}\|T_hT_{t_n}x - T_{t_n}x\| = 0.$$

Then from (3.5), we have

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

By Lemma 2.12, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Consequently, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|y_n - x_n\| = 0$$

and

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}(1-s_n)\|x_{n+1}-x_n\| = 0. \tag{3.6}$$

Next, we show that $\lim_{n\to\infty}\|x_n - T_h x_n\| = 0$ for all $h \geqslant 0$. Since

$$\begin{aligned}\|x_{n+1} - T_{t_n}x_n\| &\leqslant \alpha_n\|f(x_n) - T_{t_n}z_n\| + \beta_n\|x_n - T_{t_n}z_n\| + \gamma_n\|T_{t_n}z_n - T_{t_n}x_n\| \\ &\leqslant \alpha_n\|f(x_n) - T_{t_n}x_n\| + \beta_n\|x_n - x_{n+1}\| + \beta_n\|x_{n+1} - T_{t_n}x_n\| + \gamma_n\|z_n - x_n\|,\end{aligned}$$

and hence

$$\|x_{n+1} - T_{t_n}x_n\| \leqslant \frac{\alpha_n}{1-\beta_n}\|f(x_n) - T_{t_n}x_n\| + \frac{\beta_n}{1-\beta_n}\|x_{n+1}-x_n\| + \frac{\gamma_n}{1-\beta_n}\|z_n - x_n\| \to 0 \text{ as } n\to\infty.$$

Consequently,

$$\begin{aligned}\|z_n - T_{t_n}z_n\| &\leqslant \|z_n - x_{n+1}\| + \|x_{n+1} - T_{t_n}z_n\| \\ &\leqslant s_n\|x_n - x_{n+1}\| + \|x_{n+1} - T_{t_n}x_n\| + \|T_{t_n}x_n - T_{t_n}z_n\| \\ &\leqslant s_n\|x_n - x_{n+1}\| + \|x_{n+1} - T_{t_n}x_n\| + \|x_n - z_n\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Then, for all $h \geqslant 0$, we obtain that

$$\begin{aligned}\|z_n - T_h z_n\| &\leqslant \|z_n - T_{t_n}z_n\| + \|T_{t_n}z_n - T_h T_{t_n}z_n\| + \|T_h T_{t_n}z_n - T_h z_n\| \\ &\leqslant 2\|z_n - T_{t_n}z_n\| + \sup_{x\in\{z_n\}}\|T_{t_n}x - T_h T_{t_n}x\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

From (3.6), we also have

$$\lim_{n\to\infty}\|x_n - T_h x_n\| = 0, \quad \forall h \geqslant 0.$$

Let $u_m = \alpha_m f(u_m) + (1-\alpha_m)T_{t_m}u_m$, where $\{\alpha_m\}$ and $\{t_m\}$ satisfy the condition of Theorem 2.11. From these, we know that $\{u_m\}$ converges strongly to p, where $p \in F(\mathcal{S})$ is a unique solution of (3.2). Since

$$\begin{aligned}\|u_m - x_n\|\varphi(\|u_m - x_n\|) &= \alpha_n\langle f(u_m) - x_n, j_\varphi(u_m - x_n)\rangle + (1-\alpha_m)\langle T_{t_m}u_m - x_n, j_\varphi(u_m - x_n)\rangle \\ &= \alpha_m\langle f(u_m) - f(p) - u_m + p, j_\varphi(u_m - x_n)\rangle + \alpha_m\langle f(p) - p, j_\varphi(u_m - x_n)\rangle \\ &\quad + \alpha_m\langle u_m - x_n, j_\varphi(u_m - x_n)\rangle + (1-\alpha_m)\langle T_{t_m}u_m - T_{t_m}x_n, j_\varphi(u_m - x_n)\rangle \\ &\quad + (1-\alpha_m)\langle T_{t_m}x_n - x_n, j_\varphi(u_m - x_n)\rangle \\ &\leqslant \|u_m - x_n\|\varphi(\|u_m - x_n\|) + \|T_{t_m}x_n - x_n\|\varphi(\|u_m - x_n\|) \\ &\quad + \alpha_m(1+\rho)\varphi(\|u_m - x_n\|)\|u_m - p\| + \alpha_m\langle f(p) - p, j_\varphi(u_m - x_n)\rangle,\end{aligned}$$

which implies that

$$\langle f(p) - p, j_\varphi(x_n - u_m)\rangle \leqslant \left(\frac{\|T_{t_m}x_n - x_n\|}{\alpha_m} + (1+\rho)\|u_m - p\|\right)M_2, \tag{3.7}$$

where $M_2 = \sup_{n\geqslant 1}\{\varphi(\|u_m - x_n\|)\}$. Now, taking the upper limit as $n\to\infty$ and as $m\to\infty$, respectively in (3.7), we obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(p) - p, j_\varphi(x_n - u_m)\rangle \leqslant 0. \tag{3.8}$$

Since j_φ is norm-weak* uniformly continuous on bounded sets, as $m\to\infty$, then

$$\langle f(p) - p, j_\varphi(x_n - u_m)\rangle \to \langle f(p) - p, j_\varphi(x_n - p)\rangle.$$

Hence, for each $\epsilon > 0$, there exists $N \geqslant 1$ such that if $m > N$, for all $n \geqslant 1$ we have

$$\langle f(p) - p, j_\varphi(x_n - p)\rangle < \langle f(p) - p, j_\varphi(x_n - u_m)\rangle + \epsilon. \tag{3.9}$$

Thus taking upper limit as $n \to \infty$ and as $m \to \infty$ in both sides of (3.9), we get from (3.8) that

$$\limsup_{n\to\infty}\langle f(p) - p, j_\varphi(x_n - p)\rangle \leqslant \epsilon.$$

Since $\epsilon > 0$ is arbitrary, then we obtain

$$\limsup_{n\to\infty}\langle f(p) - p, j_\varphi(x_n - p)\rangle \leqslant 0. \tag{3.10}$$

Finally, we show that x_n converges strongly to p. We have from Lemma 2.4 that

$$\begin{aligned}
\Phi(\|x_{n+1} - p\|) &= \Phi(\|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p\|)) \\
&\leqslant \Phi(\|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + \gamma_n(T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p)\|) \\
&\quad + \alpha_n\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle \\
&\leqslant \alpha_n\Phi(\|f(x_n) - f(p)\|) + \beta_n\Phi(\|x_n - p\|) + \gamma_n\Phi(\|T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p\|) \\
&\quad + \alpha_n\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle \\
&\leqslant \alpha_n\rho\Phi(\|x_n - p\|) + \beta_n\Phi(\|x_n - p\|) + \gamma_n\Phi(\|s_n(x_n - p) + (1 - s_n)(x_{n+1} - p)\|) \\
&\quad + \alpha_n\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle \\
&\leqslant \alpha_n\rho\Phi(\|x_n - p\|) + \beta_n\Phi(\|x_n - p\|) + \gamma_n\big(s_n\Phi(\|x_n - p\|) + (1 - s_n)\Phi(\|x_{n+1} - p\|)\big) \\
&\quad + \alpha_n\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle \\
&= (\alpha_n\rho + \beta_n + \gamma_n s_n)\Phi(\|x_n - p\|) + \gamma_n(1 - s_n)\Phi(\|x_{n+1} - p\|) \\
&\quad + \alpha_n\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\Phi(\|x_{n+1} - p\|) &\leqslant \frac{\alpha_n\rho + \beta_n + \gamma_n s_n}{1 - \gamma_n(1 - s_n)}\Phi(\|x_n - p\|) + \frac{\alpha_n}{\gamma_n(1 - s_n)}\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle \\
&= \left(1 - \frac{(1 - \rho)\alpha_n}{1 - \gamma_n(1 - s_n)}\right)\Phi(\|x_n - p\|) + \frac{\alpha_n}{\gamma_n(1 - s_n)}\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle \\
&= (1 - \theta_n)\Phi(\|x_n - p\|) + \theta_n\sigma_n,
\end{aligned}$$

where $\theta_n = \frac{(1-\rho)\alpha_n}{1-\gamma_n(1-s_n)}$ and $\sigma_n = \frac{1}{1-\rho}\langle f(p) - p, j_\varphi(x_{n+1} - p)\rangle$. From (C1) and (3.10), we see that $\sum_{n=1}^{\infty}\theta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \leqslant 0$. We conclude by Lemma 2.13 that $\Phi(\|x_n - p\|) \to 0$ as $n \to \infty$. By the property of Φ, we obtain that $\{x_n\}$ converges strongly to p as $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\mathcal{S} = \{T_t\}_{t\geqslant 0} : C \to C$ be a u.a.r. nonexpansive semigroup such that $F(\mathcal{S}) := \bigcap_{t\geqslant 0} F(T_t) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{t_n}\big(s_n x_n + (1 - s_n)x_{n+1}\big), \quad \forall n \geqslant 1. \tag{3.11}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\}$, and $\{t_n\}$ be the same as in Theorem 3.1. Then, $\{x_n\}$ defined by (3.11) converges strongly to a point $p \in F(\mathcal{S})$, which also solves the variational inequality

$$\langle f(p) - x^*, z - p\rangle \leqslant 0, \quad \forall z \in F(\mathcal{S}).$$

4. Some applications

4.1. Convergence theorem for a family of mappings

Definition 4.1. Let C be a subset of a Banach space E. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition ([3]) if

$$\sum_{n=1}^{\infty} \sup_{x\in C} \|T_{n+1}x - T_n x\| < \infty. \tag{4.1}$$

Lemma 4.2 ([3]). *Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfy the AKTT-condition. Then, for any $x \in C$, $\{T_n x\}_{n=1}^{\infty}$ converges strongly to some point of C. Further, let $T : C \to C$ defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Then, $\lim_{n\to\infty} \sup_{x\in C} \|Tx - T_n x\| = 0$.*

In the sequel, we say that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3. *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a sequence of nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n\big(s_n x_n + (1 - s_n)x_{n+1}\big), \quad \forall n \geqslant 1. \tag{4.2}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{s_n\}$ be the same as in Theorem 3.1. Suppose in addition, $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}$ defined by (4.2) converges strongly to a point $p \in \bigcap_{n=1}^{\infty} F(T_n)$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z - p)\rangle \leqslant 0, \quad \forall z \in \bigcap_{n=1}^{\infty} F(T_n).$$

Proof. Following the proof line as in Theorem 3.1, we can show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, we obtain from Lemma 4.2 that

$$\begin{aligned}\|T_{n+1}z_n - T_n z_n\| &= \|T_{n+1}z_n - Tz_n\| + \|Tz_n - T_n z_n\| \\ &\leqslant \sup_{x\in\{z_n\}} \|T_{n+1}x - Tx\| + \sup_{x\in\{z_n\}} \|Tx - T_n x\| \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

On the other hand, we need to show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Again, since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, then we obtain that

$$\|x_n - Tx_n\| \leqslant \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \leqslant \|x_n - T_n x_n\| + \sup_{x\in\{x_n\}} \|T_n x - Tx\| \to 0 \ \text{ as } \ n \to \infty.$$

Some parts of the proof are also the same as the Theorem 3.1. Then, we can obtain the desired conclusion easily. This completes the proof. □

Example 4.4. Let $C = E = \mathbb{R}$ with the usual norm. For each $n \geqslant 1$, define T_n by

$$T_n x = \begin{cases} 0, & x = 0, \\ \sin x + \frac{1}{n^2}, & x \neq 0 \end{cases}$$

for all $x \in C$. It is not hard to show that $\{T_n\}_{n=1}^{\infty}$ is nonexpansive and satisfies the AKTT-condition with $\bigcap_{n=1}^{\infty} F(T_n) = F(T) = \{0\}$, where $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$.

4.2. The problem of finding zeros of accretive operators

Let $A \subset E \times E$ be an operator. We denote by $D(A)$ and $\overline{D(A)}$ the domain of A and closure of $D(A)$, respectively. We say that A is said to be *accretive* if there exists $j_\varphi(x_1 - x_2) \in J_\varphi(x_1 - x_2)$ such that $\langle y_1 - y_2, j_\varphi\rangle \geqslant 0$, where $(x_i, y_i) \in A$ for $i = 1, 2$. We say that A is said to satisfy the range condition if $\overline{D(A)} = R(I+\lambda A)$ for all $\lambda > 0$, where $R(I+\lambda A)$ is the range of $I+\lambda A$. It is well known that if A is an accretive operator which satisfies the range condition, then we can defined a single-valued mapping $J_\lambda^A : R(I+\lambda A) \to D(A)$ by $J_\lambda = (I+\lambda A)^{-1}$, which is called the *resolvent* of A. We denote by $A^{-1}0$ the set of zeros of A, i.e., $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. It is well known that J_λ is nonexpansive and $F(J_\lambda) = A^{-1}0$ (see [27]). We also know the following [16]: For each $\lambda, \mu > 0$ and $x \in R(I+\lambda A) \cap R(I+\mu A)$, it holds that

$$\|J_\lambda x - J_\mu x\| \leqslant \frac{|\lambda - \mu|}{\lambda}\|x - J_\lambda x\|.$$

Lemma 4.5 ([3])**.** *Let C be a nonempty, closed, and convex subset of a Banach space E. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{\lambda>0} R(I+\lambda A)$. Suppose that $\{\lambda_n\} \subset (0,\infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$. Then $\{J_{\lambda_n}\}$ satisfies the AKTT-condition. Consequently, for each $x \in C$, $\{J_{\lambda_n} x\}$ converges strongly to some point of C. Moreover, let $J_\lambda : C \to C$ defined by $J_\lambda x = \lim_{n\to\infty} J_{\lambda_n} x$ for all $x \in C$ and $F(J_\lambda) = \bigcap_{n=1}^{\infty} F(J_{\lambda_n})$, where $\lambda_n \to \lambda$ as $n \to \infty$. Then, $\lim_{n\to\infty} \sup_{x\in C} \|J_\lambda x - J_{\lambda_n} x\| = 0$.*

Theorem 4.6. *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{\lambda>0} R(I+\lambda A)$ and f be a contraction on C with coefficient $\rho \in (0,1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}(s_n x_n + (1-s_n)x_{n+1}), \quad \forall n \geqslant 1, \tag{4.3}$$

where $\{\lambda_n\}$ is a real sequence in $(0,\infty)$ with $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\}$ be the same as in Theorem 3.1. Then $\{x_n\}$ defined by (4.3) converges strongly to a point $p \in A^{-1}0$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z-p)\rangle \leqslant 0, \quad \forall z \in A^{-1}(0).$$

Proof. Since $(\{J_{\lambda_n}\}, J_\lambda)$ satisfies the AKTT-condition, by following the proof line in Theorem 4.3, we can conclude the desired conclusion immediately. □

5. Numerical examples

In this section, we present two numerical experiments to support the main result.

Example 5.1. Let $E = C = \mathbb{R}^2$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, where $x_i, y_i \in \mathbb{R}$ for $i = 1,2$. Let $\langle \cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the inner product defined by $\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2$ and let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be the usual norm defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\mathbf{x}) = \frac{1}{4}\mathbf{x}$. For each $t \geqslant 0$, let $T_t : \mathbb{R}^2 \to \mathbb{R}^2$ be a u.a.r. nonexpansive semigroup defined by

$$T_t\mathbf{x} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}.$$

It is not hard to see that $\bigcap_{t\geqslant 0} F(T_t) = p = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$. Let $t_n = \frac{n}{2}$, $s_n = \frac{n}{n+1}$, $\alpha_n = \frac{1}{50n+1}$, $\beta_n = \frac{n}{50n+1}$, we have $\gamma_n = \frac{49n}{50n+1}$. So our algorithm (3.1) has the following form:

$$\begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \end{pmatrix} = \frac{5n+1}{200n+4}\begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} + \frac{49n}{50n+1}\begin{pmatrix} e^{-n} & 0 \\ 0 & 1 \end{pmatrix}\left[\frac{n}{n+1}\begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} + \frac{1}{n+1}\begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \end{pmatrix}\right], \quad \forall n \geqslant 1.$$

Choose $\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ be the initial point. Then, we obtain the numerical results shown in Table 1 and Figure 1.

Table 1: Numerical results of Example 5.1.

n	$\mathbf{x}^n = (x_1^n, x_2^n)^T$	$\|\mathbf{x}_n - p\|$
1	$(2.00000000, 3.00000000)^T$	2.00000000
2	$(0.49707287, 3.00000000)^T$	0.49707287
3	$(0.05745620, 3.00000000)^T$	0.05745620
4	$(0.00337702, 3.00000000)^T$	0.00337702
5	$(0.00012027, 3.00000000)^T$	0.00012027
6	$(3.18033\text{e-}06, 3.00000000)^T$	3.18033e-06
7	$(7.26788\text{e-}08, 3.00000000)^T$	7.26788e-08
8	$(1.55817\text{e-}09, 3.00000000)^T$	1.55817e-09
9	$(3.25133\text{e-}11, 3.00000000)^T$	3.25133e-11
10	$(6.70392\text{e-}13, 3.00000000)^T$	6.70392e-13

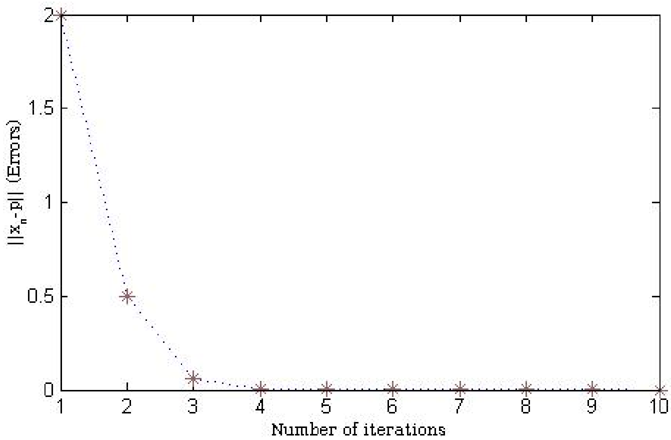

Figure 1: Behavior of convergence error values.

Example 5.2. Let $E = C = \mathbb{R}^3$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$, and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, where $x_i, y_i \in \mathbb{R}$ for $i = 1, 2, 3$. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the inner product defined by $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$ and let $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be the usual norm defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}$. For each $t \geqslant 0$, let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be a u.a.r. nonexpansive semigroup defined by

$$T_t \mathbf{x} = e^{-t} \begin{pmatrix} \cos\sqrt{2}t & \sin\sqrt{2}t & 0 \\ -\sin\sqrt{2}t & \cos\sqrt{2}t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

It is not hard to see that $\bigcap_{t \geqslant 0} F(T_t) = p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Let $t_n = 2n$, $s_n = \frac{1}{2}$, $\alpha_n = \frac{1}{2n+1}$, $\beta_n = \frac{n}{2n+1}$, we have $\gamma_n = \frac{n}{2n+1}$. So our algorithm (3.1) has the following form:

$$\begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \end{pmatrix} = \begin{pmatrix} 0.5x_1^n \\ 0.5x_2^n \\ 0.5x_3^n \end{pmatrix} + \frac{ne^{-2n}}{2n+1} \begin{pmatrix} \cos 2\sqrt{2}n & \sin 2\sqrt{2}n & 0 \\ -\sin 2\sqrt{2}n & \cos 2\sqrt{2}n & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0.5x_1^n \\ 0.5x_2^n \\ 0.5x_3^n \end{pmatrix} + \begin{pmatrix} 0.5x_1^{n+1} \\ 0.5x_2^{n+1} \\ 0.5x_3^{n+1} \end{pmatrix} \right], \quad \forall n \geqslant 1.$$

Choose $\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ be the initial point. Then, we obtain the numerical results shown in Table 2 and Figures 2 and 3, respectively.

Table 2: Numerical results of Example 5.2.

n	$\mathbf{x}^n = (x_1^n, x_2^n, x_3^n)^T$	$\lVert \mathbf{x}_n - p \rVert$
1	$(1.00000000, \quad -1.00000000, \quad 2.00000000)^T$	2.44948974
2	$(0.45843057, \quad -0.47840944, \quad 1.06922917)^T$	1.25788918
3	$(0.23280903, \quad -0.23985213, \quad 0.54051127)^T$	0.63551673
4	$(0.11614084, \quad -0.11996289, \quad 0.27068651)^T$	0.31804241
5	$(0.05808722, \quad -0.05997330, \quad 0.13537353)^T$	0.15905004
6	$(0.02904268, \quad -0.02998755, \quad 0.06768886)^T$	0.07952680
7	$(0.01452138, \quad -0.01499369, \quad 0.03384457)^T$	0.03976351
8	$(0.00726069, \quad -0.00749685, \quad 0.01692230)^T$	0.01988176
9	$(0.00363035, \quad -0.00374843, \quad 0.00846115)^T$	0.00994088
10	$(0.00181517, \quad -0.00187421, \quad 0.00423057)^T$	0.00497044
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots$
20	$(1.77263\text{e-}06, \quad -1.83029\text{e-}06, \quad 4.13142\text{e-}06)^T$	4.85395e-06

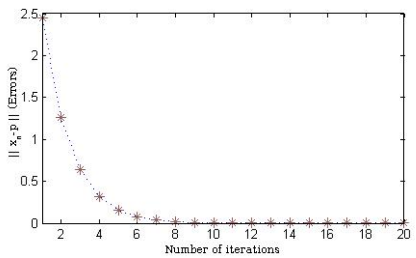

Figure 2: Behavior of convergence error values.

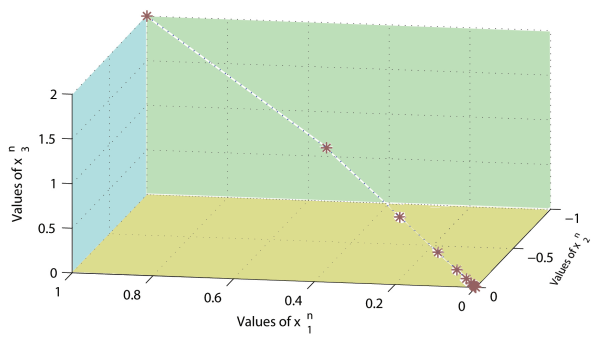

Figure 3: Behavior of convergence values of $\{x^n\}$.

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A Method for Solving the Variational Inequality Problem and Fixed Point Problems in Banach Spaces

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Abstract. The purpose of this research is to modify Halpern iteration's process for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

1 Introduction

For the last decades, fixed point theory is a very importance tool for solving the problems in economic, computer science, physics, etc. Throughout this paper, let E be a Banach space with dual space of E^* and let C be a nonempty closed convex subset of E. We use the norm of E and E^* by the same symbol $\|\cdot\|$. We denote weak and strong convergence by notations " $\rightharpoonup$ " and " $\to$ ", respectively. Let q be a given real number with $q > 1$. The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\},$$

for all $x \in E$. If $q = 2$, then $J_2 = J$ is called *normalized duality mapping.*

Remark 1. If J_q is generalized duality mapping of E into 2^{E^*}. Then the following properties are holds:

1. $J_q(tx) = t^{q-1}J_q(x)$, for all $x \in E$ and $t \in [0, \infty)$;

2. $J_q(-x) = -J_q(x)$, for all $x \in E$.

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Definition 1. Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a self-mapping. Then

1. T is called a nonexpansive mapping if
$$\|Tx - Ty\| \leq \|x - y\|,$$
for all $x, y \in C$.

2. T is called an $\eta-$strictly pseudo-contractive mapping if there exists a constant $\eta \in (0,1)$ such that
$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^2 - \eta\|(I-T)x - (I-T)y\|^2, \tag{1.1}$$
for every $x, y \in C$ and for some $j_q(x-y) \in J_q(x-y)$. It is clear that (1.1) is equivalent to the following
$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq \eta\|(I-T)x - (I-T)y\|^2, \tag{1.2}$$
for every $x, y \in C$ and for some $j_q(x-y) \in J_q(x-y)$.

Definition 2. Let $C \subseteq E$ be closed convex and Q_C be a mapping of E onto C. The mapping Q_C is said to be sunny if $Q_C(Q_Cx + t(x - Q_Cx)) = Q_Cx$, for all $x \in E$ and $t \geq 0$. A mapping Q_C is called retraction if $Q_C^2 = Q_C$. A subset C of E is called a sunny nonexpansive retraction of E if there exists a sunny nonexpansive retraction from E onto C.

For more information about (sunny) nonexpansive retraction can be found in [13].

The modulas of smootheness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by
$$\rho_E(\tau) = \{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}. \tag{1.3}$$
A Banach space E is uniformly smooth if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. It is well known that E is q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. In a Hilbert space, $L_p(l_p)$ with $1 < p < \infty$ are q-uniformly smooth. Clearly every q-uniformly smooth Banach space is uniformly smooth. If E is smooth, then J_q is a single valued which is denoted by j_q.

An operator A of C into E is said to be *accretive* if there exists $j_q(x-y) \in J_q(x-y)$ such that
$$\langle Ax - Ay, j_q(x-y)\rangle \geq 0, \ \forall x, y \in C.$$
A mapping $A : C \to E$ is said to be α*-inverse strongly accretive* if there exists $j_q(x-y) \in J_q(x-y)$ and $\alpha > 0$ such that
$$\langle Ax - Ay, j_q(x-y)\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.4}$$

Remark 2. From (1.2) and (1.4), if T is an η-strictly pseudo-contractive mapping, then $I - T$ is η-inverse strongly accretive.

Let C be a nonemty subset of q-uniformly smooth Banach space E and $A : C \to E$ be a nonlinear operator. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, J_q(y - x^*)\rangle \geq 0, \ \ \forall y \in C, \tag{1.5}$$

where J_q is generalized duality mapping from E into 2^{E^*}. The set of solutions of the variational inequality in Banach space is denoted by $S_q(C, A)$. If $q = 2$, then $S_q(C, A)$ is reduced to $S(C, A)$, where $S(C, A)$ is the set of solutions of the generalized variational inequality in Banach spaces proposed by Aoyama et. al,. [1] in 2005. Many research papers have increasingly investigated variational inequality problems in Banach spaces, see, for instance, [2], [3], and the references therein.

In 1967, Halpern [4] introduced the *Halpern's iterative method* as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \forall n \geq 1,$$

where $\alpha_n \in (0, 1)$ satisfying suitable conditions, for all $n \geq 1$. He proved that the sequence $\{x_n\}$ converges strongly to a fixed point of mapping T in a real Hilbert space, where T is a nonexpansive mapping. In the last decade, many authors have studied and modified Halpern's iterative method for various nonlinear mappings, see, for instance, [5], [6], [7], [8] and the references therein.

In a uniformly convex and 2-smooth Banach space, Aoyama *et al.* [1] introduced the iterative method for finding a solution of generalized variational inequality problem for an inverse strongly accretive operator in a uniformly convex and 2-uniformly smooth Banach space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \forall n \geq 1,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0, 1]$, Q_C is a sunny nonexpansive retraction from E onto C, A is an α-inverse strongly accretive operator. Under suitable conditions, They also proved that the sequence generated by the proposed algorithm weakly converges to a solution of $S(C, A)$.

In 2013, Kangtunyakarn [9] introduced an iterative scheme for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings and two sets of solutions of variational inequality problems in a uniformly convex and 2-smooth Banach space as follows:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S^A x_n, \forall n \geq 1,$$

where A, B are α and β-inverse strongly accretive mappings, respectively, Q_C is a sunny nonexpansive retraction, S^A is the S^A-mapping generated by a finite family of nonexpansive mappings and a finite family of strictly pseudo-contractive mappings and finite real numbers. He also proved a strong convergence theorem of sequence $\{x_n\}$ under suitable conditions of the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{\eta_n\}$.

Motivated by the results of Aoyama *et al.* [1], Kangtunyakarn [9] and by the ongoing research in this direction, we have the following question.

Question Can we prove a strong convergence theorem of two nonlinear mapping in q-uniformly smooth Banach space ?

The purpose of this manuscript is to modify Halpern iteration's process in order to answer the question above and prove a strong convergence theorem for finding a common element of the set of solutions of (1.5) and the set of fixed points of a strictly pseudo contractive mapping in q-uniformly smooth Banach space. We also introduce a new technique to prove a strong convergence theorem for a finite family of strictly pseudo contractive mappings in q-uniformly smooth Banach space. Moreover, we give a numerical result to illustrate the main theorem.

2 Preliminaries

The following lemmas are important tool to prove our main results in the next section.

Lemma 2.1. *Let E be a Banach space and let $J_q : E \to 2^{E^*}$, $1 < q < \infty$ be the generalized duality mapping. Then for any $x, y \in E$, there exists $j_q(x+y) \in J_q(x+y)$ such that $\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle$.*

Lemma 2.2. *[10] Let C be a closed and convex subset of a real uniformly smooth Banach space E and $T : C \to C$ a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansie retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty}\langle u - Q_{F(T)}u, J_q(x_n - Q_{F(T)}u)\rangle \leq 0,$$

for any given $u \in C$.

Lemma 2.3. *[11] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *[12] Let $q > 1$ be a given real number and E be a real Banach space. Then the following statements are equivalent.*

(i) *E is q-uniformly smooth.*

(ii) *There is a constant $C_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q.$$

(iii) *There exists a constant d_q such that for all $x, y \in E$ and $t \in [0, 1]$,*

$$\|(1-t)x + ty\|^q \geq (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x - y\|^q,$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^{q\cdot}$

Lemma 2.5. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a λ-strictly pseudo contractive mapping with $F(T)\bigcap F(S) \neq \emptyset$. For every $a \in (0, 1)$, defined the mapping $H : C \to C$ by $Hx = T((1-a)I + aS)x$, for all $x \in C$ and $a \in (0, \mu)$ where $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, C_q is the best q-uniformly smooth constant of E. Then $F(H) = F(T)\bigcap F(S)$.*

Proof. It is obvious that $F(T)\bigcap F(S) \subseteq F(H)$. Let $x_0 \in F(H)$ and $x^* \in F(T)\bigcap F(S)$, we have

$$\begin{aligned}
\|x_0 - x^*\|^q =& \|T((1-a)I + aS)x_0 - x^*\|^q \\
\leq& \|x_0 - x^* + a(Sx_0 - x_0)\|^q \\
\leq& \|x_0 - x^*\|^q + aq\langle Sx_0 - x_0, j_q(x_0 - x^*)\rangle + C_q a^q\|Sx_0 - x_0\|^q \\
=& \|x_0 - x^*\|^q + aq\langle Sx_0 - x^*, j_q(x_0 - x^*)\rangle + aq\langle x^* - x_0, j_q(x_0 - x^*)\rangle \\
&+ C_q a^q\|Sx_0 - x_0\|^q \\
\leq& \|x_0 - x^*\|^q + aq(\|x_0 - x^*\|^q - \lambda\|x_0 - Sx_0\|^q) - aq\|x^* - x_0\|^q \\
&+ C_q a^q\|Sx_0 - x_0\|^q \\
=& \|x_0 - x^*\|^q - a(q\lambda - C_q a^{q-1})\|x_0 - Sx_0\|^q.
\end{aligned} \tag{2.1}$$

From above it implies that $x_0 \in F(S)$. From the definition of H, we have

$$x_0 = Hx_0 = T((1-a)I + aS)x_0 = Tx_0.$$

Then $x_0 \in F(T)$. We can conclude that $x_0 \in F(S)\bigcap F(T)$. Hence $F(H) \subseteq F(S)\bigcap F(T)$. Applying (2.1), we have H is a nonexpansive mapping. $\square$

Example 1. Let $S : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $Sx = \frac{x^2}{x+1}$, for all $x \in \mathbb{R}^+$ and let $T : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $Tx = \frac{3x}{4}$, for all $x \in [0, 5]$. Define the mapping $H : \mathbb{R}^+ \to \mathbb{R}^+$ by $Hx = T(\frac{9}{10}I + \frac{1}{10}S)x$ for all $x \in \mathbb{R}^+$. From Lemma 2.5, we have $F(H) = F(S) \bigcap F(T) = \{0\}$

Lemma 2.6. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let $j, j_q : E \to E^*$ be a normalized duality mapping and generalized duality mapping, respectively. Let Q_C be a retraction from E onto C. Then the following are equivalent.*

(i) *Q_C is both sunny and nonexpansive,*

(ii) *$\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$, for all $x \in E$ and $y \in C$,*

(iii) *$\langle x - Q_C x, J_q(y - Q_C x)\rangle \leq 0$, for all $x \in E$ and $y \in C$.*

Proof. From [13], we have $(i) \Leftrightarrow (ii)$. Then we only show that (ii) equivalent to (iii). Since $J_q(x) = \|x\|^{q-1}J(x)$, for all $x \in E$. For every $x \in E$ and $y \in C$.
If $y - Q_C x \neq 0$, we have

$$\langle x - Q_C x, J_q(y - Q_C x)\rangle \leq 0 \Leftrightarrow \langle x - Q_C x, J(y - Q_C x)\rangle \leq 0.$$

If $y - Q_C x = 0$, we have

$$\langle x - Q_C x, J_q(y - Q_C x)\rangle = \langle x - Q_C x, J(y - Q_C x)\rangle = 0.$$

From above we can conclude the desire result. □

Remark 3. Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let $x \in E$, $x_0 \in C$. From Lemma 2.6, we have

$$x_0 = Q_C x \Leftrightarrow \langle x - x_0, J_q(y - x_0)\rangle \leq 0, \forall y \in C.$$

Lemma 2.7. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $A : C \to E$ be a mapping. Then $S_q(C, A) = F(Q_C(I - \lambda A))$, for all $\lambda > 0$, where $S_q(C, A) = \{u \in C : \langle Au, J_q(y - u)\rangle \geq 0, \forall y \in C\}$.*

Proof. Let $x^* \in F(Q_C(I - \lambda A))$, for all $\lambda > 0$. Then $x^* = Q_C(I - \lambda A)x^*$. From 2.6, we have

$$\langle (I - \lambda A)x^* - x^*, J_q(y - x^*)\rangle \leq 0, \forall y \in C.$$

It follows that

$$\langle Ax^*, J_q(y - x^*)\rangle \geq 0, \forall y \in C.$$

Then $x^* \in S_q(C, A)$. Hence $F(Q_C(I - \lambda A)) \subseteq S_q(C, A)$. Similarly, we can conclude that $S_q(C, A) \subseteq F(Q_C(I - \lambda A))$.

□

3 Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $S : C \to C$ be λ-strictly pseudo contractive mapping and $A : C \to E$ be a α-inverse strongly accretive operator with $\mathcal{F} = F(S) \bigcap S_q(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) Q_C(I - \rho A)(aI + (1-a)S)x_n, \forall n \in \mathbb{N}, \tag{3.1}$$

where $\alpha_n \in [0, 1]$, $a \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where C_q is the q-uniformly smooth constant of E;*

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is a unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we show that $Q_C(I - \rho A)$ is a nonexpansive mapping. Let $x, y \in C$, we have

$$\begin{aligned}
\|Q_C(I-\rho A)x - Q_C(I-\rho A)y\|^q \leq& \|x - y - \rho(Ax - Ay)\|^q \\
\leq& \|x-y\|^q - \rho q \langle Ax - Ay, j_q(x-y)\rangle + C_q \rho^q \|Ax - Ay\|^q \\
\leq& \|x-y\|^q - \rho q \alpha \|Ax - Ay\|^q + C_q \rho^q \|Ax - Ay\|^q \\
\leq& \|x-y\|^q - \rho(q\alpha - C_q \rho^{q-1}) \|Ax - Ay\|^q \\
\leq& \|x-y\|^q.
\end{aligned}$$

Then $Q_C(I - \rho A)$ is a nonexpansive mapping. Next we show that the sequence $\{x_n\}$ is bounded. Put $Wx = Q_C(I - \rho A)(aI + (1-a)S)x$, for all $x \in C$. From Lemma 2.5 and 2.7, we have

$$F(W) = F(Q_C(I - \rho A)) \bigcap F(S) = S_q(C, A) \bigcap F(S)$$

and W is a nonexpansive mapping. From (3.1), we can rewrite that

$$x_{n+1} = \alpha_n u + (1-\alpha_n) W x_n, \forall n \in \mathbb{N}. \tag{3.2}$$

Let $x^* \in \mathcal{F}$ and the definition of x_n, we have

$$\|x_{n+1} - x^*\| \leq \alpha_n \|u - x^*\| + (1-\alpha_n)\|W x_n - x^*\|$$

$$\begin{aligned}&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\|\\ &\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.\end{aligned}$$

Applying induction, we have $\{x_n\}$ is bounded. From the definition of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1} - x_n\| \leq &|\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|Wx_n - Wx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Wx_{n-1}\|\\ \leq &|\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Wx_{n-1}\|.\end{aligned}$$

Since $\{x_n\}$ is bounded sequnce, the condition (iv) and Lemma 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.3}$$

From (3.2), we have

$$x_{n+1} - x_n = \alpha_n(u - x_n) + (1-\alpha_n)(Wx_n - x_n). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\lim_{n\to\infty}\|Wx_n - x_n\| = 0. \tag{3.5}$$

From Lemma 2.2 and (3.5), we have

$$\limsup_{n\to\infty}\langle u - z_0, j_q(x_n - z_0)\rangle \leq 0, \tag{3.6}$$

where $z_0 = Q_{\mathcal{F}}u$. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - z_0\|^q \leq &\|\alpha_n(u - x^*) + (1-\alpha_n)(Wx_n - z_0)\|^q\\ \leq &(1-\alpha_n)\|x_n - z_0\|^q + q\alpha_n\langle u - z_0, j_q(x_{n+1} - z_0)\rangle.\end{aligned}$$

From Lemma 2.3 and (3.6), we have the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$. □

By using the method of proof in Theorem 3.1, we have the following theorems.

Theorem 3.2. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $S : C \to C$ be be λ-strictly pseudo contractive mapping and $T : C \to E$ be a nonexpansive mapping with $\mathcal{F} = F(S)\bigcap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(aI + (1-a)S)x_n, \forall n \in \mathbb{N}, \tag{3.7}$$

where $\alpha_n \in [0,1]$, $a \in (0,1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E*;*

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = Q_{\mathcal{F}}u$*, where* $Q_{\mathcal{F}}$ *is a unique sunny nonexpansive retraction of* C *onto* $\mathcal{F}$*.*

Proof. Applying the method of Theorem 3.1 and Lemma 2.5, we can conclude the desired result. □

4 Application

In this section, we use the main results to obtain fixed points theorems for a finite family of strictly pseuso contractive mappings in q-uniformly smooth Banach space. Before prove this theorems, we need the following results.

Lemma 4.1. *[14] Let* E *be a smooth Banach space and* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that for each* $i \in \Lambda$*,* $T_i : C \to C$ *is a* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$*. Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*, then* $\sum_{i=1}^N \eta_i T_i : C \to C$ *is a* λ_i*-strict pseudocontraction with* $\lambda = min\{\lambda_i : 1 \leq i \leq N\}$*.*

Lemma 4.2. *[14] Let* E *be a smooth Banach space and* C *be a nonempty convex subset of* E*. Given an integer* $N \geq 1$*, assume that for each* $i \in \Lambda$*,* $\{T_i\}_{i=1}^N : C \to C$ *is a finite family of* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$ *such that* $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$*. Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*. Then* $F(\sum_{i=1}^N \eta_i T_i) = F$

Theorem 4.1. *Let* C *be a nonempty closed convex subset of* q*-uniformly smooth Banach space* E *and let* Q_C *be a sunny nonexpansive retraction from* E *onto* C*. Let* $T_i : C \to C$ *is a* λ_i*-strict pseudocontraction for some* $0 \leq \lambda_i < 1$ *and* $A : C \to E$ *be a* α*-inverse strongly accretive operator with* $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \bigcap S_q(C, A) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by* $x_1, u \in C$ *and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q_C (I - \rho A)(aI + (1 - a)\sum_{i=1}^{N} \eta_i T_i) x_n, \forall n \in \mathbb{N}, \tag{4.1}$$

where $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$*,* $\alpha_n \in [0, 1]$*,* $a \in (0, 1)$ *and* $\rho > 0$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $a \in (0, \mu)$*, where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$*, where* C_q *is the* q*-uniformly smooth constant of* E*;*

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. From Theorem 3.1, Lemma 4.1 and 4.2, we can conclude the desired result. □

Lemma 4.3. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let $S : C \to C$ be κ-strictly pseudo contractive mapping with $F(S) \neq \emptyset$. Then $F(S) = S_q(C, I - S)$.*

Proof. Obvious that $F(S) \subseteq S_q(C, I - S)$. Let $x_0 \in S_q(C, I - S)$ and $x^* \in F(S)$. Then

$$\langle (I - S)x_0, j_q(y - x_0)\rangle \geq 0, \forall y \in C.$$

Put $A = I - S$. Since $S : C \to C$ is κ-strictly pseudo contractive mapping, then there exists $j_q(x_0 - x^*)$ such that

$$\begin{aligned}\langle Sx_0 - Sx^*, j_q(x_0 - x^*)\rangle =& \langle (I - A)x_0 - (I - A)x^*, j_q(x_0 - x^*)\rangle \\ =& \langle x_0 - x^*, j_q(x_0 - x^*)\rangle - \langle Ax_0 - Ax^*, j_q(x_0 - x^*)\rangle \\ =& \|x_0 - x^*\|^q - \langle (I - S)x_0, j_q(x_0 - x^*)\rangle \\ \leq& \|x_0 - x^*\|^q - \kappa\|(I - S)x_0\|^q.\end{aligned}$$

It implies that

$$\kappa\|(I - S)x_0\|^q \leq \langle (I - S)x_0, j_q(x_0 - x^*)\rangle \leq 0.$$

Then $x_0 \in F(S)$. Hence $S_q(C, I - S) \subseteq F(S)$. □

Corollary 4.2. *Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $T_i : C \to C$ is a λ_i-strictly pseudo contractive mapping for some $0 \leq \lambda_i < 1$ and $S : C \to E$ be a α-strictly pseudo contractive mapping with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \bigcap F(S) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Q_C(I - \rho(I - S))(aI + (1 - a)\sum_{i=1}^{N} \eta_i T_i)x_n, \forall n \in \mathbb{N}, \tag{4.2}$$

where $\{\eta_i\}_{i=1}^{N}$ is a positive sequence such that $\sum_{i=1}^{N} \eta_i = 1$, $\alpha_n \in [0, 1]$, $a \in (0, 1)$ and $\rho > 0$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $a \in (0, \mu)$, *where* $\mu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *where* C_q *is the* q*-uniformly smooth constant of* E;

(iii) $0 < \rho < (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converses strongly to* $z_0 = Q_{\mathcal{F}}u$, *where* $Q_{\mathcal{F}}$ *is a unique sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. From Theorem 4.1 and Lemma 4.3, we can conclude the desired result. □

5 Example and Numerical results

In this section, we give numerical results to illustrate the main theorem.

Example 2. Let $\mathbb{R}$ be a set of real number. Let $S : [0, 10] \to [0, 1]$ be a mapping defined by $Sx = \frac{2x^2}{x+2x}$, for all $x \in [0, 1]$ and let $A : [0, 10] \to \mathbb{R}$ defined by $Ax = 3x^2$ for all $x \in [0, 10]$. Suppose the sequence $\{x_n\}$ generated by (3.1), where $\alpha_n = \frac{1}{60n}$, $\rho = \frac{1}{100}$, and $a = \frac{1}{80}$. Then the sequence $\{x_n\}$ converses strongly to 0.
Solution. It is obvious that S is $\frac{1}{50}$-strictly pseudo contractive mapping and A is $\frac{1}{60}$-inverse strongly accretive operator with $F(S) \bigcap S_2(C, A) = \{0\}$. Since $\{x_n\}$ generated by (3.1), we have

$$x_{n+1} = \frac{1}{60n}u + \left(1 - \frac{1}{60n}\right) Q_{[0,10]} \left(I - \frac{1}{100}A\right) \left(\frac{1}{80}I + (1 - \frac{1}{80})S\right) x_n, \tag{5.1}$$

where $u, x_1 \in [0, 10]$. It is easy to see that α_n, for all $n \geq 1$, a, ρ satisfied all condition in Theorem 3.1. From Theorem 3.1, we have the sequence $\{x_n\}$ coonvergence strongly to 0.
Putting $u = 0.55$ and $x_1 = 0.99$ in (5.1), we have the numerical results as shown in the following Figure 1 and Table 1.

n	x_n
1	0.990000
2	0.649212
3	0.430824
4	0.287969
5	0.193551
$\vdots$	$\vdots$
46	0.000650
47	0.000635
48	0.000621
49	0.000607
50	0.000594

Table 1: The values of the sequences $\{x_n\}$ with initial values $u = 0.55$, $x_1 = 0.99$ and $n = N = 50$.

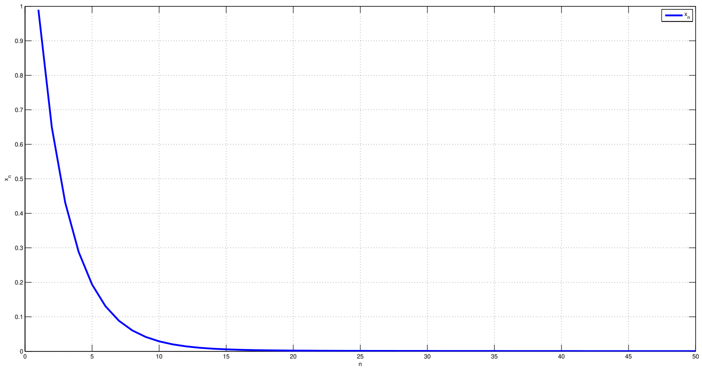

Figure 1: The behavior of the sequences $\{x_n\}$ with initial values $u = 0.55$, $x_1 = 0.99$ and $n = N = 50$.

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The Modification of Generalized Mixed Equilibrium Problems for Convergence Theorem of Variational Inequality Problems and Fixed Point Problems

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Abstract The purpose of this research, we modify generalized mixed equilibrium problems and prove a strong convergence theorem for approximating a common element of the set of such a problem and variational inequality problems and the set of fixed points of infinite family of strictly pseudo contractive mappings. Utilizing our main result, we also prove a strong convergence theorem involving generalized equilibrium problems and variational inequality problems.

MSC: 47H09; 47H10; 90C33

Keywords: strictly pseudo contractive mapping; generalized mixed equilibrium problem; inverse-strongly monotone

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1. Introduction

Throughout this article, we assume that H is a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$.

Definition 1.1. Let $T : C \to C$ be a nonlinear mapping, then

(1) T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C,$$

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(2) T is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|, \forall x \in C \text{ and } \forall p \in Fix(T),$$

(3) T is said to be *κ-strictly pseudo-contractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in C.$$

Note that the class of strictly pseudo-contractive mappings includes the class of nonexpansive mappings. The mapping T is nonexpansive if and only if T is 0-strictly pseudo contractive.

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

A mapping A is said to be *ρ-strongly monotone* if there exists a positive real number ρ such that

$$\langle Ax - Ay, x - y\rangle \geq \rho \|x - y\|^2,$$

for all $x, y \in C$.

The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, A)$. The application of the variational inequality problem has been expanded to problems from economics, finance, optimization and game theory. Many authors have studied the variational inequality problem, see for instance [1] and [2].

Let $F : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* (see [3]), is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \tag{1.2}$$

for all $y \in C$. The set of solution of (1.2) is denoted by

$$GMEP(F, \varphi, A) = \{x \in C : F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}.$$

Generalized mixed equilibrium problem has been studied by many authors, see for example [4], [5], [6] and [7]. If $\varphi = 0$, then (1.2) reduces to *the generalized equilibrium problem*, that is,

$$EP(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x\rangle \geq 0, \forall y \in C\}. \tag{1.3}$$

If $A = 0$, then problem (1.3) reduces to *the equilibrium problem*, that is,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \tag{1.4}$$

Optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem can be applied with the equilibrium problem. Many authors have introduced iterative algorithms in order to solve the equilibrium problem, see for instance [8], [9] and [10].

In 2005, Combettes and Hirstoaga [10] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. By using the viscosity approximation method, Takahashi and Takahashi [8] introduced an iteration for finding a common element of the set $EP(A)$ and $Fix(T)$ and proved a strong convergence theorem in a Hilbert space. In 2008, Takahashi and Takahashi [11] introduced another iterative scheme for finding the common element of the set $EP(F, A)$ and $Fix(T)$.

Recently, Kangtunyakarn [12] modified the set of solutions of generalized equilibrium problem as follows:

$$\begin{aligned} &EP(F, aA + (1-a)B) \\ &\quad = \{x \in C : F(x, y) + \langle (aA + (1-a)B)x, y - x\rangle \geq 0, \forall y \in C, a \in (0,1)\}. \end{aligned} \tag{1.5}$$

He introduced an iterative scheme for finding a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and the set of solution of (1.5) as follows:

$$\begin{aligned} &F(u_n, y) + \langle (aA + (1-a)B)x_n, y - x_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \gamma(I - T))u_n, \forall n \geq 1, \end{aligned} \tag{1.6}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions.

Let $D_1, D_2 : C \to H$ be two nonlinear mappings. Motivated by (1.2) and (1.5), we modify the set of solution of generalized mixed equilibrium problem as follows:

$$\begin{aligned} GMEP(F, \varphi, aD_1 + (1-a)D_2) = \{x \in C : F(x, y) + \varphi(y) - \varphi(x) \\ + \langle (aD_1 + (1-a)D_2)x, y - x\rangle \geq 0\}, \end{aligned} \tag{1.7}$$

for all $y \in C$ and $a \in (0,1)$. If $D_1 = D2$, then $GMEP(F, \varphi, aD_1 + (1-a)D_2)$ is reduced to (1.2).

In this research, we modify generalized mixed equilibrium problems and prove the strong convergence theorem for approximating a common element of the set of such a problem and variational inequality problem and the set of fixed points of infinite family of a strictly pseudo contractive mappings. Based on main result, we prove a strong convergence theorem involving generalized equilibrium problems and variational inequality problems.

2. Preliminaries

In this paper, we denote weak and strong convergence by the notations " $\rightharpoonup$ " and " $\to$ ", respectively. In a real Hilbert space H, recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

For a proof of the main theorem, we will use the following lemmas.

Lemma 2.1. [13] *Given $x \in H$ and $y \in C$, then $P_C x = y$ if and only if we have the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.2. [14] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1): $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2): $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$..

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *Let H be a real Hilbert space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.4. [13] *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I-\lambda A)u \Leftrightarrow u \in VI(C,A),$$

where P_C is the metric projection of H onto C.

Definition 2.5. [15] Let C be a nonempty convex subset of a real Hilbert space. Let $T_i, i = 1, 2, ...$ be mappings of C into itself. For each $j = 1, 2, ...,$ let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$. Define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I \\
U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I \\
U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I \\
&\vdots \\
U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I \\
U_{n,k} &= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I \\
&\vdots \\
U_{n,2} &= \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I \\
S_n &= U_{n,1} = \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I.
\end{aligned}$$

Such mapping is called *S-mapping* generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$.

Lemma 2.6. [16] *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then T satisfies the Lipschitz condition;*

$$\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x-y\|, \forall x, y \in C.$$

For finding solutions of the equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a lower semicontinuous and convex function satisfies the following conditions:
$(A1)$ $F(x,x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\left(tz + (1-t)x, y\right) \leq F(x,y);$$

$(A4)$ for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous;
$(B1)$ for each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\left\langle y_x - z, z - x\right\rangle < 0;$$

$(B2)$ C is a bounded set.

Lemma 2.7. [17] *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfies $(A1)-(A4)$, $A : C \to H$ be a continuous monotone mapping, and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$ then there exists $z \in C$ such that*

$$F(z,y) + \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle.$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{z \in C : F(z,y) + \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\} \tag{2.1}$$

for all $x \in H$. Then the following conclusions hold:

(1) For each $x \in H, T_r \neq \emptyset$;
(2) T_r is single-valued;
(3) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) $Fix(T_r) = GMEP(F, \varphi, A)$
(5) $GMEP(F, \varphi, A)$is closed and convex.

Lemma 2.8. [15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Then, for every $x \in C$ and $k \in \mathbb{N}, \lim_{n\to\infty} U_{n,k}x$ exists.*

For every $k \in \mathbb{N}$ and $x \in C$. [15] defined mapping $U_{\infty,k}$ and $S : C \to C$ as follows:

$$\lim_{n\to\infty} U_{n,k}x = U_{\infty,k}x \tag{2.2}$$

and

$$\lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x = Sx. \tag{2.3}$$

Such a mapping S is called *S-mapping* generated by $T_n, T_{n-1}, ...$ and $\alpha_n, \alpha_{n-1}, ...$.

Remark 2.9. [15] For every $n \in \mathbb{N}$, S_n is nonexpansive and $\lim_{n\to\infty} sup_{x\in D}\|S_n x - Sx\| = 0$, for every bounded subset D of C.

Lemma 2.10. [15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, ...$ and $\alpha_n, \alpha_{n-1}, ...$, respectively. Then $Fix(S) = \bigcap_{i=1}^{\infty} Fix(T_i)$.*

Lemma 2.11. [18] *Let C be a nonempty closed convex subset of a real Hilbert space H.Let A, B be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \bigcap VI(C,B) \neq \emptyset$. Then*

$$VI(C, aA + (1-a)B) = VI(C, A) \bigcap VI(C, B), \forall a \in (0,1). \tag{2.4}$$

Furthermore if $0 \leq \gamma \leq min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is nonexpansive mapping.

Remark 2.12. From Lemma (2.4) and Lemma (2.11), we have

$$VI(C, aA+(1-a)B) = VI(C, A) \bigcap VI(C, B) = Fix(P_C(I - \gamma(aA+(1-a)B))),$$

for all $a \in (0,1)$ and $\gamma > 0$.

From (1.7), we have the following result.

Lemma 2.13. *Let C be a nonempty closed convex subset of a real Hilbert space H and F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfy $A1) - A4)$ and $F(x,z) \leq F(x,y) + F(y,z)$ for all $x, y, z \in C$. Let A, B be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B) \neq \emptyset$. Then*

$$GMEP(F, \varphi, aA + (1-a)B) = GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B), \forall a \in (0,1).$$

Proof. It is obvious that $GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B) \subseteq GMEP(F, \varphi, aA + (1-a)B)$. Next, we will show that $GMEP(F, \varphi, aA + (1-a)B) \subseteq GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$. Let $x_0 \in GMEP(F, \varphi, aA + (1-a)B)$ and $x^* \in GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$, we have

$$F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, y - x_0\rangle \geq 0, \forall y \in C, \tag{2.5}$$

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C \tag{2.6}$$

and

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Bx^*, y - x^*\rangle \geq 0, \forall y \in C. \tag{2.7}$$

For every $a \in (0,1)$, we have

$$aF(x^*, y) + a\varphi(y) - a\varphi(x^*)\langle aAx^*, y - x^*\rangle \geq 0, \forall y \in C$$

and

$$(1-a)F(x^*, y) + (1-a)\varphi(y) - (1-a)\varphi(x^*) + \langle(1-a)Bx^*, y - x^*\rangle \geq 0, \forall y \in C.$$

By the monotonicity of B and $x^*, x_0 \in C$, we have

$$\begin{aligned}
aF(x_0, x^*) &+ a\varphi(x^*) - a\varphi(x_0) + \langle aAx_0, x^* - x_0\rangle \\
&= aF(x_0, x^*) + a\varphi(x^*) - a\varphi(x_0) + (1-a)\varphi(x^*) - (1-a)\varphi(x^*) \\
&\quad + (1-a)\varphi(x_0) - (1-a)\varphi(x_0) + (1-a)F(x_0, x^*) - (1-a)F(x_0, x^*) \\
&\quad + \langle aAx_0 + (1-a)Bx_0 - (1-a)Bx_0, x^* - x_0\rangle \\
&= F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \\
&\quad - (1-a)F(x_0, x^*) - (1-a)\varphi(x^*) + (1-a)\varphi(x_0) - \langle (1-a)Bx_0, x^* - x_0\rangle \\
&\geq (1-a)F(x^*, x_0) + (1-a)\varphi(x_0) - (1-a)\varphi(x^*) + (1-a)\langle Bx_0, x_0 - x^*\rangle \\
&= (1-a)(F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle Bx^*, x_0 - x^*\rangle + \langle Bx_0 - Bx^*, x_0 - x^*\rangle) \\
&\geq 0. \qquad (2.8)
\end{aligned}$$

Since $GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B) \subseteq GMEP(F, \varphi, aA + (1-a)B)$ and $x^* \in GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$, we have

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle aAx^* + (1-a)Bx^*, y - x^*\rangle \geq 0. \forall y \in C. \qquad (2.9)$$

Since $x^* \in C$ and (2.5), we have

$$F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \geq 0. \qquad (2.10)$$

From (2.9) and $x_0 \in C$, we have

$$F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle aAx^* + (1-a)Bx^*, x_0 - x^*\rangle \geq 0. \qquad (2.11)$$

Summing up (2.10), (2.11) and (A2), we have

$$\langle a(Ax^* - Ax_0) + (1-a)(Bx^* - x_0), x_0 - x^*\rangle \geq 0. \qquad (2.12)$$

Since A, B are α, β-inverse strongly monotone, respectively, and (2.12), we have

$$\begin{aligned}
0 &\leq \langle a(Ax^* - Ax_0) + (1-a)(Bx^* - Bx_0), x_0 - x^*\rangle \\
&= \langle a(Ax^* - Ax_0), x_0 - x^*\rangle + \langle (1-a)(Bx^* - Bx_0), x_0 - x^*\rangle \\
&= a\langle Ax^* - Ax_0, x_0 - x^*\rangle + (1-a)\langle Bx^* - Bx_0, x_0 - x^*\rangle \\
&\leq -a\alpha\|Ax^* - Ax_0\|^2 - (1-a)\beta\|Bx^* - Bx_0\|^2.
\end{aligned}$$

This implies that

$$0 \leq -a\alpha\|Ax^* - Ax_0\|^2.$$

It follows that

$$Ax^* = Ax_0. \qquad (2.13)$$

By using the same method as (2.13), we obtain

$$Bx^* = Bx_0. \qquad (2.14)$$

For every $y \in C$. From (2.6), (2.8), (2.13) and $x^* \in GMEP(F, \varphi, A)$, we have

$$\begin{aligned}
F(x_0, y) &+ \varphi(y) - \varphi(x_0) + \langle Ax_0, y - x_0\rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Ax_0, y - x^* + x^* - x_0\rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \varphi(x^*) - \varphi(x^*) + F(x^*, y) - F(x^*, y) \\
&\quad + \langle Ax_0, y - x^*\rangle + \langle Ax_0, x^* - x_0\rangle \\
&= F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + F(x^*, y) + \varphi(y) - \varphi(x^*) \\
&\quad + \langle Ax^*, y - x^*\rangle + \langle Ax_0, x^* - x_0\rangle \\
&\geq F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0\rangle \\
&\geq F(x_0, y) + F(y, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0\rangle \\
&\geq F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0\rangle \\
&\geq 0.
\end{aligned}$$

Then

$$x_0 \in GMEP(F, \varphi, A). \tag{2.15}$$

Since $x^*, x_0 \in C$ and (2.5), (2.13), we have

$$\begin{aligned}
(1-a)F(x_0, x^*) &+ (1-a)\varphi(x^*) - (1-a)\varphi(x_0) + \langle (1-a)Bx_0, x^* - x_0\rangle \\
&= (1-a)F(x_0, x^*) + (1-a)\varphi(x^*) - (1-a)\varphi(x_0) + aF(x_0, x^*) \\
&\quad - aF(x_0, x^*) + \langle (1-a)Bx_0 + aAx_0 - aAx_0, x^* - x_0\rangle \\
&= F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \\
&\quad - aF(x_0, x^*) + a\varphi(x_0) - a\varphi(x^*) - \langle aAx_0, x^* - x_0\rangle \\
&\geq aF(x^*, x_0) + a\varphi(x_0) - a\varphi(x^*) + \langle aAx_0, x_0 - x^*\rangle \\
&= aF(x^*, x_0) + a\varphi(x_0) - a\varphi(x^*) + a\langle Ax^*, x_0 - x^*\rangle \\
&\geq 0.
\end{aligned} \tag{2.16}$$

For every $y \in C$, from (2.7), (2.14), (2.16) and $x^* \in GMEP(F, \varphi, B)$, we have

$$\begin{aligned}
F(x_0, y) + \varphi(y) &- \varphi(x_0) + \langle Bx_0, y - x_0\rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Bx_0, y - x^*\rangle + \langle Bx_0, x^* - x_0\rangle \\
&= F(x_0, y) + \varphi(y) - \varphi(x_0) + \varphi(x^*) - \varphi(x^*) + F(x^*, y) - F(x^*, y) \\
&\quad + \langle Bx_0, y - x^*\rangle + \langle Bx_0, x^* - x_0\rangle \\
&= F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + F(x^*, y) + \varphi(y) - \varphi(x^*) \\
&\quad + \langle Bx^*, y - x^*\rangle + \langle Bx_0, x^* - x_0\rangle \\
&\geq F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0\rangle \\
&\geq F(x_0, y) + F(y, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0\rangle \\
&\geq F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0\rangle \\
&\geq 0.
\end{aligned}$$

Hence

$$x_0 \in GMEP(F, \varphi, B). \tag{2.17}$$

By (2.15) and (2.17), we have $x_0 \in GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B)$. Then

$$GMEP(F,\varphi,aA+(1-a)B) \subseteq GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B)$$

■

3. Main Result

In this section, we prove a strong convergence theorem and for the set of fixed point of strictly pseudo contractive mappings and the sets of solution of generalized mixed equilibrium problems and variational inequality problems by using Lemma 2.13.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1,F_2 be bifunctions from $C\times C$ to $\mathbb{R}$ satisfy A1)-A4) and $F_i(x,z) \le F_i(x,y)+F_i(y,z)$ for all $x,y,z \in C$ and $i=1,2$. Let $\varphi_1,\varphi_2 : C \to \mathbb{R}\bigcup\{+\infty\}$ be proper lower semicontinuous and convex function. Let A,B be α,β-inverse strongly monotone, respectively, and let D,E be L_D,L_E-Lipschitz continuous and μ,ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i)\bigcap GMEP(F_1,\varphi_1,A)\bigcap GMEP(F_1,\varphi_1,B)\bigcap GMEP(F_2,\varphi_2,A)$ $\bigcap GMEP(F_2,\varphi_2,B)\bigcap VI(C,D)\bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j,\alpha_2^j,\alpha_3^j)$ $\in I\times I\times I$ where $I=[0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$, $\alpha_1^j+\alpha_2^j \le b<1$ and $\alpha_1^j,\alpha_2^j,\alpha_3^j \in (\kappa,1)$ for all $j=1,2,\dots$. For every $n\in\mathbb{N}$, let S_n be S-mapping generated by $T_n,T_{n-1},\dots,T_1$ and $\alpha_n,\alpha_{n-1},\dots,\alpha_1$. Assume the either $B_1)$ or $B_2)$ holds. Let the sequence $\{x_n\}$ generated by $x_1,u\in C$ and*

$$\begin{cases} F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle a_nAx_n+(1-a_n)Bx_n, y-u_n\rangle \\ +\dfrac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle \ge 0, \\ F_2(v_n,y)+\varphi_2(y)-\varphi_2(v_n)+\langle a_nAx_n+(1-a_n)Bx_n, y-v_n\rangle \\ +\dfrac{1}{r_n^2}\langle y-v_n,v_n-x_n\rangle \ge 0, \forall y\in C, \\ y_n=\delta_nu_n+(1-\delta_n)v_n, \\ x_{n+1}=\alpha_nu+\beta_nx_n+\lambda_nS_nx_n+\eta_nP_C(I-\gamma_n(a_nD+(1-a_n)E))y_n, \end{cases} \tag{3.1}$$

for all $n\ge 1$, where the sequences $\{\alpha_n\},\{\beta_n\},\{\lambda_n\},\{\eta_n\},\{\delta_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\lambda_n+\eta_n=1$ for all $n\in\mathbb{N}$, $\{a_n\}\subset(0,1)$ and $\{r_n^j\}\subseteq[b,c]\subset(0,2min\{\alpha,\beta\})$ for all $j=1,2$. Suppose the following conditions hold:

(i): $\displaystyle\sum_{n=1}^{\infty}\alpha_n=\infty, \lim_{n\to\infty}\alpha_n=0, \{\beta_n\}\subseteq[d,e]\subset(0,1);$

(ii): $0<\gamma_n\le min\{\frac{2\mu}{L_D^2},\frac{2\rho}{L_E^2}\};$

(iii): $\displaystyle\lim_{n\to\infty}\delta_n=\delta\in(0,1), \sum_{n=1}^{\infty}\alpha_1^n<\infty;$

$$\textbf{(iv):} \sum_{n=1}^{\infty}\left|r_{n+1}^{j}-r_{n}^{j}\right|<\infty \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty,$$
$$\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty, \sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty,$$
$$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty, \sum_{n=1}^{\infty}|\eta_{n+1}-\eta_n|<\infty \textit{ for all } j=1,2.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$.

Proof. First, we show that D is $\frac{\mu}{L_D^2}$-inverse strongly monotone mapping. Let $x, y \in C$, we have

$$\begin{aligned}\langle x-y, Dx-Dy\rangle &\geq \mu\|x-y\|^2\\ &\geq \frac{\mu}{L_D^2}\|Dx-Dy\|^2.\end{aligned}$$

Similarly, we get E is $\frac{\rho}{L_E^2}$-inverse-strongly monotone mapping.

Next, we show that $I-\gamma_n D$ and $I-\gamma_n E$ are nonexpansive mappings. For every $x, y \in C$, we have

$$\begin{aligned}\|(I-\gamma_n D)x-(I-\gamma_n D)y\|^2 &= \|x-y\|^2+\gamma_n^2\|Dx-Dy\|^2-2\gamma_n\langle x-y, Dx-Dy\rangle\\ &\leq \|x-y\|^2+\gamma_n^2\|Dx-Dy\|^2-\frac{2\gamma_n\mu}{L_D^2}\|Dx-Dy\|^2\\ &= \|x-y\|^2+\gamma_n\left(\gamma_n-\frac{2\mu}{L_D^2}\right)\|Dx-Dy\|^2\\ &\leq \|x-y\|^2.\end{aligned}$$

Then we obtain $I-\gamma_n D$ is a nonexpansive mapping. Similarly, we can show that $I-\gamma_n E$ is also a nonexpansive mapping.

The proof of Theorem 3.1 will be divided into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded.
From (3.1) and Lemma 2.7, we have $u_n = T_{r_n^1}(I-r_n^1(a_nA+(1-a_n)B))x_n$ and $v_n = T_{r_n^2}(I-r_n^2(a_nA+(1-a_n)B))x_n$.
From Lemma 2.7 and Lemma 2.13, we have

$$\begin{aligned}F\left(T_{r_n^1}\left(I-r_n^1\left(a_nA+(1-a_n)B\right)\right)\right) &= GMEP(F_1,\varphi_1,a_nA+(1-a_n)B)\\ &= GMEP(F_1,\varphi_1,A)\bigcap GMEP(F_1,\varphi_1,B)\end{aligned}$$

and

$$\begin{aligned}F\left(T_{r_n^2}\left(I-r_n^2\left(a_nA+(1-a_n)B\right)\right)\right) &= GMEP(F_2,\varphi_2,a_nA+(1-a_n)B)\\ &= GMEP(F_2,\varphi_2,A)\bigcap GMEP(F_2,\varphi_2,B).\end{aligned}$$

Let $z \in \mathcal{F}$. From Lemma 2.4 and Lemma 2.11, we have

$$z \in VI(C, a_nD+(1-a_n)E) = Fix(P_C(I-\gamma_n(a_nD+(1-a_n)E))).$$

From the nonexpansiveness of $T_{r_n^1}$, $T_{r_n^2}$ and Lemma 2.11, we have

$$\begin{aligned}
\|x_{n+1}-z\| &\leq \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|S_nx_n-z\| \\
&\quad + \eta_n\|P_C(I-\gamma_n(a_nDx+(1-a_n)E))y_n-z\| \\
&\leq \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| + \eta_n\|y_n-z\| \\
&= \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| \\
&\quad + \eta_n\|\delta_n(u_n-z)+(1-\delta_n)(v_n-z)\| \\
&= \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| \\
&\quad + \eta_n\|\delta_n(T_{r_n^1}(I-r_n^1(a_nA+(1-a_n)B))x_n-z) \\
&\quad + (1-\delta_n)(T_{r_n^2}(I-r_n^2(a_nA+(1-a_n)B))x_n-z)\| \\
&\leq \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| \\
&\quad + \eta_n(\delta_n\|x_n-z\| + (1-\delta_n)\|x_n-z\|) \\
&= \alpha_n \|u-z\| + \beta_n\|x_n-z\| + \lambda_n\|x_n-z\| + \eta_n\|x_n-z\| \\
&= \alpha_n \|u-z\| + (1-\alpha_n)\|x_n-z\|. \qquad (3.2)
\end{aligned}$$

Put $M = max\{\|u-z\|, \|x_1-z\|\}$. From induction, we can show that $\|x_n-z\| \leq M$, for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. For every $n \in \mathbb{N}$, put $J_n = a_nD+(1-a_n)E$ and $G_n = a_nA+(1-a_n)B$. From the definition of x_n and the nonexpansivenss of $Pc(I-\gamma_nJ_n)$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + \lambda_n\|S_nx_n-S_{n-1}x_{n-1}\| + |\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\quad + \|\eta_nP_C(I-\gamma_nJ_n)y_n - \eta_{n-1}P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + \lambda_n\|S_nx_n-S_nx_{n-1}\| + \lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\| \\
&\quad + |\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| + \eta_n\|P_C(I-\gamma_nJ_n)y_n - P_C(I-\gamma_nJ_n)y_{n-1}\| \\
&\quad + \eta_n\|P_C(I-\gamma_nJ_n)y_{n-1} - P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\quad + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + \lambda_n\|x_n-x_{n-1}\| \\
&\quad + \lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\| + |\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\quad + \eta_n\|y_n-y_{n-1}\| + \eta_n\|P_C(I-\gamma_nJ_n)y_{n-1} - P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\quad + |\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|. \qquad (3.3)
\end{aligned}$$

Since $y_n = \delta_nu_n + (1-\delta_n)v_n$, we have

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|\delta_nu_n + (1-\delta_n)v_n - \delta_{n-1}u_{n-1} - (1-\delta_{n-1})v_{n-1}\| \\
&= \|\delta_n(u_n-u_{n-1}) + (\delta_n-\delta_{n-1})u_{n-1} + (1-\delta_n)(v_n-v_{n-1}) \\
&\quad + (\delta_{n-1}-\delta_n)v_{n-1}\| \\
&\leq \delta_n\|u_n-u_{n-1}\| + |\delta_n-\delta_{n-1}|\|u_{n-1}\| + (1-\delta_n)\|v_n-v_{n-1}\| \\
&\quad + |\delta_n-\delta_{n-1}|\|v_{n-1}\|. \qquad (3.4)
\end{aligned}$$

From the nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C(I-\gamma_nJ_n)y_{n-1}-P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| &\\ \leq \|(I-\gamma_nJ_n)y_{n-1}-(I-\gamma_{n-1}J_{n-1})y_{n-1}\| &\\ = \|\gamma_nJ_ny_{n-1}-\gamma_{n-1}J_{n-1}y_{n-1}\| &\\ \leq \gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\|+a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\| &\\ +\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\|+(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\|. &\end{aligned} \tag{3.5}$$

Substitute (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \leq{}& |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|\\ &+\lambda_n\|x_n-x_{n-1}\|\\ &+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\|\\ &+\eta_n\delta_n\|u_n-u_{n-1}\|+\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n\|v_n-v_{n-1}\|\\ &+\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\|\\ &+\eta_na_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\|\\ &+\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\|\\ &+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|.\end{aligned} \tag{3.6}$$

By the same method as Theorem 3.1 in [18], we have

$$\|S_nx_{n-1}-S_{n-1}x_{n-1}\| \leq \alpha_1^n\frac{2}{1-\kappa}\|x_{n-1}-z\|. \tag{3.7}$$

Since $u_n=T_{r_n^1}(I-r_n^1G_n)x_n$ where $G_n=a_nA+(1-a_n)B$. From the definition of T_{r_n}, we have

$$F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle G_nx_n,y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle \geq 0 \tag{3.8}$$

and

$$\begin{aligned}F_1(u_{n+1},y)+\varphi_1(y)-\varphi_1(u_{n+1})+\langle G_{n+1}x_{n+1},y-u_{n+1}\rangle&\\ +\frac{1}{r_{n+1}^1}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle \geq 0,&\end{aligned} \tag{3.9}$$

for all $y\in C$.
From (3.8) and (3.9), we have

$$\begin{aligned}F_1(u_n,u_{n+1})+\varphi_1(u_{n+1})-\varphi_1(u_n)+\langle G_nx_n,u_{n+1}-u_n\rangle&\\ +\frac{1}{r_n^1}\langle u_{n+1}-u_n,u_n-x_n\rangle \geq 0.&\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}F_1(u_{n+1},u_n)+\varphi_1(u_n)-\varphi_1(u_{n+1})+\langle G_{n+1}x_{n+1},u_n-u_{n+1}\rangle&\\ +\frac{1}{r_{n+1}^1}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle \geq 0.&\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$F_1(u_n, u_{n+1}) + \varphi_1(u_{n+1}) - \varphi_1(u_n) + \frac{1}{r_n^1}\langle u_{n+1} - u_n, u_n - x_n + r_n^1 G_n x_n\rangle \geq 0 \tag{3.12}$$

and

$$\begin{aligned} &F_1(u_{n+1}, u_n) + \varphi_1(u_n) - \varphi_1(u_{n+1}) \\ &\qquad + \frac{1}{r_{n+1}^1}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} + r_{n+1}^1 G_{n+1} x_{n+1}\rangle \geq 0. \end{aligned} \tag{3.13}$$

Summing up (3.12) and (3.13), we have

$$\begin{aligned} &\frac{1}{r_n^1}\langle u_{n+1} - u_n u_n - x_n + r_n^1 G_n x_n\rangle \\ &\qquad + \frac{1}{r_{n+1}^1}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} + r_{n+1}^1 G_{n+1} x_{n+1}\rangle \geq 0. \end{aligned}$$

It follows that

$$\langle u_{n+1} - u_n, \frac{u_n - (I - r_n^1 G_n)x_n}{r_n^1} - \frac{u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}}{r_{n+1}^1}\rangle \geq 0.$$

This implies that

$$\begin{aligned} 0 &\leq \langle u_{n+1} - u_n, u_n - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1}\rangle \\ &\quad + \langle u_{n+1} - u_n, u_{n+1} - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, u_{n+1} - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle \\ &= \langle u_{n+1} - u_n, (I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n \\ &\quad + \left(1 - \frac{r_n^1}{r_{n+1}^1}\right)(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\big(\|(I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n\| \\ &\quad + \left|1 - \frac{r_n^1}{r_{n+1}^1}\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\|\big). \end{aligned}$$

Then

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \|(I-r^1_{n+1}G_{n+1})x_{n+1}-(I-r^1_nG_n)x_n\| \\
&\quad+\frac{1}{r^1_{n+1}}\left|r^1_{n+1}-r^1_n\right|\|u_{n+1}-(I-r^1_{n+1}G_{n+1})x_{n+1}\| \\
&\le \|(I-r^1_{n+1}G_{n+1})x_{n+1}-(I-r^1_{n+1}G_{n+1})x_n\| \\
&\quad+\|(I-r^1_{n+1}G_{n+1})x_n-(I-r^1_nG_n)x_n\| \\
&\quad+\frac{1}{r^1_{n+1}}\left|r^1_{n+1}-r^1_n\right|\|u_{n+1}-(I-r^1_{n+1}G_{n+1})x_{n+1}\| \\
&\le \|x_{n+1}-x_n\|+\|r^1_{n+1}G_{n+1}x_n-r^1_nG_nx_n\| \\
&\quad+\frac{1}{r^1_{n+1}}\left|r^1_{n+1}-r^1_n\right|\|u_{n+1}-(I-r^1_{n+1}G_{n+1})x_{n+1}\| \\
&\le \|x_{n+1}-x_n\| \\
&\quad+\|r^1_{n+1}(a_{n+1}A+(1-a_{n+1})B)x_n-r^1_{n+1}(a_nA+(1-a_n)B)x_n\| \\
&\quad+|r^1_{n+1}-r^1_n|\|G_nx_n\|+\frac{1}{r^1_{n+1}}\left|r^1_{n+1}-r^1_n\right|\|u_{n+1}-(I-r^1_{n+1}G_{n+1})x_{n+1}\| \\
&= \|x_{n+1}-x_n\| \\
&\quad+\|r^1_{n+1}(a_{n+1}-a_n)Ax_n+r^1_{n+1}((1-a_{n+1})-(1-a_n))Bx_n\| \\
&\quad+|r^1_{n+1}-r^1_n|\|G_nx_n\|+\frac{1}{r^1_{n+1}}\left|r^1_{n+1}-r^1_n\right|\|u_{n+1}-(I-r^1_{n+1}G_{n+1})x_{n+1}\| \\
&\le \|x_{n+1}-x_n\|+r^1_{n+1}|a_{n+1}-a_n|\|Ax_n\|+r^1_{n+1}|a_{n+1}-a_n|\|Bx_n \\
&\quad+|r^1_{n+1}-r^1_n|\|G_nx_n\|+\frac{1}{b}\left|r^1_{n+1}-r^1_n\right|\|u_{n+1}-(I-r^1_{n+1}G_{n+1})x_{n+1}\|.
\end{aligned} \tag{3.14}$$

From (3.14), we have

$$\begin{aligned}
\|u_n-u_{n-1}\| &\le \|x_n-x_{n-1}\|+r^1_n|a_n-a_{n-1}|\|Ax_{n-1}\|+r^1_n|a_n-a_{n-1}|\|Bx_{n-1}\| \\
&\quad+|r^1_n-r^1_{n-1}|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r^1_n-r^1_{n-1}\right|\|u_n-(I-r^1_nG_n)x_n\|.
\end{aligned} \tag{3.15}$$

By using the same method as (3.15), we have

$$\begin{aligned}
\|v_n-v_{n-1}\| &\le \|x_n-x_{n-1}\|+r^2_n|a_n-a_{n-1}|\|Ax_{n-1}\|+r^2_n|a_n-a_{n-1}|\|Bx_{n-1}\| \\
&\quad+|r^2_n-r^2_{n-1}|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r^2_n-r^2_{n-1}\right|\|v_n-(I-r^2_nG_n)x_n\|.
\end{aligned} \tag{3.16}$$

Substitute (3.7), (3.15) and (3.16) into (3.6), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| \leq\; & |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
& +\lambda_n\|x_n-x_{n-1}\|+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
& +\eta_n\delta_n\|u_n-u_{n-1}\|+\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n\|v_n-v_{n-1}\| \\
& +\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\| \\
& +\eta_na_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\| \\
& +\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\| \\
& +|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
\leq\; & |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\lambda_n\|x_n-x_{n-1}\| \\
& +\lambda_n\left(\alpha_1^n\frac{2}{1-\kappa}\|x_{n-1}-z\|\right)+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
& +\eta_n\delta_n(\|x_n-x_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Bx_{n-1}\| \\
& +|r_n^1-r_{n-1}^1|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|\|u_n-(I-r_n^1G_n)x_n\|) \\
& +\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n(\|x_n-x_{n-1}\| \\
& +r_n^2|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^2|a_n-a_{n-1}|\|Bx_{n-1}\| \\
& +|r_n^2-r_{n-1}^2|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|\|v_n-(I-r_n^2G_n)x_n\|) \\
& +\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\| \\
& +\eta_na_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\| \\
& +\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\| \\
& +|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
\leq\; & (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+|\beta_n-\beta_{n-1}|M_1 \\
& +\alpha_1^n\frac{2}{1-\kappa}M_1+|\lambda_n-\lambda_{n-1}|M_1 \\
& +c|a_n-a_{n-1}|M_1+c|a_n-a_{n-1}|M_1+|r_n^1-r_{n-1}^1|M_1+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|M_1 \\
& +|\delta_n-\delta_{n-1}|M_1+c|a_n-a_{n-1}|M_1+c|a_n-a_{n-1}|\|M_1+|r_n^2-r_{n-1}^2|M_1 \\
& +\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|M_1+|\delta_n-\delta_{n-1}|M_1+|a_n-a_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_1 \\
& +|a_n-a_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_1+|\eta_n-\eta_{n-1}|M_1,
\end{aligned}$$

where $M_1 := max_{n\in\mathbb{N}}\{\|u\|,\|x_n\|,\|x_n-z\|,\|S_nx_n\|,\|Ax_n\|,\|Bx_n\|,\|G_nx_n\|,$ $\|\|u_n-(I-r_n^1G_n)x_n\|,\|u_n\|,\|v_n-(I-r_n^2G_n)x_n\|,\|v_n\|,\|Dy_n\|,\|Ey_n\|,$ $\|P_C(I-\gamma_nJ_n)y_n\|\}$. From the conditions (ii), (iv) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.17}$$

Step 3. We show that $\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|v_n-x_n\|=\lim_{n\to\infty}\|y_n-x_n\|=$ $\lim_{n\to\infty}\|P_C(I-\gamma_nJ_n)y_n-x_n\|=0$. Let $z\in\mathcal{F}$. Since $u_n=T_{r_n^1}(I-r_n^1G_n)x_n$, $v_n=$

$T_{r_n^2}(I-r_n^2G_n)x_n$ and T_{r_n} is a firmly nonexpensive mapping, we have

$$\begin{aligned}
\|T_{r_n^1}(I-r_n^1G_n)x_n))-z\|^2 &= \|T_{r_n^1}(I-r_n^1G_n)x_n - T_{r_n^1}(I-r_n^1G_n)z\|^2\\
&\le \langle (I-r_n^1G_n)x_n-(I-r_n^1G_n)z, u_n-z\rangle\\
&= \frac{1}{2}(\|(I-r_n^1G_n)x_n-(I-r_n^1G_n)z\|^2+\|u_n-z\|^2\\
&\quad -\|(I-r_n^1G_n)x_n-(I-r_n^1G_n)z-u_n+z\|^2)\\
&\le \frac{1}{2}(\|x_n-z\|^2+\|u_n-z\|^2-\|(x_n-u_n)-r_n^1(G_nx_n-G_nz)\|^2)\\
&= \frac{1}{2}(\|x_n-z\|^2+\|u_n-z\|^2-\|(x_n-u_n)\|^2-(r_n^1)^2\|G_nx_n-G_nz\|^2\\
&\quad +2r_n^1\langle x_n-T_{r_n^1}(I-r_n^1G_n)x_n, G_nx_n-G_nz\rangle)\\
&\le \frac{1}{2}(\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2-(r_n^1)^2\|G_nx_n-G_nz\|^2\\
&\quad +2r_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_n-z\|^2 &\le \|x_n-z\|^2-\|x_n-u_n\|^2-(r_n^1)^2\|G_nx_n-G_nz\|^2\\
&\quad +2r_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|. \qquad (3.18)
\end{aligned}$$

Applying (3.18) and $v_n=T_{r_n^2}(I-r_n^2G_n)x_n$, we have

$$\begin{aligned}
\|v_n-z\|^2 &\le \|x_n-z\|^2-\|x_n-v_n\|^2-(r_n^2)^2\|G_nx_n-G_nz\|^2\\
&\quad +2r_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|. \qquad (3.19)
\end{aligned}$$

For every $x,y\in C$, we have

$$\begin{aligned}
\langle G_nx_n-G_nz, x_n-z\rangle &= \langle (a_nA+(1-a_n)B)\,x_n-(a_nA+(1-a_n)B)\,z, x_n-z\rangle\\
&= \langle a_n\,(Ax_n-Az)+(1-a_n)\,(Bx_n-Bz)\,, x_n-z\rangle\\
&= a_n\,\langle Ax_n-Az, x_n-z\rangle+(1-a_n)\,\langle Bx_n-Bz, x_n-z\rangle\\
&\ge a_n\alpha\,\|Ax_n-Az\|^2+(1-a_n)\beta\,\|Bx_n-Bz\|^2. \qquad (3.20)
\end{aligned}$$

From the definition of u_n and (3.20), we have

$$\begin{aligned}
\|u_n-z\|^2 &= \|T_{r_n^1}(I-r_n^1G_n)x_n-T_{r_n^1}(I-r_n^1G_n)z\|^2\\
&\le \|(I-r_n^1G_n)x_n-(I-r_n^1G_n)z\|^2\\
&= \|x_n-z\|^2-2r_n^1\langle x_n-z, G_nx_n-G_nz\rangle+(r_n^1)^2\|(G_nx_n-G_nz)\|^2\\
&\le \|x_n-z\|^2-2r_n^1a_n\alpha\,\|Ax_n-Az\|^2-2r_n^1(1-a_n)\beta\,\|Bx_n-Bz\|^2\\
&\quad +(r_n^1)^2\|a_n(Ax_n-Az)+(1-a_n)(Bx_n-Bz)\|^2\\
&\le \|x_n-z\|^2-2r_n^1a_n\alpha\,\|Ax_n-Az\|^2-2r_n^1(1-a_n)\beta\,\|Bx_n-Bz\|^2\\
&\quad +(r_n^1)^2a_n\|Ax_n-Az\|^2+(1-a_n)(r_n^1)^2\|Bx_n-B)\|^2\\
&\le \|x_n-z\|^2-r_n^1a_n(2\alpha-r_n^1)\,\|Ax_n-Az\|^2\\
&\quad -r_n^1(1-a_n)(2\beta-r_n^1)\,\|Bx_n-Bz\|^2. \qquad (3.21)
\end{aligned}$$

Applying (3.21) and $v_n = T_{r_n^2}(I - r_n^2 G_n)x_n$, we have

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - r_n^2 a_n(2\alpha - r_n^2)\|Ax_n - Az\|^2 - r_n^2(1 - a_n)(2\beta - r_n^2)\|Bx_n - Bz\|^2. \tag{3.22}$$

From the definition of x_n, (3.21) and (3.22), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|S_n x_n - z\|^2 \\ &\quad + \eta_n\|P_C(I - \gamma_n(a_n D + (1 - a_n)E))y_n - z)\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 + \eta_n\|y_n - z\|^2 \\ &= \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n\|\delta_n(u_n - z) + (1 - \delta_n)(v_n - z)\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n(\delta_n\|u_n - z\|^2 + (1 - \delta_n)\|v_n - z\|^2)\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n(\delta_n(\|x_n - z\|^2 - r_n^1 a_n(2\alpha - r_n^1)\|Ax_n - Az\|^2 \\ &\quad - r_n^1(1 - a_n)(2\beta - r_n^1)\|Bx_n - Bz\|^2) + (1 - \delta_n)(\|x_n - z\|^2 \\ &\quad - r_n^2 a_n(2\alpha - r_n^2)\|Ax_n - Az\|^2 - r_n^2(1 - a_n)(2\beta - r_n^2)\|Bx_n - Bz\|^2)) \\ &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \eta_n a_n(r_n^1\delta_n(2\alpha - r_n^1) + r_n^2(1 - \delta_n)(2\alpha - r_n^2))\|Ax_n - Az\|^2 \\ &\quad - (1 - a_n)\eta_n(r_n^1\delta_n(2\beta - r_n^1) + r_n^2(1 - \delta_n)(2\beta - r_n^2))\|Bx_n - Bz\|^2.\end{aligned} \tag{3.24}$$

From (3.24), we have

$$\begin{aligned}\eta_n a_n(r_n^1\delta_n(2\alpha - r_n^1) &+ r_n^2(1 - \delta_n)(2\alpha - r_n^2))\|Ax_n - Az\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (\|x_n - z\|^2 + \|x_{n+1} - z\|)(\|x_{n+1} - x_n\|).\end{aligned}$$

From the condition (i) and (3.17), we have

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.25}$$

By using the same method as (3.25), we have

$$\lim_{n\to\infty}\|Bx_n - Bz\| = 0. \tag{3.26}$$

Since $G_n = a_n A + (1 - a_n)B$, we obtain

$$\|G_n x_n - G_n z\| \leq a_n\alpha\|Ax_n - Az\|^2 + (1 - a_n)\beta\|Bx_n - Bz\|^2.$$

From (3.25) and (3.26), we have

$$\lim_{n\to\infty}\|G_n x_n - G_n z\| = 0. \tag{3.27}$$

From (3.21), (3.22), (3.23) and the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2\\
&\quad +\eta_n(\delta_n\|u_n-z\|^2+(1-\delta_n)\|v_n-z\|^2)\\
&\le \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2\\
&\quad +\eta_n(\delta_n(\|x_n-z\|^2-\|x_n-u_n\|^2-(r_n^1)^2\|G_nx_n-G_nz\|^2\\
&\quad +2r_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|)\\
&\quad +(1-\delta_n)(\|x_n-z\|^2-\|x_n-v_n\|^2-(r_n^2)^2\|G_nx_n-G_nz\|^2\\
&\quad +2r_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|))\\
&\le \alpha_n\|u-z\|^2+\|x_n-z\|^2-\eta_n\delta_n\|x_n-u_n\|^2-(1-\delta_n)\eta_n\|x_n-v_n\|^2\\
&\quad -\eta_n(\delta_n(r_n^1)^2+(1-\delta_n)(r_n^2)^2\|G_nx_n-G_nz\|^2\\
&\quad +2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|\\
&\quad +2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\eta_n\delta_n\|u_n-x_n\|^2 &\le \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\
&\quad +2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|\\
&\quad +2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|\\
&\le \alpha_n\|u-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)(\|x_{n+1}-x_n\|)\\
&\quad +2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|\\
&\quad +2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|.
\end{aligned}$$

From the condition (i), (3.17) and (3.27), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.28}$$

By using the same method as (3.28), we have

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \tag{3.29}$$

From the definition of y_n, we have

$$\begin{aligned}
\|y_n-x_n\| &= \|\delta_nu_n+(1-\delta_n)v_n-x_n\|\\
&\le \delta_n\|u_n-x_n\|+(1-\delta_n)\|v_n-x_n\|.
\end{aligned}$$

From (3.28) and (3.29), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.30}$$

From the nonexpansiveness of P_C and $z \in \mathcal{F}$, we have

$$\begin{aligned}
\|P_C(I-\gamma_n J_n)y_n - z\|^2 &= \|P_C(I-\gamma_n J_n)y_n - P_C(I-\gamma_n J_n)z\|^2 \\
&\le \|(I-\gamma_n J_n)y_n - (I-\gamma_n J_n)z\|^2 \\
&= \|y_n - z - \gamma_n(J_n y_n - J_n z)\|^2 \\
&= \|y_n - z\|^2 - 2\gamma_n \langle y_n - z, J_n y_n - J_n z\rangle + \gamma_n^2 \|J_n y_n - J_n z\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma_n \langle y_n - z, J_n y_n - J_n z\rangle + \gamma_n^2 \|J_n y_n - J_n z\|^2 .
\end{aligned} \tag{3.31}$$

For every $x, y \in C$, we have

$$\begin{aligned}
\langle J_n x - J_n y, x - y\rangle &= a_n \langle Dx - Dy, x - y\rangle + (1-a_n)\langle Ex - Ey, x - y\rangle \\
&\ge a_n \frac{\mu}{L_D^2}\|Dx - Dy\|^2 + (1-a_n)\frac{\rho}{L_E^2}\|Ex - Ey\|^2 .
\end{aligned} \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\begin{aligned}
\|P_C(I-\gamma_n J_n)y_n - z\|^2 &\le \|x_n - z\|^2 - 2\gamma_n \langle y_n - z, J_n y_n - J_n z\rangle + \gamma_n^2 \|J_n y_n - J_n z\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma_n a_n \frac{\mu}{L_D^2}\|Dy_n - Dz\|^2 \\
&\quad - 2\gamma_n(1-a_n)\frac{\rho}{L_E^2}\|Ey_n - Ez\|^2 \\
&\quad + \gamma_n^2 \|a_n(Dy_n - Dz) + (1-a_n)(Ey_n - Ez)\|^2 \\
&\le \|x_n - z\|^2 - 2\gamma_n a_n \frac{\mu}{L_D^2}\|Dy_n - Dz\|^2 \\
&\quad - 2\gamma_n(1-a_n)\frac{\rho}{L_E^2}\|Ey_n - Ez\|^2 \\
&\quad + \gamma_n^2 a_n \|Dy_n - Dz\|^2 + \gamma_n^2(1-a_n)\|Ey_n - Ez\|^2 \\
&= \|x_n - z\|^2 - \gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n\right)\|Dy_n - Dz\|^2 \\
&\quad - \gamma_n(1-a_n)\left(\frac{2\rho}{L_E^2} - \gamma_n\right)\|Ey_n - Ez\|^2 .
\end{aligned} \tag{3.33}$$

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \lambda_n \|x_n - z\|^2 \\
&\quad + \eta_n \|P_C(I - \gamma_n J_{n+1})y_n - z\|^2 \\
&\le \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \lambda_n \|x_n - z\|^2 \\
&\quad + \eta_n(\|x_n - z\|^2 - \gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n\right)\|Dy_n - Dz\|^2 \\
&\quad - \gamma_n(1-a_n)\left(\frac{2\rho}{L_E^2} - \gamma_n\right)\|Ey_n - Ez\|^2)
\end{aligned}$$

$$\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \eta_n \gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n \right) \|Dy_n - Dz\|^2$$
$$- \eta_n \gamma_n (1-a_n) \left(\frac{2\rho}{L_E^2} - \gamma_n \right) \|Ey_n - Ez\|^2 .$$

It implies that

$$\begin{aligned}
\eta_n \gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n \right) \|Dy_n - Dz\|^2 &\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad - \eta_n \gamma_n (1-a_n) \left(\frac{2\rho}{L_E^2} - \gamma_n \right) \|Ey_n - Ez\|^2 \\
&\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| .
\end{aligned}$$

From the condition (i) and (3.17), we have

$$\lim_{n\to\infty} \|Dy_n - Dz\| = 0. \tag{3.34}$$

By using the same method as (3.35), we have

$$\lim_{n\to\infty} \|Ey_n - Ez\| = 0. \tag{3.35}$$

From the definition of J_n, we have

$$\|J_n y_n - J_n z\| \leq a_n \|Dy_n - Dz\| + (1-a_n) \|Ey_n - Ez\| . \tag{3.36}$$

From (3.34), (3.35) and (3.36), we have

$$\lim_{n\to\infty} \|J_n y_n - J_n z\| = 0. \tag{3.37}$$

From the definition of $P_C (I - \gamma_n J)$ and Lemma 2.11, it implies that

$$\begin{aligned}
\|P_C (I-\gamma_n J_n) y_n - z\|^2 =\ & \|P_C (I-\gamma_n J_n) y_n - P_C (I-\gamma_n J_n) z\|^2 \\
\leq\ & \langle (I-\gamma_n J_n) y_n - (I-\gamma_n J_n) z, P_C (I-\gamma_n J_n) y_n - z \rangle \\
=\ & \frac{1}{2} \Big[\|(I-\gamma_n J_n) y_n - (I-\gamma_n J_n) z\|^2 + \|P_C (I-\gamma_n J_n) y_n - z\|^2 \\
& - \|(I-\gamma_n J_n) y_n - (I-\gamma_n J_n) z - (P_C (I-\gamma_n J_n) y_n - z)\|^2 \Big] \\
\leq\ & \frac{1}{2} (\|y_n - z\|^2 + \|P_C (I-\gamma J_n) y_n - z\|^2 \\
& - \|y_n - P_C (I-\gamma_n J_n) y_n - \gamma_n (J_n y_n - J_n z)\|^2) \\
\leq\ & \frac{1}{2} (\|x_n - z\|^2 + \|P_C (I-\gamma_n J) y_n - z\|^2 \\
& - \|y_n - P_C (I-\gamma_n J_n) y_n\|^2 - \gamma_n^2 \|J_n y_n - J_n z\|^2 \\
& + 2\gamma_n \langle y_n - P_C (I-\gamma_n J_n) y_n, J_n y_n - J_n z \rangle) \\
\leq\ & \frac{1}{2} (\|x_n - z\|^2 + \|P_C (I-\gamma_n J_n) y_n - z\|^2 \\
& - \|y_n - P_C (I-\gamma_n J_n) y_n\|^2 - \gamma_n^2 \|J_n y_n - J_n z\|^2 \\
& + 2\gamma_n \|y_n - P_C (I-\gamma_n J_n) y_n\| \|J_n y_n - J_n z\|).
\end{aligned}$$

It follows that

$$\|P_C (I-\gamma_n J_n) y_n - z\|^2 \le \|x_n - z\|^2 - \|y_n - P_C (I-\gamma_n J_n) y_n\|^2 + 2\gamma \|y_n - P_C (I-\gamma_n J_n) y_n\| \|J_n y_n - J_n z\| . \quad (3.38)$$

From the definition of x_n and (3.38), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n \|u-z\|^2 + \beta_n \|x_n - z\|^2 + \lambda_n \|x_n - z\|^2 + \eta_n \|P_C(I-\gamma_n J_n)y_n - z\|^2 \\ &\le \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \lambda_n \|x_n - z\|^2 + \eta_n(\|x_n - z\|^2 \\ &\quad - \|y_n - P_C(I-\gamma_n J_n) y_n\|^2 + 2\gamma_n \|y_n - P_C(I-\gamma_n J_n)y_n\| \|J_n y_n - J_n z\|) \\ &\le \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \eta_n \|y_n - P_C (I-\gamma_n J_n) y_n\|^2 \\ &\quad + 2\eta_n\gamma_n \|y_n - P_C (I-\gamma_n J_n) y_n\| \|J_n y_n - J_n z\| .\end{aligned}$$

It implies that

$$\begin{aligned}\eta_n \|y_n - P_C (I-\gamma_n J_n) y_n\|^2 &\le \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\quad + 2\eta_n\gamma_n \|y_n - P_C (I-\gamma_n J_n) y_n\| \|J_n y_n - J_n z\| \\ &\le \alpha_n \|u-z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|) \|x_{n+1} - x_n\| \\ &\quad + 2\eta_n\gamma_n \|y_n - P_C (I-\gamma_n J_n) y_n\| \|J_n y_n - J_n z\| .\end{aligned} \quad (3.39)$$

From the condition (i), (3.17), (3.37) and (3.39), we obtain

$$\lim_{n\to\infty} \|y_n - P_C (I-\gamma_n J_n) y_n\| = 0. \quad (3.40)$$

Since

$$\|x_n - P_C (I-\gamma_n J_n) y_n\| \le \|x_n - y_n\| + \|y_n - P_C (I-\gamma_n J_n) y_n\| ,$$

from (3.30) and (3.40), we have

$$\lim_{n\to\infty} \|x_n - P_C (I-\gamma_n J_n) y_n\| = 0. \quad (3.41)$$

Step 4. We show that $\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0\rangle \le 0$ where $z_0 = P_{\mathcal{F}} u$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0\rangle . \quad (3.42)$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$ where $\omega \in C$. From (3.30), we obtain $y_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (3.28), we have $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Assume $\omega \notin VI(C,D) \bigcap VI(C,E)$. From Lemma 2.11 and Lemma 2.4, we have

$$VI(C,D) \bigcap VI(C,E) = VI(C, J_{n_k}) = Fix(P_C(I-\gamma_{n_k} J_{n_k})).$$

From the nonexpansiveness of $P_C (I-\gamma_{n_k} J_{n_k})$, (3.41) and Opial's condition , we obtain

$$\begin{aligned}\lim_{k\to\infty} \inf \|y_{n_k} - \omega\| &< \lim_{k\to\infty} \inf \|y_{n_k} - P_C(I-\gamma_{n_k} J_{n_k})\omega\| \\ &\le \lim_{k\to\infty} \inf \|y_{n_k} - P_C(I-\gamma_{n_k} J_{n_k})y_{n_k}\| \\ &\quad + \lim_{k\to\infty} \inf \|P_C(I-\gamma_n J_{n_k})y_{n_k} - P_C(I-\gamma_{n_k} J_{n_k})\omega\| \\ &\le \lim_{k\to\infty} \inf \|y_{n_k} - \omega\| .\end{aligned}$$

This is a contradiction. Hence

$$\omega \in VI(C,D)\bigcap VI(C,E). \tag{3.43}$$

From the definition of x_n, we have

$$x_{n+1}-x_n=\alpha_n(u-x_n)+\lambda_n(S_nx_n-x_n)+\eta_n(P_C(I-\gamma_nJ_n)y_n-x_n).$$

From the condition (i), (3.15) and (3.41), we have

$$\lim_{n\to\infty}\|S_nx_n-x_n\|=0. \tag{3.44}$$

Assume $\omega\notin\bigcap_{i=1}^{\infty}Fix(T_i)$. From Lemma 2.10 , we have $Fix(S)=\bigcap_{i=1}^{\infty}Fix(T_i)$. Then $\omega\notin Fix(S)$. From Remark 2.9, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-S\omega\|\\ &\le \liminf_{k\to\infty}(\|x_{n_k}-S_{n_k}x_{n_k}\|+\|S_{n_k}x_{n_k}-S_{n_k}\omega\|+\|S_{n_k}\omega-S\omega\|)\\ &\le \liminf_{k\to\infty}\|x_{n_k}-\omega\|.\end{aligned}$$

This is a contradiction. Then

$$\omega\in Fix(S)=\bigcap_{i=1}^{\infty}Fix(T_i). \tag{3.45}$$

Since

$$F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle G_nx_n,y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle\ge 0,\forall y\in C,$$

from (A2), we have

$$\varphi_1(y)-\varphi_1(u_n)+\langle G_nx_n,y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle\ge F_1(y,u_n),\forall y\in C.$$

In particular

$$\varphi_1(y)-\varphi_1(u_{n_i})+\langle G_{n_i}x_{n_i},y-u_{n_i}\rangle+\frac{1}{r_{n_i}^1}\langle y-u_{n_i},u_{n_i}-x_{n_i}\rangle\ge F_1(y,u_{n_i}),\forall y\in C.$$

It follows that

$$\varphi_1(y)-\varphi_1(u_{n_i})+\langle G_{n_i}x_{n_i},y-u_{n_i}\rangle+\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle\ge F_1(y,u_{n_i}). \tag{3.46}$$

For $0<t\le 1$ and $y\in C$, let $y_t=ty+(1-t)\omega$. From (3.46), we have

$$\begin{aligned}&\varphi_1(y_t)-\varphi_1(u_{n_i})+\langle y_t-u_{n_i},G_{n_i}y_t\rangle\\ &\qquad\ge \langle y_t-u_{n_i},G_{n_i}y_t\rangle-\langle y_t-u_{n_i},G_{n_i}x_{n_i}\rangle\\ &\qquad\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle+F_1(y_t,u_{n_i})\\ &\qquad=\langle y_t-u_{n_i},G_{n_i}y_t-G_{n_i}u_{n_i}+G_{n_i}u_{n_i}\rangle-\langle y_t-u_{n_i},G_{n_i}x_{n_i}\rangle\\ &\qquad\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle+F_1(y_t,u_{n_i})\\ &\qquad=\langle y_t-u_{n_i},G_{n_i}y_t-G_{n_i}u_{n_i}\rangle+\langle y_t-u_{n_i},G_{n_i}u_{n_i}-G_{n_i}x_{n_i}\rangle\\ &\qquad\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}^1}\rangle+F_1(y_t,u_{n_i}).\end{aligned}$$

Since $\|u_{n_i}-x_{n_i}\| \to 0$, we have $\|G_{n_i}u_{n_i}-G_{n_i}x_{n_i}\| \to 0$. Since $\frac{u_{n_i}-x_{n_i}}{r^1_{n_i}} \to 0$ and $\langle y_t-u_{n_i}, G_{n_i}y_t-G_{n_i}u_{n_i}\rangle \geq 0$ and (A4), we have

$$\varphi_1(y_t)-\varphi_1(\omega)+\langle y_t-\omega, G_{n_i}y_t\rangle \geq F_1(y_t,\omega). \tag{3.47}$$

Form (A1), (A4) and (3.47), we have

$$\begin{aligned}
0 &= F_1(y_t,y_t)+\varphi_1(y_t)-\varphi_1(y_t)\\
&\leq tF_1(y_t,y)+(1-t)F_1(y_t,\omega)+t\varphi_1(y)+(1-t)\varphi_1(\omega)-\varphi_1(y_t)\\
&\leq tF_1(y_t,y)+(1-t)\varphi_1(y_t)-(1-t)\varphi_1(\omega)+(1-t)\langle y_t-\omega, G_{n_i}y_t\rangle\\
&\quad +t\varphi_1(y)+(1-t)\varphi_1(\omega)-\varphi_1(y_t)\\
&= tF_1(y_t,y)+t\varphi_1(y)-t\varphi_1(y_t)+(1-t)\langle ty+(1-t)\omega-\omega, G_{n_i}y_t\rangle\\
&= tF_1(y_t,y)+t\varphi_1(y)-t\varphi_1(y_t)+(1-t)t\langle y-\omega, G_{n_i}y_t\rangle.
\end{aligned}$$

Dividing by t, we have

$$0 \leq F_1(y_t,y)+\varphi_1(y)-\varphi_1(y_t)+(1-t)\langle y-\omega, G_{n_i}y_t\rangle.$$

Letting $t \to 0$, it follows from (A3), we have

$$0 \leq F_1(\omega,y)+\varphi_1(y)-\varphi_1(\omega)+\langle y-\omega, G_{n_i}\omega\rangle, \forall y \in C. \tag{3.48}$$

From Lemma 2.13, we have

$$\omega \in GMEP(F_1,\varphi_1,a_{n_i}A+(1-a_{n_i})B) = GMEP(F_1,\varphi_1,A)\bigcap GMEP(F_1,\varphi_1,B).$$

By using the same method as (3.48), we have

$$\omega \in GMEP(F_2,\varphi_2,A)\bigcap GMEP(F_2,\varphi_2,B).$$

Hence $\omega \in \mathcal{F}$. Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathcal{F}$, we have

$$\lim_{n\to\infty}\sup\,\langle u-z_0, x_n-z_0\rangle = \lim_{k\to\infty}\langle u-z_0, x_{n_k}-z_0\rangle = \langle u-z_0, \omega-z_0\rangle \leq 0. \tag{3.49}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{\mathcal{F}}u$. From the nonexpansiveness of $P_C(I-\gamma J_n)$, we have

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &= \|\alpha_n(u-z_0)+\beta_n(x_n-z_0)+\lambda_n(S_nx_n-z_0)+\eta_n(P_C(I-\gamma_nJ_n)y_n-z_0)\|^2\\
&\leq \|\beta_n(x_n-z_0)+\lambda_n(S_nx_n-z_0)+\eta_n(P_C(I-\gamma_nJ_n)y_n-z_0)\|^2\\
&\quad +2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle\\
&= (1-\alpha_n)^2\|x_n-z_0\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle\\
&\leq (1-\alpha_n)\|x_n-z_0\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle.
\end{aligned}$$

Applying Lemma 2.2 and (3.49), we have the sequence $\{x_n\}$ converse strongly to $z_0 = P_{\mathcal{F}}u$. This complete the proof. ■

Using our main theorem (Theorem 3.1), we obtain the following results.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfy $A_1) - A_4)$ and $F_i(x,z) \leq F_i(x,y) + F_i(y,z)$ for all $x,y,z \in C$ and $i = 1,2$. Let A,B be α,β-inverse strongly monotone, respectively and D,E be L_D, L_E-Lipschitz continuous and μ,ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap EP(F_1,A) \bigcap EP(F_1,B) \bigcap EP(F_2,A) \bigcap EP(F_2,B) \bigcap VI(C,D) \bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\dots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$. Assume the either $B_1)$ or $B_2)$ holds and let the sequence $\{x_n\}$ generated by $x_1, u \in C$ and*

$$\begin{aligned}
&F_1(u_n,y) + \langle a_n A x_n + (1-a_n) B x_n, y - u_n\rangle + \frac{1}{r_n^1}\langle y - u_n, u_n - x_n\rangle \geq 0,\\
&F_2(v_n,y) + \langle a_n A x_n + (1-a_n) B x_n, y - v_n\rangle + \frac{1}{r_n^2}\langle y - v_n, v_n - x_n\rangle \geq 0,\\
&y_n = \delta_n u_n + (1-\delta_n) v_n,\\
&x_{n+1} = \alpha_n u + \beta_n x_n + \lambda_n S_n x_n + \eta_n P_C(I - \gamma_n(a_n D + (1-a_n)E)) y_n, \forall n \geq 1.
\end{aligned}$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \lambda_n + \eta_n = 1$ for all $n \in \mathbb{N}$, $\{a_n\} \subset (0,1)$ and $\{r_n^j\} \subseteq [b,c] \subset (0, 2min\{\alpha,\beta\})$for all $j = 1,2$. Suppose the following conditions hold:

(i): $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,\ \{\beta_n\} \subseteq [d,e] \subset (0,1),$

(ii): $0 < \gamma_n \leq min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\},$

(iii): $\lim_{n\to\infty} \delta_n = \delta \in (0,1), \sum_{n=1}^{\infty} \alpha_1^n < \infty,$

(iv): $\sum_{n=1}^{\infty} \left| r_{n+1}^j - r_n^j \right| < \infty \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$ *for all* $j = 1,2.$

Then, the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. Put $\varphi_1 \equiv \varphi_2 \equiv 0$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. ∎

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A,B be α,β-inverse strongly monotone, respectively and D,E be L_D, L_E-Lipschitz continuous and μ,ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap VI(C,A) \bigcap VI(C,B) \bigcap VI(C,D) \bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\dots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$. Let the* *sequence* $\{x_n\}$ *generated by* $x_1, u \in C$ *and*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_n x_n + \eta_n P_C(I - \gamma_n(a_n D + (1-a_n)E))P_C(I - r_n^1(a_n A + (1-a_n)B))x_n, \forall n \geq 1.$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n + \eta_n = 1$ *for all* $n \in \mathbb{N}$ *,* $\{a_n\} \subset (0,1)$ *and* $\{r_n^1\} \subseteq [b,c] \subset (0, 2min\{\alpha,\beta\})$ *for all* $j = 1,2$. *Suppose the following conditions hold:*

(i): $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \{\beta_n\} \subseteq [d,e] \subset (0,1),$

(ii): $0 < \gamma_n \leq min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\},$

(iii): $\sum_{n=1}^{\infty} \alpha_1^n < \infty,$

(iv): $\sum_{n=1}^{\infty} |r_{n+1}^1 - r_n^1| < \infty \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$$

$$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty.$$

Then, the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$.

Proof. Putting $F_1 \equiv F_2 \equiv \varphi_1 \equiv \varphi_2 \equiv 0, r_n^1 = r_n^2$ and $v_n = u_n$ in Theorem 3.1, we have.

$$\langle y - u_n, x_n - r_n^1(a_n A x_n + (1-a_n)B x_u) - u_n \rangle, \forall y \in C.$$

It implies that

$$u_n = P_C(I - r_n^1(a_n A + (1-a_n)B))x_n.$$

So, from Theorem 3.1 and Remark 2.12, we obtain the desired result. ■

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The Rectangular Quasi-Metric Space and Common Fixed Point Theorem for ψ-Contraction and ψ-Kannan Mappings

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Abstract : In this work, we extend and improve rectangular metric spaces to rectangular quasi-metric spaces by using the concept of quasi-metric spaces. Next, we obtain fixed point theorems in rectangular quasi-metric spaces. Moreover, we present some examples to illustrate and support our results.

Keywords : fixed point; quasi-metric space; rectangular metric space; rectangular quasi-metric space.

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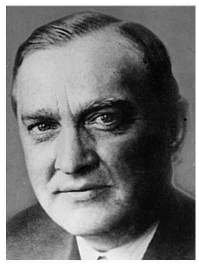
1 Introduction and Preliminaries

In 1922, Banach [1] proved a fixed point theorem for metric spaces, which later on came to be known as the famous "Banach contraction principle".



Stefan Banach

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a *contraction mapping* on X, if there exists $q \in [0, 1)$ such that

$$d(T(x), T(y)) \leq qd(x, y)$$

for all x, y in X. If (X, d) is a complete metric space with a contraction mapping $T : X \to X$, then T admits a unique fixed-point $x*$ in X. Furthermore, We can to find $x*$ as follows: We start x_0 in X and define a sequence x_n by $x_n = T(x_{n-1})$, then $x_n \to x*$. After that, we well-known to Banach Fixed Point Theorem.

Now, we recall definition of metric spaces was introduced by Frechet [2] as follows :

Definition 1.1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies :

$(MS1)$ $\quad d(x, y) = 0 \; if \; and \; only \; if \; x = y,$

$(MS2)$ $\quad d(x, y) = d(y, x) \; for \; all \; x, y \in X,$

$(MS3)$ $\quad d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X.$

If d satisfying (MS1)-(MS3), then d is called a metric on X and (X, d) is called a metric space.

Example 1.2. Let $X = \mathbb{R}$ and defined $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$. Then (X, d) is metric spaces.

In 1931, Wilson [3] introduced quasi-metric spaces as follows :

Definition 1.3. Let X be a nonempty set. Suppose that the mapping $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

$(QS1)$ $\quad d(x, y) = 0 \; if \; and \; only \; if \; x = y;$

$(QS2)$ $\quad d(x, y) \leq d(x, z) + d(z, y) \; for \; all \; x, y, z \in X.$

If d satisfies conditions (QS1) and (QS2), then d is called a quasi-metric on X and (X, d) is called a quasi-metric space.

Example 1.4. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}\}$ and $B = [1,5]$. Define the generalized metric d on X as follows :
$d(\frac{1}{2}, \frac{1}{3}) = 0.3$, $\quad d(\frac{1}{3}, \frac{1}{2}) = 0.2$, $\quad d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = 0$, and $d(x,y) = |x-y|$.
If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$,
then (X,d) is a quasi-metric space, but it is not metric space.

In 2000, Branciari [4] introduced rectangular metric spaces as follows :

Definition 1.5. Let X be a none-mpty set and Suppose that the mapping $d : X \times X \to [0,\infty)$ satisfies:

$(RMS1)$ $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
$(RMS2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;
$(RMS3)$ $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all $x, y, z \in X$ and all distinct point $u, v \in X \backslash \{x,y\}$.

Then d is called a rectangular metric on X and (X,d) is called a rectangular metric space.

Example 1.6 ([5])**.** Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$. Define the generalized metric d on X as follows :

$$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3, \qquad d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2,$$

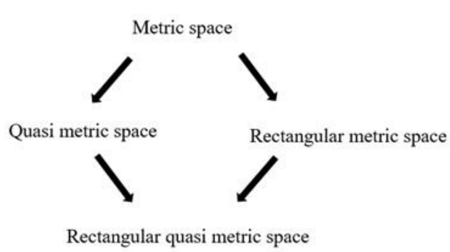
$$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6, \qquad d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0$$

and $d(x,y) = |x-y|$ if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$.
It is clear that d does not satisfy the triangle inequality in metric space,

$$0.6 = d(\frac{1}{2}, \frac{1}{4}) \geq d(\frac{1}{2}, \frac{1}{3}) + d(\frac{1}{3}, \frac{1}{4}) = 0.5.$$

Then d is a rectangular metric, but it is not a metric.

In this work, we extend and improve rectangular metric spaces to rectangular quasi-metric spaces by using the concept of quasi-metric spaces. Next, we obtain fixed point theorems in rectangular quasi-metric spaces. Moreover, we present some examples to illustrate and support our results.i.e,



2 Main Results

In this section, we introduce rectangular quasi-metric spaces and prove fixed point theorems. Likewise, we present some examples to illustrate and support our results.

Definition 2.1. Let X be a non-empty set and Suppose that the mappings $d : X \times X \longrightarrow [0, \infty)$ satisfies :
(RQMS1) $d(x, y) = 0$ if and only if $x = y$;
(RQMS2) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct points $u, v \in X \backslash \{x, y\}$.
Then d is called a rectangular quasi-metric on X and (X, d) is called a rectangular quasi-metric space.

Example 2.2. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$. Define the generalized metric d on X as follows :

$$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3, \qquad d(\frac{1}{3}, \frac{1}{2}) = d(\frac{1}{5}, \frac{1}{4}) = 0.1,$$
$$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6, \qquad d(\frac{1}{4}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{5}) = 0.4,$$
$$d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2, \qquad d(\frac{1}{5}, \frac{1}{2}) = d(\frac{1}{4}, \frac{1}{3}) = 0.5,$$
$$d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0,$$

and

$$d(x, y) = |x - y| \text{ if } x, y \in B \text{ or } x \in A, y \in B \text{ or } x \in B, y \in A .$$

It is clear that d does not satisfy the triangle inequality A

$$0.6 = d(\frac{1}{2}, \frac{1}{4}) \geq d(\frac{1}{2}, \frac{1}{3}) + d(\frac{1}{3}, \frac{1}{4}) = 0.5.$$

We see that d is not a rectangular metrics, because $d(\frac{1}{2}, \frac{1}{4}) \neq d(\frac{1}{4}, \frac{1}{2})$. So d is a rectangular quasi-metric. Indeed,
(RMQ1)
($\Rightarrow$) Suppose that $d(x, y) = 0$.
Case(I) If $x, y \in A$, then $x = y$.
Case(II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$ then $d(x, y) = |x - y| = 0$, so $x = y$.
($\Leftarrow$) Suppose that $x = y$.
To show that $d(x, y) = 0$. we prove by two case.
Case(I) If $x, y \in A$ then $d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0$.
Case(II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$ then $x - y = 0$.
Thus $d(x, y) = |x - y| = 0$.

This is a proof of (RQM1)
(RQM2)
Case (I) If $x, y \in A$ then
$d(x,y) = d(\frac{1}{2}, \frac{1}{3}) = 0.3 \le d(\frac{1}{2}, u) + d(u,v) + d(v, \frac{1}{3})$ when $u, v \in \{\frac{1}{4}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{3}, \frac{1}{2}) = 0.1 \le d(\frac{1}{3}, u) + d(u,v) + d(v, \frac{1}{2})$ when $u, v \in \{\frac{1}{4}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{3}, \frac{1}{4}) = 0.2 \le d(\frac{1}{3}, u) + d(u,v) + d(v, \frac{1}{4})$ when $u, v \in \{\frac{1}{2}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{4}, \frac{1}{3}) = 0.2 \le d(\frac{1}{4}, u) + d(u,v) + d(v, \frac{1}{3})$ when $u, v \in \{\frac{1}{2}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{4}, \frac{1}{5}) = 0.3 \le d(\frac{1}{4}, u) + d(u,v) + d(v, \frac{1}{5})$ when $u, v \in \{\frac{1}{2}, \frac{1}{3}\}$
$d(x,y) = d(\frac{1}{5}, \frac{1}{4}) = 0.1 \le d(\frac{1}{5}, u) + d(u,v) + d(v, \frac{1}{4})$ when $u, v \in \{\frac{1}{2}, \frac{1}{3}\}$.
Case (II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$, then

$$\begin{aligned} d(x,y) &= |x-y| \\ &\le |x-u| + |u-y| \\ &\le |x-u| + |u-v| + |v-y|, \end{aligned}$$

for all distinct points $u, v \in X \backslash \{x, y\}$.

Now, we introduce a definition of a convergent, cauchy, complete rectangular quasi-metric space as follows : For any $x \in X$, we define the open ball with centre x and radius $r > 0$ by

$$B_r(x); = \{y \in X | \max\{d(x,y), d(y,x)\} < r\}.$$

Definition 2.3. Let (X, d) be a rectangular quasi-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ in X is called convergence to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n)$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

(b)The sequence $\{x_n\}$ in X is called cauchy sequence in (X, d) if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p}, x_n)$, for all $p > 0$.

(c) (X, d) is called complete rectangular quasi metric space if every Cauchy sequence in X convergence to some $x \in X$.

Next, we present main theorems as follows :

Theorem 2.4. *Let (X, d) be a complete rectangular quasi-metric space. A mapping $g : X \to X$ satisfies:*

$$d(g(x), g(y)) \le \psi(d(x,y)), \tag{2.1}$$

for all $x, y \in X$, where
(i) $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and continuous functions,
(ii) $\sum_{i=n}^{\infty} \psi^i(t) + \psi^m(t^*) < \infty$ for $t, t^* > 0$ and for $m, n \in \mathbb{N}$,
(iii) $\psi(0) = 0$ and $\psi(t) < t$ for $0 < t$.
Then g has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitraty. We define a sequence $\{x_n\}$ by $x_{n+1} = gx_n$ for all $n = 0, 1, 2, \ldots,$. We will show that $\{x_n\}$ is Cauchy sequence, i.e., $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p}, x_n)$ for all $p > 0$. If $x_n = x_{n+1}$ then x_n is fixed point of g, i.e., $x_n = gx_n$. So, suppose that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \ldots$.
We consider

$$
\begin{aligned}
e_n := \ d(x_n, x_{n+1}) &= d(gx_{n-1}, gx_n) \\
&\leq \psi(d(x_{n-1}, x_n)) \\
&= \psi(d(gx_{n-2}, gx_{n-1})) \\
&\leq \psi^2(d(x_{n-2}, x_{n-1})) \\
&= \psi^2(d(gx_{n-3}, gx_{n-2})) \\
&\vdots \\
&\leq \psi^n(d(x_0, x_1)) \\
&= \psi^n(e_0), \qquad (2.2)
\end{aligned}
$$

and,

$$
\begin{aligned}
l_n := \ d(x_{n+1}, x_n) &= d(gx_n, gx_{n-1}) \\
&\leq \psi(d(x_n, x_{n-1})) \\
&= \psi(d(gx_{n-1}, gx_{n-2})) \\
&\leq \psi^2(d(x_{n-1}, x_{n-2})) \\
&= \psi^2(d(gx_{n-2}, gx_{n-3})) \\
&\vdots \\
&\leq \psi^n(d(x_1, x_0)) \\
&= \psi^n(l_0). \qquad (2.3)
\end{aligned}
$$

Since (2.2) and (2.3), we have $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$ and $d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0))$.
We consider

$$
\begin{aligned}
e_n^* := d(x, x_{n+2}) &= d(gx_{n-1}, gx_{n+1}) \\
&\leq \psi(d(x_{n-1}, x_{n+1})) \\
&= \psi(d(gx_{n-2}, gx_n)) \\
&\leq \psi^2(d(x_{n-2}, x_n)) \\
&\vdots \\
&\leq \psi^n(d(x_0, x_2)) \\
&= \psi^n(e_0^*), \qquad (2.4)
\end{aligned}
$$

and,

$$\begin{aligned}
l_n^* := \ d(x_{n+2}, x_n) &= d(gx_{n+1}, gx_{n-1}) \\
&\leq \psi(d(x_{n+1}, x_{n-1})) \\
&= \psi(d(gx_n, gx_{n-2})) \\
&\leq \psi^2(d(x_n, x_{n-2})) \\
&\vdots \\
&\leq \psi^n(d(x_2, x_0)) \\
&= \psi^n(l_0^*). \qquad (2.5)
\end{aligned}$$

Now, if p is odd say $2m + 1$ then we obtain that

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1}) \\
&\leq e_n + e_{n+1} + [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\
&\leq e_n + e_{n+1} + e_{n+2} + ... + e_{n+2m} \\
&\leq \psi^n(e_0) + \psi^{n+1}(e_0) + \psi^{n+2}(e_0) + ... + \psi^{n+2m}(e_0) \\
&= \sum_{i=n}^{n+2m} \psi^i(e_0) \leq \sum_{i=n}^{\infty} \psi^i(e_0) < \infty. \qquad (2.6)
\end{aligned}$$

If p is even say $2m$ then we obtain that

$$\begin{aligned}
d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m}) \\
&\leq e_n + e_{n+1} + [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
&\leq e_n + e_{n+1} + e_{n+2} + ... + d(x_{n+2m-2}, x_{n+2m}) \\
&= e_n + e_{n+1} + ... + e^*_{n+2m-2} \\
&\leq \psi^n(e_0) + \psi^{n+1}(e_0) + ... + \psi^{n+2m-2}(e_0^*) \\
&= \sum_{i=n}^{n+2m-2} \psi^i(e_0) + \psi^{n+2m-n}(e_0^*) \\
&\leq \sum_{i=n}^{\infty} \psi^i(e_0) + \psi^{n+2m-n}(e_0^*) < \infty. \qquad (2.7)
\end{aligned}$$

Similarly, if p is odd say $2m+1$ then we get that

$$\begin{aligned} d(x_{n+2m+1},x_n) \leq & d(x_{n+2m+1},x_{n+2m}) + d(x_{n+2m},x_{n+2m-1}) + d(x_{n+2m-1},x_n) \\ \leq & l_{n+2m+1} + l_{n+2m} + [d(x_{n+2m-1},x_{n+2m-2}) \\ & + d(x_{n+2m-2},x_{n+2m-3}) + d(x_{n+2m-3},x_n)] \\ \leq & \psi^{n+2m+1}(l_0) + \psi^{n+2m}(l_0) + ... + \psi^{n-1}(l_0) \\ = & \sum_{i=n-1}^{n+2m+1} \psi^i(l_0) \leq \sum_{i=n-1}^{\infty} \psi^i(l_0) < \infty. \end{aligned} \tag{2.8}$$

Similarly, if p is even say $2m$ then we get that

$$\begin{aligned} d(x_{n+2m},x_n) \leq & d(x_{n+2m},x_{n+2m-1}) + d(x_{n+2m-1},x_{n+2m-2}) + d(x_{n+2m-2},x_n) \\ \leq & l_{n+2m} + l_{n+2m-1} + [d(x_{n+2m-2},x_{n+2m-3}) \\ & + d(x_{n+2m-3},x_{n+2m-4}) + d(x_{n+2m-4},d(x_n)] \\ \leq & \psi^{n+2m}(l_0) + \psi^{n+2m-2}(l_0) + ... + \psi^{n-2}(l_0^*) \\ = & \sum_{i=n-2}^{n+2m} \psi^i(l_0) + \psi^{n-2}(l_0^*) \\ \leq & \sum_{i=n-2}^{\infty} \psi^i(l_0) + \psi^{n-2}(l_0^*) < \infty \end{aligned} \tag{2.9}$$

It follows from (2.6), (2.7), (2.8) and (2.9) that $\lim_{n\to\infty} d(x_n,x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p},x_n)$ for all $p > 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X,d). By completeness of (X,d) there exists a $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We will show that u is a fixed point of g. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,gu) &\leq d(u,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},gu) \\ &= d(u,x_n) + e_n + d(gx_n,gu) \\ &\leq d(u,x_n) + e_n + \psi(d(x_n,u)). \end{aligned} \tag{2.10}$$

And, we get that

$$\begin{aligned} d(gu,u) &\leq d(gu,x_{n+1}) + d(x_{n+1},x_n) + d(x_n,u) \\ &= d(gu,gx_n) + l_n + d(x_n,u) \\ &\leq \psi(d(u,x_n)) + l_n + d(x_n,u). \end{aligned} \tag{2.11}$$

Using (2.10) and (2.11) it follows that $d(u,gu) = 0 = d(gu,u)$. So $gu = u$. Thus u is a fixed point of g. For uniqueness, let v be another a fixed point of g. Then it follows that $d(u,v) = d(gu,gv) \leq \psi(d(u,v)) < d(u,v)$ and $d(v,u) = d(gv,gu) \leq \psi(d(v,u)) < d(v,u)$, which is a contradiction. Therefore, we must have $d(u,v) = 0 = d(v,u)$. So $u = v$. Thus u is a fixed point of g. □

Next, we obtain corollary by set $\psi(t) = \exists r(t), \forall t \in [0,\infty), r \in [0,1)$.

Corollary 2.1. Let (X, d) be a complete rectangular quasi-metric space.Suppose that $T : X \longrightarrow X$ $x, y \in X$

$$d(gx, gy) \leq rd(x, y)$$

for all $x, y \in X$ where $r \in [0, 1)$. Then g has a unique fixed point in X.

Example 2.5. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$. Define the generalized metric d on X as follows :

$$\begin{array}{ll} d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3, & d(\frac{1}{3}, \frac{1}{2}) = d(\frac{1}{5}, \frac{1}{4}) = 0.1, \\ d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6, & d(\frac{1}{4}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{5}) = 0.4, \\ d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2, & d(\frac{1}{5}, \frac{1}{2}) = d(\frac{1}{4}, \frac{1}{3}) = 0.5, \\ d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0, & \end{array}$$

and

$$d(x, y) = |x - y| \text{ if } x, y \in B \text{ or } x \in A, y \in B \text{ or } x \in B, y \in A \ .$$

Then (X, d) is a complete rectangular quasi-metric space.

Next, let $g : X \longrightarrow X$ by

$$gx = \begin{cases} \frac{1}{5} & x \in A, \\ \frac{x}{6} & x \in B, \end{cases}$$

where $\psi(t) = \frac{t}{2}; \ \ \forall t \in [0, \infty)$. Then g satisfy Theorem 2.4, and we see that $\frac{1}{5}$ is a fixed point of g. Indeed,

Case(I) If $x, y \in A$, then $d(gx, gy) = d(\frac{1}{5}, \frac{1}{5}) = 0 \leq \frac{d(x,y)}{2} = \psi(d(x, y))$.

Case (II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$, then

$$\begin{aligned} d(gx, gy) &= |gx - gy| \\ &= |\frac{x}{6} - y|; \ (set \ x \in B) \\ &\leq \frac{1}{2}|x - y| \\ &= \frac{d(x, y)}{2} \\ &= \psi(d(x, y)). \end{aligned} \tag{2.12}$$

In 1982, Sessa [6] introduced a common fixed point theorem for a selfmapping of a complete metric space as follows :

Definition 2.6. Two self-mappings S and T of metric space (X, d) are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx), \qquad \forall x \in X.$$

It is clear that two commuting mappings are weakly commuting

In 1986, Jungck [7] introduced a compatible mappings and common fixed points as follows :

Definition 2.7. Let T and S be two self-mappings of a metric space (X, d). S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

It is easy to see that two compatible maps are weakly compatible.

In 2002, Aamri and El Moutawakil [8] defined a new property called the (E.A) property which generalizes the concept of non-compatible mappings and proved some common fixed point theorems.

Definition 2.8. Let S and T be two self-mappings of a rectangular quasi-metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Example 2.9. (1) Let $X = [0, +\infty]$.Define $T, S : X \longrightarrow X$ by

$$Tx = \frac{x^2}{4} \text{ and } Sx = \frac{3x^2}{4}\ ,\ \forall x \in X.$$

Consider the sequence $x_n = 1/n$. Clearly $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$. Then T and S satisfy (E.A).

(2) Let $X = [2, +\infty]$. Define $T, S : X \longrightarrow X$ by

$$Tx = x + 1 \text{ and } Sx = 2x + 1\ ,\ \forall x \in X.$$

Suppose that property (E.A) hold,Then there exists a $\{x_n\}$ in X sequence satisfying

$$\lim_{n\to\infty} Tx = \lim_{n\to\infty} Sx = t, \qquad \text{for some } t \in X.$$

Therefore

$$\lim_{n\to\infty} x_n = t - 1 \text{ and } \lim_{n\to\infty} x_n = \frac{t-1}{2}\ .$$

then $t = 1$, which is a contradiction $1 \notin X$. Hence T and S do not satisfy (E.A).

Theorem 2.2. Let S and T be two weakly compatible self-mappings of a rectangular quasi-metric spaces (X, d) such that

(i) T and S satisfy the property (E.A),

(ii) $d(Tx, Ty) < \max\{d(Sx, Sy)\frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\}$, $\forall x \neq y \in X$,

(iii) $TX \subset SX$,

(iv) SX or TX is complete subspace of X.

Then T and S have a unique common fixed point.

Proof. Since T and S satisfy the property (E.A), there exists a sequence $\{x_n\}$ in X satifying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

Suppose that SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$. Also $\lim_{n\to\infty} Tx_n = Sa$. We show that $Ta = Sa$. Suppose that $Ta \neq Sa$. Condition (ii) implies

$$\begin{aligned} d(Ta, Tx_n) < \max\{&d(Sa, Sx_n), [d(Ta, Sa) + d(Tx_n, Sx_n)]/2. \\ &[d(Tx_n, Sa) + d(Ta, Sx_n)]/2\}. \end{aligned} \tag{2.13}$$

Letting $n \to +\infty$ yields

$$\begin{aligned} d(Ta, Sa) \leq \max\{&d(Sa, Sa), [d(Ta, Sa) + d(Sa, Sa)]/2, \\ &[d(Sa, Sa) + d(Ta, Sa)]/2\} \\ \leq\ & d(Ta, Sa)/2; \end{aligned} \tag{2.14}$$

a contradiction. Hence $Ta = Sa$.

Since T and S are a weakly compatible, $STa = TSa$ and $TTa = TSa = STa = SSa$.

Finally, we show that Ta is a common fixed point of T and S. Suppose that $Ta \neq TTa$. Then

$$\begin{aligned} d(Ta, TTa) < \max\{&d(Sa, STa), [d(Ta, Sa) + d(TTa, STa)]/2, \\ &[d(TTa, Sa) + d(Ta, STa)]/2\} \\ < \max\{&d(Ta, TTa), [d(TTa, Ta) + d(Ta, TTa)]/2\} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} d(TTa, Ta) < \max\{&d(STa, Sa), [d(TTa, STa) + d(Ta, Sa)]/2, \\ &[d(Ta, STa) + d(TTa, Sa)]/2\} \\ < \max\{&d(TTa, Ta), [d(Ta, TTa) + d(TTa, Ta)]/2\}. \end{aligned} \tag{2.16}$$

Since (2.15) and (2.16) we have
$d(Ta, TTa) + d(TTa, Ta) < \max\{d(Ta, TTa), [d(TTa, Ta) + d(Ta, TTa)]/2\} + \max\{d(TTa, Ta), [d(Ta, TTa)+d(TTa, Ta)]/2\} = d(Ta, TTa)+d(TTa, Ta)$, where $\max\{d(Ta, TTa), [d(TTa, Ta)+d(Ta, TTa)]/2\} \neq d(Ta, TTa)$ and $< \max\{d(TTa, Ta), [d(Ta, TTa) + d(TTa, Ta)]/2\} \neq d(TTa, Ta)$;
which is a contradiction. Hence $TTa = Ta$ and $STa = TTa = Ta$. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$. Uniquness of the common fixed point, suppose that a, b are distinct common fixed

point of S and T .

$$\begin{aligned} d(a,b) = d(Ta,Tb) &< \max\{d(Sa,Sb), \frac{[d(Ta,Sa)+d(Tb,Sb)]}{2}, \\ &\frac{[d(Tb,Sa)+d(Ta,Sb)]}{2}\}, \\ &= \frac{d(Tb,Sa)+d(Ta,Sb)}{2} = \frac{d(b,a)+d(a,b)}{2} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} d(b,a) = d(Tb,Ta) &< \max\{d(Sb,Sa), \frac{[d(Tb,Sb)+d(Ta,Sa)]}{2}, \\ &\frac{[d(Ta,Sb)+d(Tb,Sa)]}{2}\}, \\ &= \frac{d(Ta,Sb)+d(Tb,Sa)}{2} = \frac{d(a,b)+d(b,a)}{2}. \end{aligned} \tag{2.18}$$

Since (2.17) and (2.18) we get that $d(a,b)+d(b,a) < \frac{d(b,a)+d(a,b)}{2} + \frac{d(a,b)+d(b,a)}{2}$ $\square$

Example 2.3. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$. Define the generalized metric d on X as follows :
$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3$, $\quad d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2$,
$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6$, $\quad d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0$,
such that $d(x,y) = d(y,x)$ and
$d(x,y) = |x-y|$ if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$. Define $T, S : X \longrightarrow X$ by

$$Tx = \frac{3x}{4} \text{ and } Sx = \frac{x^2}{2}, \qquad \forall x \in X.$$

Then

(1) T and S satisfy the property (E.A) for the sequence $x_n = 1 + 1/n, n = 1, 2, ...,$

(2) S and T are weakly compatible,

(3) T and S satisfy for all $x \neq y$,

(4) $T1 = S1 = 1$.

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Research Article

Fixed Point Theorems for a Demicontractive Mapping and Equilibrium Problems in Hilbert Spaces

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Abstract. In this research, we introduce some properties of demicontractive mapping and the combination of equilibrium problem. Then, we prove a strong convergence for the iterative sequence converging to a common element of fixed point set of demicontractive mapping and a common solution of equilibrium problems. Finally, we give a numerical example for the main theorem to support our results.

Keywords. The combination of equilibrium problem; Fixed point; Demicontractive mapping

MSC. 47H09; 47H10; 90C33

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be bifunction. *The equilibrium problem* for F is to determine its equilibrium point, i.e., the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall\, y \in C\}. \tag{1.1}$$

Equilibrium problems were introduced by [1] in 1994 where such problems have had a significant impact and influence in the development of several branches of pure and applied sciences. Various problems in physics, optimization, and economics are related to seeking some elements of $EP(F)$ (see [1,3]). Many authors have been investigated iterative algorithms for the equilibrium problems (see, for example, [3,5,6,15]).

In 2013, Suwannaut and Kangtunyakarn [15] introduced *the combination of equilibrium problem* which is to find $u \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x,y) \geq 0, \quad \forall\, y \in C, \tag{1.2}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1,2,\dots,N$. The set of solution (1.2) is denoted by

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i).$$

Remark 1.1. Very recently, in the work of Suwannaut and Kangtunyakarn [14], Khuangsatung and Kangtunyakarn [7] and Bnouhachem [2], they give the numerical examples for main theorems and show that their iteration for the combination of equilibrium problem converges faster than their iteration for the classical equilibrium problem.

The fixed point problem for the mapping $T : C \to C$ is to find $x \in C$ such that

$$x = Tx. \tag{1.3}$$

We denote the set of solutions of (1.3) by $Fix(T)$. It is well known that $Fix(T)$ is closed and convex and $P_{Fix(T)}$ is well-defined.

Definition 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H.

(i) A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x,y \in C.$$

(ii) A mapping $T : C \to C$ is called *quasi-nonexpansive* if $Fix(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \quad \forall\, x \in C \text{ and } y \in Fix(T).$$

(iii) A mapping $T : C \to C$ is said to be κ-*strictly pseudo-contractive* if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I-T)x - (I-T)y\|^2, \quad \forall\, x,y \in C. \tag{1.4}$$

In a real Hilbert space, the inequality (1.4) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2} \|(I-T)x - (I-T)y\|^2, \quad \forall\, x,y \in C. \tag{1.5}$$

Definition 1.2. A mapping T is said to be *demicontractive* if $Fix(T) \neq \emptyset$ and there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + \kappa \|(I-T)x\|^2, \quad \forall\, x \in C \text{ and } y \in Fix(T). \tag{1.6}$$

Observe that the class of demicontractive mapping includes various types of nonlinear mappings such as nonexpansive mapping, quasi-nonexpansive mapping and strictly pseudo-contractive mapping.

Using the same method of proof of (1.5), we obtain that if $T : C \to C$ is demicontractive mapping, then (1.6) is equivalent to the following inequality

$$\langle Tx - y, x - y\rangle \le \|x - y\|^2 - \frac{1-\kappa}{2}\|(I - T)x\|^2, \quad \forall\, x \in C \text{ and } y \in Fix(T). \tag{1.7}$$

In 1977, Maruster [10] introduced the condition (A) of a mapping T.

Definition 1.3 ([10]). The mapping T is said to satisfy the condition (A) if $Fix(T)$ is nonempty and if there exists a real positive number λ such that

$$\langle x - Tx, x - y\rangle \ge \lambda\|x - Tx\|^2, \quad \forall\, x \in C, y \in Fix(T).$$

In 2015, Maruster [9] studied a strong convergence theorem of a κ-demicontractive mapping as follows:

Theorem 1.2 ([9]). *Suppose that T is κ-demicontractive on C and satisfies the condition* (A). *Then the sequence $\{x_n\}$ generated by the Mann iteration with control sequence t_k satisfying the condition $0 < a \le t_n \le b < 1 - \kappa$, and for suitable starting point x_0, converges strongly to p.*

In 2013, Mongkolkeeha, Cho and Kumam [11] defined the new iterative scheme for two κ-demicontractive mapings as follows:

$$\begin{cases} x_1 \in C \text{ arbitrary chosen,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n Sx_n + (1 - \beta_n)Tx_n), \quad \forall\, n \in \mathbb{N}, \end{cases}$$

where $S, T : C \to C$ be two κ-demicontractive mappings such that $I - S$ is demiclosed at zero with $Fix(S) \cap Fix(T) \ne \emptyset$, $\{\alpha_n\} \subset [\kappa, 1]$ and $\{\beta_n\} \subset [0, 1]$ are the sequences satisfying some control conditions. Then the sequence $\{x_n\}$ converges strongly to a point $v \in Fix(S) \cap Fix(T)$.

Question. Is it possible to prove a strong convergence theorem for a demicontractive mapping and equilibrium problems without using the condition (A) and the control sequences that are not depended on the constant κ?

Motivated by the related research described above, we introduce the Halpern's iterative method modified for demicontractive mapping and a finite family of equilibrium problems. Then, under some appropriate conditions, we prove a strong convergence theorem for the combination of equilibrium problem and a fixed point set of demicontractive mapping. Finally, we give a numerical example for our main result in space of real numbers.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations "$\rightharpoonup$" and "$\to$", respectively. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \le \|x - y\|,'' y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 2.1 ([16]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0,'' v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, '' x, y \in H.$$

Lemma 2.2 ([12]). *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.3 ([17]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall\ n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim\limits_{n\to\infty} s_n = 0$.

Lemma 2.4 ([13]). *Let H be a real Hilbert space. Then the following results hold:*

(i) *For all $x, y \in H$ and $\alpha \in [0,1]$,*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$, *for each $x, y \in H$.*

Lemma 2.5. *Let $T : C \to C$ be a κ-demicontractive mapping with $\kappa \leq \delta$ and $Fix(T) \neq \emptyset$. Define $S : C \to C$ by $Sx := \lambda Tx + (1-\lambda)x$, where $\lambda \in (0,\sigma)$ and $\delta + \sigma < 1$. Then, there hold the following statement:*

(i) $Fix(T) = Fix(S)$;

(ii) *S is a quasi-nonexpansive mapping, that is,*

$$\|Sx - y\| \leq \|x - y\|, \quad \text{for every } x \in C \text{ and } y \in Fix(T).$$

Proof. It is obvious that $Fix(T) = Fix(S)$ due to the fact that $Sx - x = \lambda(Tx - x)$, $\forall\ x \in C$. To prove (ii), let $x \in C$ and $y \in Fix(T)$. Then, by (1.6) and (1.7), we obtain

$$\begin{aligned}
\|Sx - y\|^2 &= \|\lambda(Tx - y) + (1-\lambda)(x-y)\|^2 \\
&\leq \lambda^2\|Tx - y\|^2 + (1-\lambda)^2\|x-y\|^2 + 2\lambda(1-\lambda)\langle Tx - y, x - y\rangle \\
&\leq \lambda^2\left(\|x-y\|^2 + \kappa\|x - Tx\|^2\right) + (1-\lambda)^2\|x-y\|^2 + 2\lambda(1-\lambda)\left(\|x-y\|^2 - \frac{1-\kappa}{2}\|x - Tx\|^2\right) \\
&= \left(\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\right)\|x-y\|^2 + \left(\lambda^2\kappa - \lambda(1-\lambda)(1-\kappa)\right)\|x - Tx\|^2 \\
&= \|x-y\|^2 + \lambda(\kappa + \lambda - 1)\|x - Tx\|^2 \\
&\leq \|x-y\|^2 + \lambda(\delta + \sigma - 1)\|x - Tx\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Therefore, S is a quasi-nonexpansive mapping. □

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F and C satisfy the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x) \le 0$ for all $x,y \in C$;

(A3) For each $x,y,z \in C$, $\lim\limits_{t\to 0^+} F(tz+(1-t)x,y) \le F(x,y)$;

(A4) For each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Remark 2.6. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4). Then, $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4), where $a_i \in (0,1)$ for every $i = 1,2,\dots,N$ and $\sum_{i=1}^{N} a_i = 1$.

Proof. For every $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $x,y,z \in C$ and $a_i \in (0,1)$ for all $i = 1,2,\dots,N$ and $\sum_{i=1}^{N} a_i = 1$.
To prove (A1), we get

$$\sum_{i=1}^{N} a_i F_i(x,x) = a_1F_1(x,x)+a_2F_2(x,x)+\ldots+a_NF_N(x,x) = 0.$$

Since

$$\sum_{i=1}^{N} a_i F_i(x,y) + \sum_{i=1}^{N} a_i F_i(y,x) = \sum_{i=1}^{N} a_i \left(F_i(x,y)+F_i(y,x)\right) \le 0,$$

we have $\sum_{i=1}^{N} a_i F_i$ satisfies (A2).
Let $t \in [0,1]$, then we have

$$\lim_{t\to 0^+} \sum_{i=1}^{N} a_i F_i(tz+(1-t)x,y) = \sum_{i=1}^{N} a_i \lim_{t\to 0^+} F_i(tz+(1-t)x,y) = \sum_{i=1}^{N} a_i F_i(x,y).$$

Thus (A3) holds.
To prove (A4), we first let $\alpha \in (0,1)$. Therefore, we get

$$\begin{aligned}\sum_{i=1}^{N} a_i F_i(x,\alpha z+(1-\alpha)y) &\le \sum_{i=1}^{N} a_i \left(\alpha F_i(x,z)+(1-\alpha)F_i(x,y)\right)\\ &= \alpha \sum_{i=1}^{N} a_i F_i(x,z) + (1-\alpha)\sum_{i=1}^{N} a_i F_i(x,y).\end{aligned}$$

It follows that $\sum_{i=1}^{N} a_i F_i$ is convex. Next, let $\{y_n\} \subset C$ with $y_n \to y$ as $n \to \infty$. Thus we obtain

$$\liminf_{n\to\infty} \sum_{i=1}^{N} a_i F_i(x,y_n) \ge \sum_{i=1}^{N} a_i \liminf_{n\to\infty} F_i(x,y_n) \ge \sum_{i=1}^{N} a_i F_i(x,y).$$

Then $\sum_{i=1}^{N} a_i F_i$ is lower semicontinuous. This implies that (A4) holds.
We can conclude that $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4). □

Lemma 2.7 ([15]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,\dots,N$, let $F_i : C\times C \to \mathbb{R}$ be bifunctions satisfying* (A1)-(A4) *with* $\bigcap_{i=1}^{N} EP(F_i)\neq\emptyset$. *Then,*

$$EP\left(\sum_{i=1}^{N} a_iF_i\right)=\bigcap_{i=1}^{N} EP(F_i),$$

where $a_i\in(0,1)$ for every $i=1,2,\dots,N$ and $\sum_{i=1}^{N} a_i=1$.

Lemma 2.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,\dots,N$, let $F_i : C\times C\to\mathbb{R}$ be bifunctions satisfying* (A1)-(A4). *Let the sequences $\{x_n\}\subseteq H$, $\{u_n\}\subseteq C$ and $\{r_n\}\subseteq(0,1)$ satisfying the following condition:*

$$\sum_{i=1}^{N} a_iF_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall\ y\in C.$$

Therefore, if $u_{n_k}\rightharpoonup\omega$ as $k\to\infty$ and $\|u_n-x_n\|\to 0$ as $n\to\infty$, then $\omega\in\bigcap_{i=1}^{N} EP(F_i)$.

Proof. Due to the fact that F_i is bifunctions satisfying (A1)-(A4), for all $i=1,2,\dots,N$, then, by Remark 2.6, we have $\sum_{i=1}^{N} a_iF_i$ satisfies the conditions (A1)-(A4). Since

$$\sum_{i=1}^{N} a_iF_i(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\forall y\in C,$$

and $\sum_{i=1}^{N} a_iF_i$ satisfies the conditions (A1)-(A4), we obtain

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq\sum_{i=1}^{N} a_iF_i(y,u_n),\quad \forall\ y\in C.$$

In particular, it follows that

$$\left\langle y-u_{n_k},\frac{u_{n_k}-x_{n_k}}{r_{n_k}}\right\rangle\geq\sum_{i=1}^{N} a_iF_i\left(y,u_{n_k}\right),\quad \forall\ y\in C. \tag{2.1}$$

From $\|u_n-x_n\|\to 0$ as $n\to\infty$, (2.1) and (A4), we have

$$\sum_{i=1}^{N} a_iF_i(y,\omega)\leq 0,\quad \forall\ y\in C. \tag{2.2}$$

Put $y_t:=ty+(1-t)\omega$, $t\in(0,1]$, we have $y_t\in C$. By using (A1), (A4) and (2.2), we have

$$\begin{aligned}
0&=\sum_{i=1}^{N} a_iF_i(y_t,y_t)\\
&=\sum_{i=1}^{N} a_iF_i(y_t,ty+(1-t)\omega)\\
&\leq t\sum_{i=1}^{N} a_iF_i(y_t,y)+(1-t)\sum_{i=1}^{N} a_iF_i(y_t,\omega)\\
&\leq t\sum_{i=1}^{N} a_iF_i(y_t,y).
\end{aligned}$$

It implies that

$$\sum_{i=1}^{N} a_i F_i(ty+(1-t)\omega, y) \geq 0, \quad \forall\, t \in (0,1] \text{ and } \forall y \in C. \tag{2.3}$$

From (2.3), taking $t \to 0^+$ and using (A3), we can conclude that

$$0 \leq \sum_{i=1}^{N} a_i F_i(\omega, y), \quad \forall\, y \in C.$$

Therefore, $\omega \in EP\left(\sum_{i=1}^{N} a_i F_i\right)$. By Lemma 2.7, we obtain $EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i)$. It follows that

$$\omega \in \bigcap_{i=1}^{N} EP(F_i). \tag{2.4}$$

□

Lemma 2.9 ([1]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall\, y \in C.$$

Lemma 2.10 ([3]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall\, y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x-y\rangle;$$

(iii) *$Fix(T_r) = EP(F)$;*

(iv) *$EP(F)$ is closed and convex.*

Remark 2.11 ([15]). By Remark 2.6, we have $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4). By using Lemma 2.7 and Lemma 2.10, we obtain

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$, for each $i = 1,2,\dots,N$, and $\sum_{i=1}^{N} a_i = 1$.

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $T : C \to C$ be a demicontractive mapping with coefficient $\kappa \leq \theta_1$ and let a mapping $S_n : C \to C$ be defined by $S_n x := (1-\lambda_n)x + \lambda_n Tx$ with $\lambda_n < \theta_2$ and $\theta_1 + \theta_2 < 1$. Assume that $\Theta = \bigcap_{i=1}^{N} EP(F_i) \cap Fix(T) \neq \emptyset$.*

Let the sequences $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\ y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)u_n) + (1-\beta_n)S_n x_n, & \forall\ n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0,1)$ *and* $0 \leq a_i \leq 1$ *for every* $i = 1,2,\ldots,N$, *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(iv) $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum_{i=1}^{N} a_i = 1$;

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = P_\Theta u$.

Proof. The proof of this theorem will be divided into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Since $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\ y \in C,$$

by Lemma 2.10 and Remark 2.11, we have $u_n = T_{r_n} x_n$ and $Fix(T_{r_n}) = \bigcap_{i=1}^{N} EP(F_i)$.

Let $z \in \Theta$. From Lemma 2.5 and Lemma 2.10 , we obtain

$$\begin{aligned} \|x_{n+1} - z\| &= \left\|\beta_n(\alpha_n u + (1-\alpha_n)u_n) + (1-\beta_n)S_n x_n - z\right\| \\ &= \left\|\beta_n(\alpha_n(u-z) + (1-\alpha_n)(u_n - z)) + (1-\beta_n)(S_n x_n - z)\right\| \\ &\leq \beta_n \|\alpha_n(u-z) + (1-\alpha_n)(u_n - z)\| + (1-\beta_n)\|S_n x_n - z\| \\ &\leq \beta_n(\alpha_n\|u - z\| + (1-\alpha_n)\|u_n - z\|) + (1-\beta_n)\|x_n - z\| \\ &\leq \beta_n(\alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\|) + (1-\beta_n)\|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_1 - z\|\}. \end{aligned}$$

By induction, we get $\|x_n - z\| \leq \max\{\|u - z\|, \|x_1 - z\|\}, \forall n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $u_n = T_{r_n} x_n$, by utilizing the definition of T_{r_n}, we obtain

$$\sum_{i=1}^{N} a_i F_i(T_{r_n} x_n, y) + \frac{1}{r_n}\left\langle y - T_{r_n} x_n, T_{r_n} x_n - x_n\right\rangle \geq 0, \quad \forall\ y \in C, \tag{3.2}$$

and

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_{n+1}}x_{n+1}, y\right) + \frac{1}{r_{n+1}}\left\langle y - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0, \quad \forall\ y \in C. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_n}x_n, T_{r_{n+1}}x_{n+1}\right) + \frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \geq 0, \tag{3.4}$$

and

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_{n+1}}x_{n+1}, T_{r_n}x_n\right) + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0. \tag{3.5}$$

From (3.4) and (3.5) and the fact that $\sum_{i=1}^{N} a_i F_i$ satisfies (A2), we have

$$\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \geq 0,$$

which implies that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \geq 0. \tag{3.6}$$

From (3.6), we obtain

$$\begin{aligned}
\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|^2 &\leq \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \\
&= \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \\
&\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left\|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\| \\
&\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
&= \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
&\leq \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right],
\end{aligned}$$

which follows that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{\epsilon}|r_{n+1} - r_n| \, \|u_{n+1} - x_{n+1}\|. \tag{3.7}$$

From (3.7), we have

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}| \, \|u_n - x_n\|. \tag{3.8}$$

First, we let $y_n = \alpha_n u + (1 - \alpha_n)u_n$. From (3.8), we derive that

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq \beta_n \|y_n - y_{n-1}\| &+ \left|\beta_n - \beta_{n-1}\right| \|y_{n-1}\| + \left(1 - \beta_n\right) \|S_n x_n - S_{n-1}x_{n-1}\| \\
&+ \left|\beta_n - \beta_{n-1}\right| \|S_{n-1}x_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n \Big[|\alpha_n - \alpha_{n-1}| \, \|u\| + (1-\alpha_n)\|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \, \|u_{n-1}\| \Big] \\
&\quad + \big|\beta_n - \beta_{n-1}\big| \, \|y_{n-1}\| + \big(1-\beta_n\big) \Big[(1-\lambda_n)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \, \|x_{n-1}\| \\
&\quad + \lambda_n \|Tx_n - Tx_{n-1}\| + |\lambda_n - \lambda_{n-1}| \, \|Tx_{n-1}\| \Big] + \big|\beta_n - \beta_{n-1}\big| \, \|S_{n-1}x_{n-1}\| \\
&\le \beta_n(1-\alpha_n) \left[\|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}| \, \|u_n - x_n\| \right] + \big(1-\beta_n\big)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|u\| + \|u_{n-1}\|) + \big|\beta_n - \beta_{n-1}\big|(\|y_{n-1}\| + \|S_{n-1}x_{n-1}\|) \\
&\quad + |\lambda_n - \lambda_{n-1}|(\|x_{n-1}\| + \|Tx_{n-1}\|) + \lambda_n \|Tx_n - Tx_{n-1}\| \\
&\le \big(1-\alpha_n\beta_n\big)\|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}| \, \|u_n - x_n\| + |\alpha_n - \alpha_{n-1}|(\|u\| + \|u_{n-1}\|) \\
&\quad + \big|\beta_n - \beta_{n-1}\big|(\|y_{n-1}\| + \|S_{n-1}x_{n-1}\|) + |\lambda_n - \lambda_{n-1}|(\|x_{n-1}\| + \|Tx_{n-1}\|) \\
&\quad + \lambda_n \|Tx_n - Tx_{n-1}\|.
\end{aligned}
$$

By Lemma 2.3 and the conditions (i), (ii), (vi), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Step 3. Prove that $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|S_n x_n - x_n\| = 0$.

To show this, let $z \in \Theta$. Since $u_n = T_{r_n} x_n$ and T_{r_n} is firmly nonexpansive mapping, then we obtain

$$
\begin{aligned}
\big\|z - T_{r_n}x_n\big\|^2 &= \big\|T_{r_n}z - T_{r_n}x_n\big\|^2 \\
&\le \big\langle T_{r_n}z - T_{r_n}x_n, z - x_n \big\rangle \\
&= \frac{1}{2}\Big(\big\|T_{r_n}x_n - z\big\|^2 + \|x_n - z\|^2 - \big\|T_{r_n}x_n - x_n\big\|^2 \Big),
\end{aligned}
$$

which follows that

$$\big\|T_{r_n}x_n - z\big\|^2 \le \|x_n - z\|^2 - \big\|T_{r_n}x_n - x_n\big\|^2.$$

That is,

$$\|u_n - z\|^2 \le \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.10}$$

By the definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \big\|\beta_n(\alpha_n(u-z) + (1-\alpha_n)(u_n - z)) + \big(1-\beta_n\big)(S_n x_n - z)\big\|^2 \\
&\le \beta_n \|\alpha_n(u-z) + (1-\alpha_n)(u_n - z)\|^2 + \big(1-\beta_n\big)\|S_n x_n - z\|^2 \\
&\le \beta_n \left[\alpha_n \|u-z\|^2 + (1-\alpha_n)\|u_n - z\|^2\right] + \big(1-\beta_n\big)\|x_n - z\|^2 \\
&\le \beta_n \left[\alpha_n \|u-z\|^2 + (1-\alpha_n)\big(\|x_n - z\|^2 - \|u_n - x_n\|^2\big)\right] + \big(1-\beta_n\big)\|x_n - z\|^2 \\
&\le \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \beta_n(1-\alpha_n)\|u_n - x_n\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\beta_n(1-\alpha_n)\|u_n - x_n\|^2 &\le \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\le \alpha_n \|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.
\end{aligned}
$$

From (i), (ii) and 3.9, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.11}$$

Since

$$x_{n+1} - x_n = \beta_n(\alpha_n(u - x_n) + (1-\alpha_n)(u_n - x_n)) + \left(1-\beta_n\right)(S_n x_n - x_n),$$

then we get

$$\left(1-\beta_n\right)\|S_n x_n - x_n\| \le \beta_n\alpha_n\|u - x_n\| + \beta_n(1-\alpha_n)\|u_n - x_n\| + \|x_{n+1} - x_n\|.$$

This follows by (i), (ii) and 3.11 that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.12}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle u - q, x_n - q\rangle \le 0$, where $q = P_\Theta u$.

To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle = \lim_{k\to\infty}\left\langle u - z, x_{n_k} - z\right\rangle. \tag{3.13}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. From (3.11), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

From

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\ y \in C,$$

and $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, by Lemma 2.8, we can conclude that

$$\omega \in EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i). \tag{3.14}$$

Next, we will show that $\omega \in Fix(T)$.

By Lemma 2.5, we have $Fix(S_n) = Fix(T)$. Assume that $\omega \neq S_n\omega$. Using Opial's condition, (3.12) and the condition (iii), then we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\| &< \liminf_{k\to\infty}\left\|x_{n_k} - S_{n_k}\omega\right\| \\
&\le \liminf_{k\to\infty}\left(\left\|x_{n_k} - S_{n_k}x_{n_k}\right\| + \left\|S_{n_k}\omega - S\omega\right\|\right) \\
&\le \liminf_{k\to\infty}\left(\left\|x_{n_k} - S_{n_k}x_{n_k}\right\| + \left\|x_{n_k} - \omega\right\| + \lambda_{n_k}\left\|(I-T)x_{n_k} - (I-T)\omega\right\|\right) \\
&\le \liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in Fix(S_n) = Fix(T). \tag{3.15}$$

From (3.14) and (3.15), we can deduce that $\omega \in \Theta$.

Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, $q = P_\Theta u$ and $\omega \in \Theta$, then, by Lemma 2.1, we can conclude that

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - q, x_n - q\rangle &= \lim_{k\to\infty}\left\langle u - q, x_{n_k} - z\right\rangle \\
&= \langle u - q, \omega - q\rangle \\
&\le 0.
\end{aligned} \tag{3.16}$$

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $q = P_\Theta u$.
By the definition of x_n and Lemma 2.4, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|\beta_n(\alpha_n(u-q)+(1-\alpha_n)(u_n-q))+\left(1-\beta_n\right)(S_nx_n-q)\right\|^2 \\
&\le \left\|\beta_n(1-\alpha_n)(u_n-q)+\left(1-\beta_n\right)(S_nx_n-q)\right\|^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&\le \left(\beta_n(1-\alpha_n)\|u_n-q\|+\left(1-\beta_n\right)\|S_nx_n-q\|\right)^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&\le \left(\beta_n(1-\alpha_n)\|x_n-q\|+\left(1-\beta_n\right)\|x_n-q\|\right)^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&= \left(1-\alpha_n\beta_n\right)^2\|x_n-q\|^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&\le \left(1-\alpha_n\beta_n\right)\|x_n-q\|^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle .
\end{aligned}$$

From (3.16), the conditions (i), (ii) and Lemma 2.3, we can conclude that $\{x_n\}$ converges strongly to $q = P_\Theta u$. By (3.11), we have $\{u_n\}$ converges strongly to $q = P_\Theta u$. This completes the proof. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4)*. Let $T : C \to C$ be a demicontractive mapping with coefficient $\kappa \le \theta_1$ and let a mapping $S_n : C \to C$ be defined by $S_nx := (1-\lambda_n)x+\lambda_nTx$ with $\lambda_n < \theta_2$ and $\theta_1+\theta_2 < 1$. Assume that $\Theta = EP(F)\cap Fix(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, & \forall\, y\in C,\\ x_{n+1}=\beta_n(\alpha_nu+(1-\alpha_n)u_n)+\left(1-\beta_n\right)S_nx_n, & \forall\, n\ge 1, \end{cases} \tag{3.17}$$

where $\{\alpha_n\},\{\beta_n\},\{\lambda_n\}\subseteq(0,1)$ and $0\le a_i\le 1$ for every $i = 1,2,\dots,N$, satisfying the following conditions:

(i) $\lim\limits_{n\to\infty}\alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \tau \le \beta_n \le \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $0 < \rho \le \lambda_n < \theta_2 < 1$, *for some* $\rho, \theta_2 > 0$ *and* $\sum\limits_{n=1}^{\infty}\lambda_n < \infty$;

(iv) $0 < \epsilon \le r_n \le \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty}\left|\beta_{n+1}-\beta_n\right| < \infty$, $\sum\limits_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$, $\sum\limits_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta u$.

Proof. Take $F = F_i$, $\forall\, i = 1,2,\dots,N$ in Theorem 3.1. Then we obtain the desired result. □

4. Applications

In this section, we obtain our additional results for fixed point problem of a nonspreading mapping and a quasi-nonexpansive mapping.

In 2008, Kohsaka and Takahashi [8] introduced *the nonspreading mapping T* in Hilbert space H as follows:

$$2\|Tu-Tv\|^2 \le \|Tu-v\|^2+\|u-Tv\|^2, \quad \forall\, u,v\in C. \tag{4.1}$$

In 2009, it is shown by Iemoto and Takahashi [4] that (4.1) is equivalent to the following equation.

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + 2\langle u - Tu, v - Tv\rangle, \quad \forall\, u, v \in C.$$

In 2014, Suwannaut and Kangtunyakarn [14] obtain the following main results for a nonspraeding mapping on C.

Lemma 4.1 ([14]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonspreading mapping with $Fix(T) \neq \emptyset$. Then there hold the following statement:*

(i) $Fix(T) = VI(C, I - T)$;

(ii) *For every $u \in C$ and $v \in Fix(T)$,*

$$\|P_C(I - \lambda(I - T))u - v\| \leq \|u - v\|, \text{ where } \lambda \in (0,1),$$

that is, a mapping $P_C(I - \lambda(I - T))$ is quasi-nonexpansive.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $T : C \to C$ be a nonspreading mapping and let a mapping $W_n : C \to C$ be defined by $W_n x := (1 - \rho_n)x + \rho_n Tx$. Assume that $\Theta = \bigcap_{i=1}^{N} EP(F_i) \cap Fix(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1 - \alpha_n)u_n) + \left(1 - \beta_n\right) W_n x_n, & \forall\, n \geq 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ and $0 \leq a_i \leq 1$ for every $i = 1, 2, \ldots, N$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \rho_n < \infty$;

(iv) $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum_{i=1}^{N} a_i = 1$;

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \left|\beta_{n+1} - \beta_n\right| < \infty$, $\sum_{n=1}^{\infty} \left|\rho_{n+1} - \rho_n\right| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta u$.

Proof. Applying Lemma 4.1 and the same proof of Theorem 3.1, we obtain the desired results. □

Observe that every a nonspreading mapping T with $Fix(T) \neq \emptyset$ is quasi-nonexpansive. Then we also have the following result.

Lemma 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping with $Fix(T) \neq \emptyset$. Then the following results are true:*

(i) $Fix(T) = VI(C, I-T)$;

(ii) *For every $u \in C$ and $v \in Fix(T)$,*

$$\|P_C(I-\lambda(I-T))u - v\| \leq \|u-v\|, \text{ where } \lambda \in (0,1).$$

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $T : C \to C$ be a quasi-nonexpansive mapping and let a mapping $W_n : C \to C$ be defined by $W_n x := (1-\rho_n)x + \rho_n Tx$. Assume that $\Theta = \bigcap_{i=1}^N EP(F_i) \cap Fix(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} \sum\limits_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)u_n) + \big(1-\beta_n\big) W_n x_n, & \forall\, n \geq 1, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ and $0 \leq a_i \leq 1$ for every $i = 1,2,\dots,N$, satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum\limits_{n=1}^{\infty} \rho_n < \infty$;

(iv) $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum\limits_{i=1}^{N} a_i = 1$;

(vi) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} \big|\beta_{n+1} - \beta_n\big| < \infty$, $\sum\limits_{n=1}^{\infty} \big|\rho_{n+1} - \rho_n\big| < \infty$, $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta u$.

Proof. Using Lemma 4.3 and Theorem 3.1, we get the result of Theorem 4.4. □

5. A Numerical Example

In this section, we give numerical examples to support our main theorem.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers. For every $i = 1,2,\dots,N$, let $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \frac{-7x}{5},$$

$$F_i(x,y) = i(-5x^2 + xy + 4y^2), \text{ for all } x, y \in \mathbb{R}.$$

Put $a_i = \frac{2}{7^i} + \frac{1}{N7^N}$, for every $i = 1,2,\dots,N$. Let $\alpha_n = \frac{1}{100n}$, $\beta_n = \frac{3n}{5n+3}$, $r_n = \frac{5n+6}{8n+9}$ and $\lambda_n = \frac{1}{n^2+2}$ for every $n \in \mathbb{N}$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Solution. Obviously, T is κ-demicontractive mapping with $\kappa = \frac{1}{6}$ and $Fix(T) = \{0\}$. If we choose $\theta_1 = \frac{1}{5}$ and $\theta_2 = \frac{1}{2}$, then we obtain $\theta_1 + \theta_2 = \frac{7}{10} < 1$. This implies by Lemma 2.5 that a mapping S_n is quasi-nonexpansive mapping.

Since $a_i = \frac{2}{7^i} + \frac{1}{N7^N}$, we obtain

$$\begin{aligned}\sum_{i=1}^{N} a_i F_i(x,y) &= \sum_{i=1}^{N}\left(\frac{2}{7^i} + \frac{1}{N7^N}\right) i(-5x^2 + xy + 4y^2)\\ &= \xi(-5x^2 + xy + 4y^2),\end{aligned}$$

where $\xi = \sum\limits_{i=1}^{N}\left(\frac{2}{7^i} + \frac{1}{N7^N}\right) i$. It is clear to check that $\sum\limits_{i=1}^{N} a_i F_i$ satisfies all conditions (A1)-(A4) and $0 \in EP\left(\sum\limits_{i=1}^{N} a_i \Phi_i\right) = \bigcap_{i=1}^{N} EP(\Phi_i)$. Then we have

$$Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}.$$

Observe that

$$\begin{aligned}0 &\le \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle\\ &= \xi(-5u_n^2 + u_n y + 4y^2) + \frac{1}{r_n}(y - u_n)(u_n - x_n)\\ &\Leftrightarrow\\ 0 &\le r_n\xi(-5u_n^2 + u_n y + 4y^2) + (y - u_n)(u_n - x_n)\\ &= 4\xi r_n y^2 + (u_n + r_n u_n \xi - x_n)y - u_n^2 - 5\xi r_n(u_n)^2 + u_n x_n.\end{aligned} \tag{5.1}$$

Let $G(y) = 4\xi r_n y^2 + (u_n + r_n u_n\xi - x_n)y - u_n^2 - 5\xi r_n(u_n)^2 + u_n x_n$. Then $G(y)$ is a quadratic function of y with coefficients $a = 4\xi r_n$, $b = u_n + r_n u_n \xi - x_n$, and $c = -u_n^2 - 5\xi r_n(u_n)^2 + u_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (u_n + r_n u_n \xi - x_n)^2 - 4(4\xi r_n)\left(-u_n^2 - 5\xi r_n(u_n)^2 + u_n x_n\right)\\ &= (u_n)^2 + 18\xi r_n(u_n)^2 + 81\xi^2(r_n)^2(u_n)^2 - 18\xi r_n u_n x_n + x_n^2\\ &= (u_n + 9\xi r_n u_n - x_n)^2.\end{aligned}$$

From (5.1), we have $G(y) \ge 0$, for every $y \in \mathbb{R}$. If $G(y)$ has most one solution in $\mathbb{R}$, thus we have $\Delta \le 0$. This implies that

$$u_n = \frac{x_n}{1 + 9\xi r_n}, \tag{5.2}$$

where $\xi = \sum\limits_{i=1}^{N}\left(\frac{2}{7^i} + \frac{1}{N7^N}\right) i$.

It is clear to see that the sequences $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and $\{\lambda_n\}$ satisfy all the conditions of Theorem 3.1. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Table 1 and Figures 1-2 show the values of sequences $\{x_n\}$ and $\{u_n\}$ where $u = x_1 = -5$ and $u = x_1 = 8$ and $n = N = 20$.

Remark 5.2. From the previous example, we can conclude that

(i) Table 1, Figure 1 and Figure 2 show that the sequences $\{u_n\}$ and $\{x_n\}$ converge to 0, where $\{0\} = Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i)$.

(ii) The convergence of $\{u_n\}$ and $\{x_n\}$ can be guaranteed by Theorem 3.1.

Table 1. The values of $\{u_n\}$ and $\{x_n\}$ with $n = N = 20$

	$u = x_1 = -5$		$u = x_1 = 8$	
n	u_n	x_n	u_n	x_n
1	−1.531532	−5.000000	2.450450	8.000000
2	−0.374176	−1.212331	0.598682	1.939730
3	−0.178184	−0.575048	0.285094	0.920077
4	−0.099990	−0.321920	0.159984	0.515072
5	−0.059731	−0.191994	0.095570	0.307190
⋮	⋮	⋮	⋮	⋮
10	−0.007125	−0.022817	0.011401	0.036508
⋮	⋮	⋮	⋮	⋮
16	−0.001899	−0.006073	0.003039	0.009716
17	−0.001702	−0.005442	0.002724	0.008707
18	−0.001552	−0.004960	0.002483	0.007936
19	−0.001433	−0.004578	0.002292	0.007325
20	−0.001335	−0.004266	0.002136	0.006826

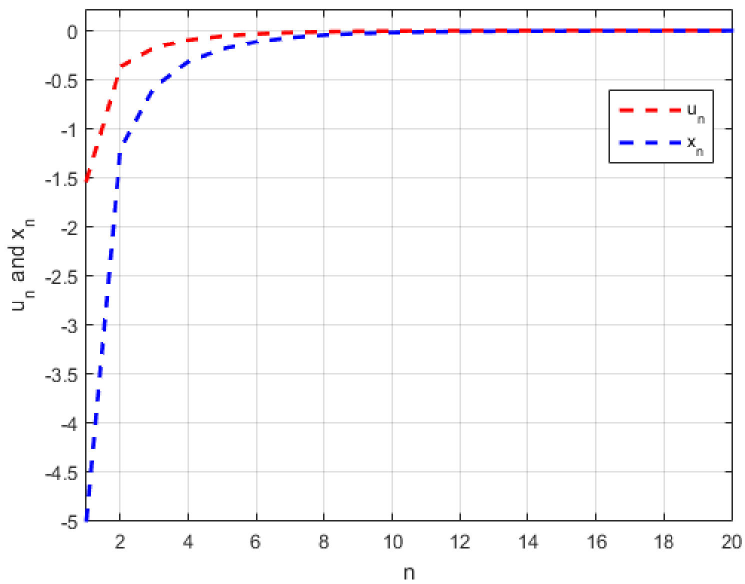

Figure 1. The convergence comparison of $\{u_n\}$ and $\{x_n\}$ with different $u = x_1 = -5$

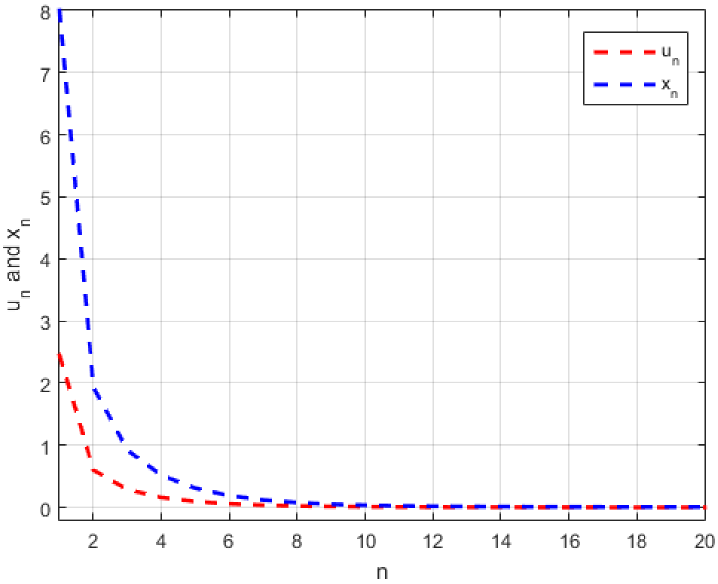

Figure 2. The convergence comparison of $\{u_n\}$ and $\{x_n\}$ with different $u = x_1 = 8$

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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THE METHOD FOR SOLVING FIXED POINT PROBLEM OF G-NONEXPANSIVE MAPPING IN HILBERT SPACES ENDOWED WITH GRAPHS AND NUMERICAL EXAMPLE

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The main aim of this paper is to study a strong convergence theorem of viscosity approximation method for G-nonexpansive mapping defined on a Hilbert space endowed with a directed graph. By using our main result, we give a numerical expample to approximate the value of π.

Key words : G-nonexpansive mappings; viscosity approximation; edge-preserving.

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1. Introduction

The fixed point theory plays an important role in nonlinear functional analysis and is a very useful tool in various fields. In particular, fixed point theorem has been applied in many branches of sciences. For a recent trend of fixed point problem, one of the most interesting problems is the combination of fixed point theory and graph theory. In the past few years, many researchers have studied fixed point theorems in a metric space endowed with a graphs; see [1-4] and references cited therein.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be contraction if there is $0 < k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

The set of all fixed points of a mapping T is denoted by $F(T)$, i.e., $x \in F(T)$ if and only if $x = Tx$.

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ be a set of its edges, assume that G has no parallel edges, we denote G^{-1} as the directed graph obtained from G by reversing the direction of edges. That is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=1}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a (directed) path between any two vertices of G.

For studying contractive-type mappings, the Banach contraction mapping principle, which was firstly introduced by Banach [5] in 1922, has been an important source for solving existence problems in fixed point theory. Some of the contractive-type mapping were studied in many directions, see [6, 7]. In 2008, Jachymski [8] combined the concept of fixed point theory and graph theory in a metric space to generalized Banach contraction mapping principle in a metric space endowed with a directed graph. He also introduced a contractive-type mapping with a directed graph as follows.

Definition 1.1 — [8]. Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$.

We say that a mapping $f : X \to X$ is a G-contraction if f preserves edges of G, *i.e.*,

$$x, y \in X, (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)$$

and there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

In the past few years, many authors have studied a concept of G-contraction in order to improve and extend the above definition, see for instance [9-12] and references cited therein. Let C be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \to C$, then T is said to be G-nonexpansive if the following conditions hold:

(1) T is edge-preserving, *i.e.*, for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;

(2) $\|Tx - Ty\| \leq \|x - y\|$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.

This mapping was introduced by Tiammee *et al.* [13] in 2015.

We know that Halpern iteration process is an important tool in fixed point problem and it can generate a strongly convergent sequence provided that the underlying space is smooth enough. So, in order to prove a strong convergence of the Halpern iteration process in a Hilbert space endowed with a directed graph , Tiammee *et al*. [13] introduced Property G and proved strong convergence of the Halpern iteration process for finding the set of fixed point of G-nonexpansive mappings in a Hilbert space endowed with a directed graph as the following theorem.

***Theorem* 1.2** — *Let C be a nonempty closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence satisfying*

$$x_0 \in C, x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \geq 0. \tag{1}$$

Let $\{x_n\}$ be a sequence defined by Halpern iteration, where $u = x_0$. If $\{x_n\}$ is dominated by Px_0 and $\{x_n\}$ dominates x_0, then $\{x_n\}$ converges strongly to Px_0, where P is the metric projection on $F(T)$.

One of the most interesting iteration processes is the viscosity approximation method introduced by Moudafi [15]. In 2004, Xu [14] studied the such method for a nonexpansive mapping in a Hilbert space and introduced an iterative scheme for finding the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$x_0 \in C, x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0, \tag{2}$$

where $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, $f : C \to C$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Then, they proved a strong convergence theorem under suitable conditions of the parameters $\{\alpha_n\}$.

In this paper, motivated by [13] and [14], we prove a strong convergence theorem for finding the set of fixed point of G-nonexpansive mapping in a Hilbert space endowed with a directed graph. By using our main result, we give a numerical expample to approximate the value of π.

2. Preliminaries

In this paper, we denote " weak and strong convergence " by notations " $\rightharpoonup$" and " $\to$", respectively. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition* [19], *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas are needed to prove the main theorem.

Definition 2.1 — A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to $x \in H$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. In this case, we write $x_n \rightharpoonup x$.

***Theorem* 2.2** — *[16]. Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset C of X is weakly compact,* i.e.*, every bounded sequence in C has a weakly convergent subsequence.*

Lemma 2.3 — [17]. Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$(1)\ : \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2)\ : \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=0}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 — [18]. Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.5 — Let H be a real Hilbert space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x, y \in H$.

Property G : [13]. Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$, where $V(G) = C$, be a directed graph. Then C is said to have Property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

The following basic definitions of domination in graphs [20, 21] are needed to prove the main theorem.

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $z \in V(G) \setminus X$ there exists $x \in X$ such that $(x, z) \in E(G)$ and we say that x dominates z or z is dominated by x. Let $z \in V$, a set $X \subseteq V$ is dominated by z if $(z, x) \in E(G)$ for any $x \in X$ and we say that X dominates z if $(x, z) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops.

***Theorem* 2.6** — *[13]. Let X be a normed space and $G = (V(G), E(G))$ a directed graph with $V(G) = X$. Suppose $T : X \to X$ is a G-nonexpansive mapping. If X has a Property G, then T is continuous.*

***Theorem* 2.7** — *[13]. Let X be a Hilbert space and C be a subset of X having Property G. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose $T : C \to C$ is a G-nonexpansive mapping and $F(T) \times F(T) \subseteq E(G)$. Then $F(T)$ is closed and convex.*

Definition 2.8 — [13]. Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

3. Main Result

In this section, we prove a strong convergence theorem of viscosity approximation methods for G-nonexpansive mapping in Hilbert spaces endowed with a directed graph.

The following Proposition is needed to prove the main theorem.

Proposition 3.1 — Let C be a convex subset of a vector space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let G be transitive, $T : C \to C$ be edge-preserving, and $f : C \to C$ be a G-contraction mapping. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, n \geq 0, \end{cases}$$

where $(f(x_0), Tx_0)$ and $(x_0, f(x_0))$ are in $E(G)$. If $\{x_n\}$ dominates x_0, then (x_n, x_{n+1}), (x_0, x_n), (x_0, Tx_n), and (x_n, Tx_n) are in $E(G)$ for all $n \in \mathbb{N}$.

PROOF : We prove by induction. By transitivity of G and since $(x_0, f(x_0))$ and $(f(x_0), Tx_0)$ are in $E(G)$, we have $(x_0, Tx_0) \in E(G)$. Since $E(G)$ is convex, $(x_0, f(x_0))$ and (x_0, Tx_0) are in $E(G)$, we obtain

$$(\alpha_0 x_0 + (1-\alpha_0)x_0, \alpha_0 f(x_0) + (1-\alpha_0)Tx_0) = (x_0, x_1) \in E(G).$$

Since T is edge-preserving, f is G-contraction mapping and $(x_0, x_1) \in E(G)$, then $(Tx_0, Tx_1) \in E(G)$ and $((f(x_0), f(x_1))) \in E(G)$, respectively. By transitivity of G and since (x_0, Tx_0) and (Tx_0, Tx_1) are in $E(G)$, we obtain $(x_0, Tx_1) \in E(G)$. By assumption, $(x_1, x_0) \in E(G)$. Then, by transitivity of G and $(x_0, Tx_1) \in E(G)$, we get $(x_1, Tx_1) \in E(G)$. By transitivity of G and since $(f(x_0), Tx_0)$ and (Tx_0, Tx_1) are in $E(G)$, we obtain $(f(x_0), Tx_1) \in E(G)$. Since $E(G)$ is convex, $(f(x_0), Tx_1)$ and $(f(x_0), f(x_1))$ are in $E(G)$, we obtain

$$(\alpha_1 f(x_0) + (1-\alpha_1)f(x_0), \alpha_1 f(x_1) + (1-\alpha_1)Tx_1) = (f(x_0), x_2) \in E(G).$$

By transitivity of G and since (x_1, x_0) and $(x_0, f(x_0))$ are in $E(G)$, we obtain $(x_1, f(x_0)) \in E(G)$. Again, by transitivity of G and since $(x_1, f(x_0))$ and $(f(x_0), x_2)$ are in $E(G)$, we obtain $(x_1, x_2) \in E(G)$.

Next, assume that (x_k, x_{k+1}), (x_0, x_k), (x_0, Tx_k), and (x_k, Tx_k) are in $E(G)$. Since T is edge-preserving and $(x_k, x_{k+1}) \in E(G)$, then $(Tx_k, Tx_{k+1}) \in E(G)$. By transitivity of G, and (x_0, Tx_k), (Tx_k, Tx_{k+1}) are in $E(G)$, we have $(x_0, Tx_{k+1}) \in E(G)$. Since $\{x_n\}$ dominates x_0, we have $(x_{k+1}, x_0) \in E(G)$. By transitivity of G, and (x_{k+1}, x_0), (x_0, Tx_{k+1}) are in $E(G)$, we have $(x_{k+1}, Tx_{k+1}) \in E(G)$. By transitivity of G, and (x_0, x_k), (x_k, x_{k+1}) are in $E(G)$, we get $(x_0, x_{k+1}) \in E(G)$. Since T is edge-preserving, f is G-contraction mapping, and $(x_0, x_{k+1}) \in E(G)$, we have (Tx_0, Tx_{k+1}), $(f(x_0), f(x_{k+1}))$ are in $E(G)$, respectively. By transitivity of G and since $(f(x_0), Tx_0)$ and (Tx_0, Tx_{k+1}) are in $E(G)$, we obtain $(f(x_0), Tx_{k+1}) \in E(G)$. Since $E(G)$ is convex, $(f(x_0), Tx_{k+1})$ and $(f(x_0), f(x_{k+1}))$ are in $E(G)$, we obtain

$$\begin{aligned}(\alpha_{k+1} f(x_0) + (1-\alpha_{k+1})f(x_0), \alpha_{k+1} f(x_{k+1}) + (1-\alpha_{k+1})Tx_{k+1})\\ = (f(x_0), x_{k+2}) \in E(G).\end{aligned}$$

By transitivity of G and since (x_{k+1}, x_0) and $(x_0, f(x_0))$ are in $E(G)$, we obtain $(x_{k+1}, f(x_0)) \in E(G)$. Again, by transitivity of G and since $(x_{k+1}, f(x_0))$ and $(f(x_0), x_{k+2})$ are in $E(G)$, we obtain $(x_{k+1}, x_{k+2}) \in E(G)$.

So, by induction, we can conclude that (x_n, x_{n+1}), (x_0, x_n), and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$. □

***Theorem* 3.2** — *Let C be a nonempty closed convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Let $f : C \to C$ be a G-contraction mapping with coefficient $\alpha \in (0,1)$. Assume that there exists $x_0 \in C$ such that $(f(x_0), Tx_0)$ and $(x_0, f(x_0))$ are in $E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\} \subseteq (0,1)$ satisfies

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0 \text{ , (ii)} \sum_{n=0}^{\infty} \alpha_n = \infty \text{ , (iii)} \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

If $\{x_n\}$ dominates $P_{F(T)}f(x_0)$ and $\{x_n\}$ dominates x_0, then the sequence $\{x_n\}$ converge strongly to $x_0 = P_{F(T)}f(x_0)$.

PROOF : We divide the proof into five steps:

Step 1 : We show that the sequence $\{x_n\}$ is bounded. Let $x^* = P_{F(T)}f(x_0)$. From Proposition 3.1, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Since $x^* \in F(T)$ and $x^* = P_{F(T)}f(x_0)$ is dominated by $\{x_n\}$, we have $(x_n, x^*) \in E(G)$. From the definition of x_n, we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n\|f(x_n) - x^*\| + (1-\alpha_n)\|Tx_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n\alpha\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\ &= (1-\alpha_n(1-\alpha))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\|. \end{aligned}$$

By mathematical induction, we obtain that

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\right\}, \forall n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is bounded and so are $\{Tx_n\}$ and $\{f(x_n)\}$.

Step 2 : We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. From the definition of x_n and $(x_n, x_{n+1}) \in E(G)$, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)Tx_n - \alpha_{n-1} f(x_{n-1}) \\
&\quad -\ (1-\alpha_{n-1})Tx_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1}) \\
&\quad +\ (1-\alpha_n)Tx_n - (1-\alpha_n)Tx_{n-1} + (1-\alpha_n)Tx_{n-1} \\
&\quad -\ (1-\alpha_{n-1})Tx_{n-1}\| \\
&= \|\alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) f(x_{n-1}) \\
&\quad +\ (1-\alpha_n)(Tx_n - Tx_{n-1}) + (\alpha_{n-1} - \alpha_n)Tx_{n-1}\| \\
&\le \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\quad +\ (1-\alpha_n)\|Tx_n - Tx_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\
&\le \alpha_n \alpha \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\quad +\ (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\| \\
&= (1 - \alpha_n(1-\alpha))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\quad +\ |\alpha_n - \alpha_{n-1}| \|Tx_{n-1}\|. \qquad (4)
\end{aligned}$$

Applying Lemma 2.3, (4), and the conditions (i), (ii), (iii), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (5)$$

Step 3 : We show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Tx_n\|.
\end{aligned}$$

Because $\{Tx_n\}$ and $\{f(x_n)\}$ are bounded, from the condition (i), (ii), and (5), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (6)$$

Step 4 : We show that $\lim_{n\to\infty} \sup \langle f(z_0) - z_0, x_n - z_0 \rangle \le 0$ where $z_0 = P_{F(T)} f(z_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0 \rangle = \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0 \rangle . \qquad (7)$$

Because all the $\{x_{n_k}\}$ lie in the weakly compact set C and C has Property G, we may assume without loss of generality that $\{x_{n_k}\} \rightharpoonup \omega$ for some $\omega \in C$ and $(x_{n_k}, \omega) \in E(G)$. Suppose $\omega \notin F(T)$, then $\omega \neq T\omega$. By G-nonexpansiveness of T, (6), and the Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - T\omega\| \\
&\leq \liminf_{k\to\infty} \big(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - T\omega\|) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\|.
\end{aligned}$$

This is a contradiction. Then $\omega \in F(T)$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in F(T)$. By (7) and Lemma 2.4, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle &= \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle \\
&= \langle f(z_0) - z_0, \omega - z_0\rangle \\
&\leq 0. \qquad (8)
\end{aligned}$$

Step 5 : Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{F(T)} f(z_0)$. By G-nonexpansiveness of T and $(z_0, x_n) \in E(G)$, and Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + (1-\alpha_n)(Tx_n - z_0)\|^2 \\
&\leq \|(1-\alpha_n)(Tx_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&= (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\|f(x_n) - f(z_0)\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\alpha\|x_n - z_0\|^2 + \alpha_n\alpha\|x_{n+1} - z_0\|^2 \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

It implies that

$$\|x_{n+1} - z_0\|^2 \leq \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle$$

$$= \left(1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)\|x_n - z_0\|^2 + \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left(\frac{\alpha_n}{2(1-\alpha)}\|x_n - z_0\|^2.\right.$$
$$\left. + \frac{1}{1-\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\right).$$

From the conditions (i), (ii), (8), and Lemma 2.3, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}f(z_0)$. This completes the proof. $\square$

4. Numerical Results

The purpose of this section we give a numerical example to support our some result. The following example is given for supporting Theorem 3.2.

Example 4.1 : Let $H = \mathbb{R}$ and $C = [0,1]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x,y) : x, y \in [0, \frac{3}{5}]$ such that $|x-y| \leq \frac{1}{5}\}$. Let $f : C \to C$ be defined by $f(x) = \frac{x}{9}$, for all $x \in [0,1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{10}x & \text{if } x \in [0,1), \\ \frac{8}{5} & \text{if } x = 1. \end{cases}$$

Solution : We observe that $F(T) = \{0\}$. Choose $x_0 = \frac{1}{5}$, then $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x,y) \in E(G)$. Then $x, y \in [0, \frac{3}{5}]$ and $|x - y| \leq \frac{1}{5}$. It implies that

$$|Tx - Ty| \leq \frac{1}{10}|x - y| \leq |x - y| \leq \frac{1}{5}.$$

Then, we have $(Tx, Ty) \in E(G)$ and $\|Tx - Ty\| \leq \|x - y\|$. Thus T is G-nonexpansive. For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{2(n+1)}$. We rewrite (3) as follows:

$$x_{n+1} = \left(\frac{1}{2(n+1)}\right)\left(\frac{x_n}{9}\right) + \left(1 - \frac{1}{2(n+1)}\right)\left(\frac{x_n}{10}\right). \tag{9}$$

Since $x_0 = \frac{1}{5} \in [0, \frac{1}{5}]$, from (9), we have

$$x_1 = \left(\frac{1}{2(1)}\right)\left(\frac{x_0}{9}\right) + \left(1 - \frac{1}{2(1)}\right)\left(\frac{x_0}{10}\right).$$

By the convexity, we have $x_1 \in [0, \frac{1}{5}]$. Since $x_1 \in [0, \frac{1}{5}]$ and (9), we have

$$x_2 = \left(\frac{1}{2(2)}\right)\left(\frac{x_2}{9}\right) + \left(1 - \frac{1}{2(2)}\right)\left(\frac{x_2}{10}\right).$$

By the convexity, we have $x_2 \in [0, \frac{1}{5}]$. By continuing in this way, we have $x_n \in [0, \frac{1}{5}]$, for all $n \in \mathbb{N}$. It implies that $x_n \leq \frac{1}{5}$ for all $n \in \mathbb{N}$. It follows that $(x_n, P_{F(T)}f(x_0)) = (x_n, 0) \in E(G)$. That is. $P_{F(T)}f(x_0)$ is dominated by $\{x_n\}$. It can be observed that parameters satisfy all the conditions of Theorem 3.2 and $C = [0, 1]$ satisfy Property G. Hence, the sequence $\{x_n\}$ converges strongly to 0.

Next, we show that T is not a nonexpansive mapping. Choose $x = 1$ and $y = \frac{3}{5}$, we have

$$\left|T(1) - T\left(\frac{3}{5}\right)\right| = \left|\frac{8}{5} - \frac{3}{50}\right| = \frac{77}{50} > \frac{2}{5} = \left|1 - \frac{3}{5}\right|.$$

Mathematicians know that the number π is an important mathematical constant. For the previous decades, many researcher have been trying to approximate the value of π; see [22, 23] and the references therein. By using our main result, we introduce the new method to approximate the value of π as shown in the following example.

Example 4.2 : Let $H = \mathbb{R}$ and $C = [3, 4]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x, y) : x, y \in [3, \frac{18}{5}]$ such that $|x - y| \leq \frac{16}{5}\}$. Let $f : C \to C$ be defined by $f(x) = \frac{1}{5}x + \frac{4}{5}(\pi)$, for all $x \in [3, 4]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{3}x + \frac{2}{3}(\pi) & \text{if } x \in [3, 4), \\ \frac{56}{35} & \text{if } x = 4. \end{cases}$$

Solution : We observe that $F(T) = \{\pi\}$. Choose $x_0 = \frac{16}{5}$, then $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x, y) \in E(G)$. Then $x, y \in [3, \frac{18}{5}]$ and $|x - y| \leq \frac{16}{5}$. It implies that

$$|Tx - Ty| = \left|\frac{1}{3}x + \frac{2}{3}(\pi) - \frac{1}{3}y - \frac{2}{3}(\pi)\right| \leq \frac{1}{3}|x - y| \leq |x - y| \leq \frac{16}{5}.$$

Then, we have $(Tx, Ty) \in E(G)$ and $\|Tx - Ty\| \leq \|x - y\|$. Thus T is G-nonexpansive. For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{2(n+1)}$. We rewrite (3) as follows:

$$x_{n+1} = \left(\frac{1}{2(n+1)}\right)\left(\frac{1}{5}x_n + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(n+1)}\right)\left(\frac{1}{3}x_n + \frac{2}{3}(\pi)\right). \tag{10}$$

Since $x_0 = \frac{16}{5} \in [3, \frac{16}{5}]$, from (10), we have

$$x_1 = \left(\frac{1}{2(1)}\right)\left(\frac{1}{5}x_0 + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(1)}\right)\left(\frac{1}{3}x_0 + \frac{2}{3}(\pi)\right).$$

By the convexity, we have $x_1 \in [3, \frac{16}{5}]$. Since $x_1 \in [3, \frac{16}{5}]$ and (10), we have

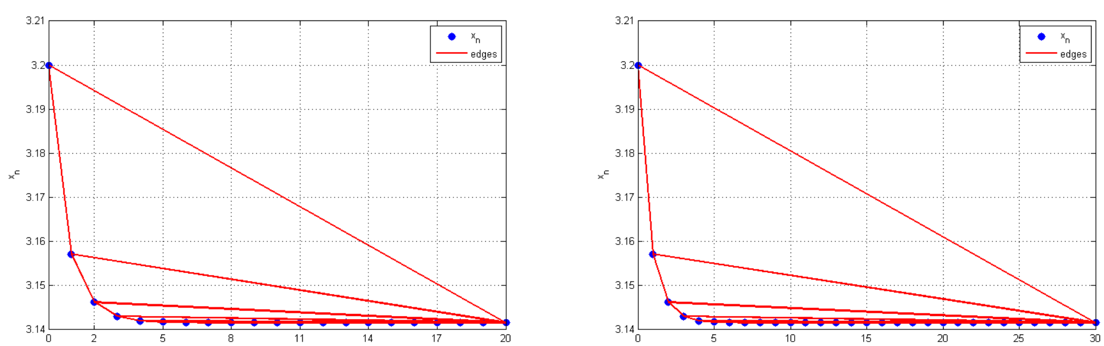
$$x_2 = \left(\frac{1}{2(2)}\right)\left(\frac{1}{5}x_1 + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(2)}\right)\left(\frac{1}{3}x_1 + \frac{2}{3}(\pi)\right).$$

By the convexity, we have $x_2 \in (3, \frac{16}{5}]$. By continuing in this way, we have $x_n \in [3, \frac{16}{5}]$, for all $n \in \mathbb{N}$. It implies that $3 \leq x_n \leq \frac{16}{5}$ for all $n \in \mathbb{N}$. Then $|x_n - \pi| \leq \frac{16}{5}$ for all $n \in \mathbb{N}$. It follows that $(x_n, P_{F(T)}f(\pi)) = (x_n, \pi) \in E(G)$. That is. $P_{F(T)}f(\pi)$ is dominated by $\{x_n\}$. It can be observed that parameters satisfy all the conditions of Theorem 3.2 and $C = [3, 4]$ satisfy Property G. Since $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to π.

Next, we show that T is not a nonexpansive mapping. Choose $x = 4$ and $y = \frac{18}{5}$, we have

$$\begin{aligned}\left|T(4) - T\left(\frac{18}{5}\right)\right| &= \left|\frac{56}{35} - \left(\frac{1}{3}\left(\frac{18}{5}\right) + \frac{2}{3}(\pi)\right)\right| \\ &\approx \left|\frac{56}{35} - \left(\frac{1}{3}\left(\frac{18}{5}\right) + \frac{2}{3}\left(\frac{22}{7}\right)\right)\right| \\ &= \frac{178}{105} \\ &> \frac{2}{5} \\ &= \left|4 - \frac{18}{5}\right|.\end{aligned}$$

Using the algorithm (10) and choosing $x_0 = \frac{16}{5}$ with $n = 20$ and $n = 30$, we have the numerical result to approximate the value of π as shown in Table 1 and Figure 1.



(A) n=20 (B) n=30

Figure 1: The convergence of $\{x_n\}$ with initial values $x_0 = \frac{16}{5}$.

$n=20$		$n=30$	
n	x_n	n	x_n
0	3.200000000000000	0	3.200000000000000
1	3.157167945965848	1	3.157167945965848
2	3.146265241302610	2	3.146265241302610
3	3.143046347544892	3	3.143046347544892
4	3.142052990008907	4	3.142052990008907
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	3.141593185425522	15	3.141592659742157
$\vdots$	$\vdots$	$\vdots$	$\vdots$
16	3.141592654255843	26	3.141592653589803
17	3.141592653809197	27	3.141592653589797
18	3.141592653662115	28	3.141592653589794
19	3.141592653613647	29	3.141592653589793
20	3.141592653597665	30	3.141592653589793

Table 1: The values of the sequences $\{x_n\}$ with initial value $x_0 = \frac{16}{5}$.

5. Conclusion

In this work, we introduce a viscosity approximation method of G-nonexpansive mapping defined on a Hilbert space endowed with a directed graph. We obtain a strong convergence theorem for the sequence generated by the proposed method under suitable conditions. However, we should like remark the following:

1. In Theorem 3.2, we use the concept of a viscosity approximation method and our result is proved with an assumption on a directed graph, which is a different result from Xu [14].
2. From Theorem 3.2, we can conclude that the sequence $\{x_n\}$, in Example 4.2, converges to π.
3. In Example 4.2, the sequence $\{x_n\}$ converges to π as shown in the Table 1 and Figure 1.
4. In order to gain more accuracy of π, the iterative approximation is depended on the number of n as shown in the Table 1.

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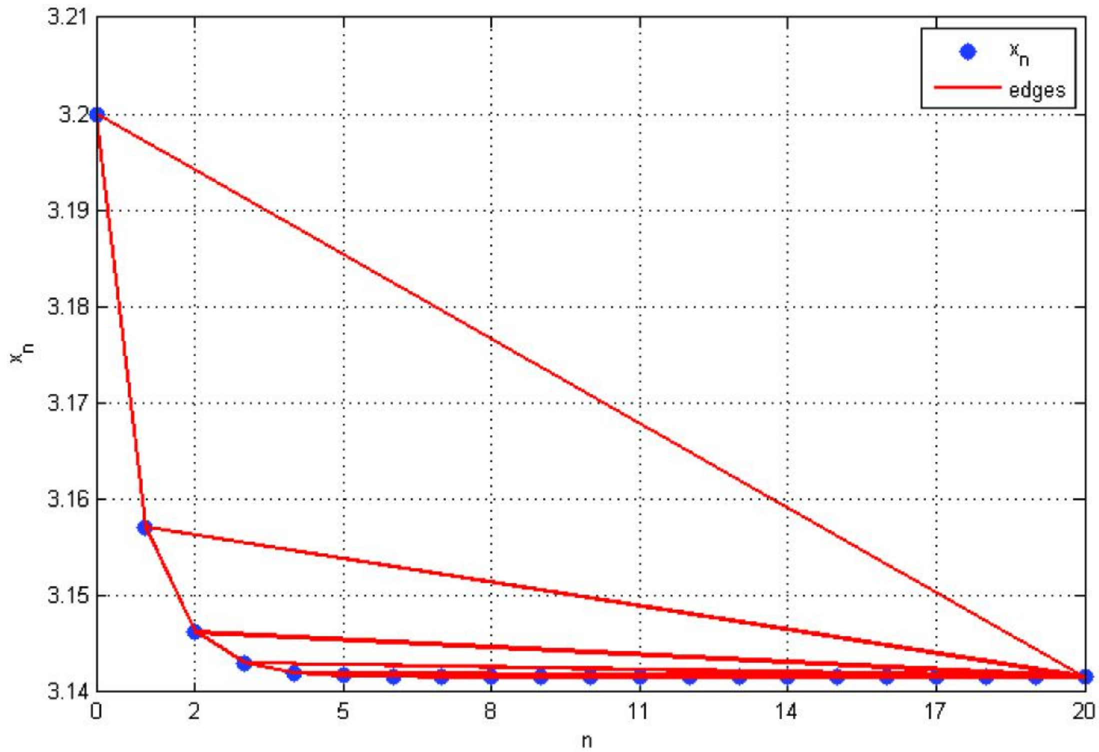
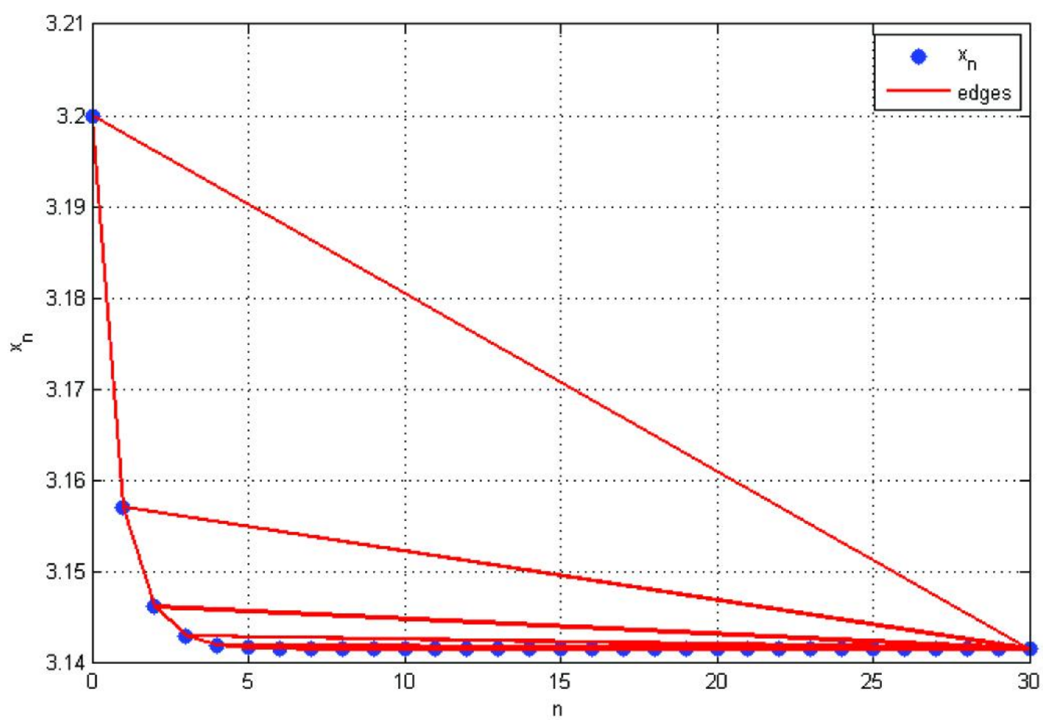

3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
0
2
5
8
11
14
17
20
n
x_n
x_n
edges


3.21
3.2
3.19
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3.17
3.16
3.15
3.14
0
5
10
15
20
25
30
n
x_n
x_n
edges

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The Convergence Theorem for a Square α-Nonexpansive Mapping in a Hyperbolic Space

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Abstract In this paper, we prove Δ-convergence theorems of the generalized Picard normal S_5-iterative process to approximate a fixed point for square α-nonexpansive mappings. Moreover, we obtain some properties of such mappings on a nonempty subset of a hyperbolic space.

MSC: 47H09; 47H10

Keywords: fixed point set; square α-nonexpansive mapping; generalized Picard normal S_5-iterative; hyperbolic spaces

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1. Introduction

Let X be a metric space and let M be a nonempty closed convex subset of X. A mapping $T : M \to M$ is said to be nonexpansive, if $d(Tx, Ty) \leq d(x, y)$, for each $x, y \in M$. In 2011, Aoyama and Kohsaka [1] introduced the class of α-nonexpansive mappings in Banach spaces as follow: Let X be a Banach space and M be a nonempty closed and convex subset of X. A mapping $T : M \to M$ is said to be α-nonexpansive if for all $x, y \in M$ and $\alpha < 1$, $\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|x - Ty\|^2 + (1 - 2\alpha) \|x - y\|^2$. This class contains the class of nonexpansive mappings and is related to the class of firmly nonexpansive mappings in Banach spaces. Then $F(T)$ is nonempty if and only if there exists $x \in M$ such that $\{T^n x\}$ is bounded, where X is a uniformly convex Banach

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space, and M is a nonempty, closed and convex subset of X, and $T : M \to M$ is an α-nonexpansive mapping for some real number α such that $\alpha < 1$.

In 2013, Naraghirad *et al.* [2] considered appropriate Ishihawa iterate algorithms ensure weak and strong convergence to a fixed point of such a mapping. Their theorems are also extended to CAT(0) spaces as follow : Let $\{x_n\}$ be a sequence with $\{x_1\}$ in M defined by

$$\begin{cases} y_n = \beta_n T x_n \oplus (1-\beta_n)x_n, \\ x_{n+1} = \gamma_n T y_n \oplus (1-\gamma_n)x_n. \end{cases}$$

In 2016, Song *et al.* [3] introduced the concept of monotone α-nonexpansive mappings in an ordered Banach space E with the partial order $\leq$, which contains monotone α-nonexpansive mappings as special case. With the help of the Mann iteration. In 2017, Shukla *et al.* [4] introduced some existence and convergence results for monotone α-nonexpansive mappings in partially ordered hyperbolic metric spaces as follow : Let $\{u_n\}$ be defined by

$$\begin{cases} u_1 \in K, \\ v_n = \gamma_n T(u_n) \oplus (1-\gamma_n)u_n, \\ u_{n+1} = \beta_n T(v_n) \oplus (1-\beta_n)T(u_n). \end{cases}$$

In 2018, Mebawondu and Izuchukwu [5] introduced some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces as follow : Suppose that the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in C, \\ z_n = W(x_n, Tx_n, \beta_n), \\ y_n = W(z_n, Tz_n, \gamma_n), \\ x_{n+1} = W(Ty, 0, 0). \end{cases}$$

Recently, there are some works that relate to hyperbolic spaces such as CAT(0) spaces that appeared (see [6–17]).

In this paper, we prove convergence and Δ-convergence theorems of the generalized Picard normal S_5-iterative process to approximate a fixed point for α-nonexpansive mappings. Moreover, we prove some properties of such mappings on a nonempty subset of a hyperbolic space.

2. Preliminaries

Throughout this paper, we work in the setting of hyperbolic spaces which were introduced by Kohlenbach [18].

Definition 2.1. A hyperbolic space is a metric space (X, d) with a mapping $W : X^2 \times [0,1] \to X$ satisfying the following conditions.
(i) $d(u, W(x,y,\alpha)) \leq (1-\alpha)d(u,x) + \alpha d(u,y)$;
(ii) $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta| d(x,y)$;
(iii) $W(x,y,\alpha) = W(y,x,1-\alpha)$;
(iv) $d(W(x,z,\alpha), W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$.
for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$.

Some definitions on hyperbolic space are considered as follow:

Definition 2.2. [19] Let X be hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$. A nonempty subset $M \subseteq X$ is said to be convex, if $W(x, y, \alpha) \in M$ for all $x, y \in M$ and $\alpha \in [0, 1]$. A hyperbolic space is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$

$$d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r,$$

provided $d(x, u) \leq r, d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$. A map $\eta : (0,\infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X. η is said to be monotone, if it decreases with r (for a fixed ϵ), i.e., $\forall \epsilon > 0$, $\forall r_1 \geq r_2 > 0$ $[\eta(r_2, \epsilon) \leq \eta(r_1, \epsilon)]$. We denote the unit sphere and the closed unit ball centered at the origin of M by S_M and B_M, respectively. We also denote the closed ball with radius $r > 0$ centered at the origin of M by rB_M.

Definition 2.3. [20] Let $\{x_n\}$ be a bounded sequence in a hyperbolic space (X, d). For $x \in X$, we define a continuous functional $r(\cdot, x_n) : X \to [0,\infty)$ by

$$r(x, x_n) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\}.$$

The asymptotic center $A_M(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $M \subseteq X$ is the set

$$A_M(\{x_n\}) = \{x \in X : r(x, x_n) \leq r(y, x_n), \quad \forall y \in M\}.$$

This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot, x_n)$ in M. If the asymptotic center is taken with respect to X, then it is simply denoted by $A_M(\{x_n\})$. It is known that uniformly convex hyperbolic spaces enjoy the property that ounded sequences have unique asymptotic centers with respect to closed convex subsets.

Definition 2.4. Recall that a sequence $\{x_n\}$ in X is said to be Δ-convergent which converges to a point $x \in X$ if x is the unique asymptotic centers of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$. Moreover, if $x_n \to x$, then $\Delta - \lim_{n\to\infty} x_n = x$ (see [18],[21]).

Lemma 2.5. [20] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unqiue asymtotic center with respect to any nonempty closed convex subset M of X.*

Lemma 2.6. [20] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n, p) \leq c, \limsup_{n\to\infty} d(y_n, p) \leq c$ *and* $\limsup_{n\to\infty} d(W(x_n, y_n, \alpha_n)$
$, p) = c$*, for some $c \geq 0$. Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.7. ([21–23]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in M has a unique asymptotic center in M.*

Lemma 2.8. [5] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A_M(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A_M(\{x_{n_k}\}) = \{x_1\}$. and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

Definition 2.9. Let M be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to M if for all $x \in M$ and $n \geq 1$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Next, we defined Picard Normal S_5-iteration process (PNS_5) in hyperbolic spaces as follow : Let M be a nonempty closed convex subset of a hyperbolic space X and $T : M \to M$ be a mapping which asymptotically Suzuki-generalized nonexpansive, for any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Tu_n, 0, 0) \\ u_n = W(v_n, Tv_n, \beta_n) \\ v_n = W(y_n, Ty_n, \gamma_n) \\ y_n = W(z_n, Tz_n, \delta_n) \\ z_n = W(x_n, Tx_n, \zeta_n), \ n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\zeta_n\}$ in $(0, 1)$.

3. Main Results

In this section, we will prove some properties for class of α-nonexpansive mappings in hyperbolic spaces.

Definition 3.1. Let (X, d) be a metric space and M be nonempty subset of X. Then $T : M \to M$ is said to be a square α-nonexpansive mapping (or α-nonexpansive mapping), if $\alpha < 1$ such that

$$d^2(Tx, Ty) \leq \alpha d^2(Tx, y) + \alpha d^2(x, Ty) + (1 - 2\alpha)d^2(x, y),$$

for all $x, y \in M$.

Now, we give example for a square α-nonexpansive mapping as follows :

Example 3.2. Let M be a nonempty closed and convex subset of a complete hyperbolic space X, and let $S, T : M \to M$ be firmly nonexpansive mappings such that $S(M)$ and $T(M)$ are contained by rB_M for some positive real number r. Let α and δ be real numbers such that $0 < \alpha \leq 1$ and $\delta \geq (1 + 2/\sqrt{\alpha})r$. Then the mapping $U : M \to M$ is defined by

$$Ux = \begin{cases} Sx & (x \in \delta B_M); \\ Tx & (\text{otherwise}), \end{cases} \tag{3.1}$$

then U is a square α-nonexpansive (See [1]).

From lemma of Naraghirad [2], we obtain the lemma as follow :

Lemma 3.3. *Let M be a nonempty subset of a hyperbolic space X. Let $T : M \to M$ be a square α-nonexpansive mapping for some $\alpha < 1$. Let $x, y \in M$, then the following assertions hold* *(i) If* $0 \le \alpha < 1$*, then*

$$d^2(x,Ty) \le \frac{1+\alpha}{1-\alpha} d^2(x,Tx) + \frac{2}{1-\alpha}(\alpha d(x,y) + d(Tx,Ty))d(x,Tx) + d^2(x,y)$$

(ii) If $\alpha < 0$*, then*

$$d^2(x,Ty) \le d^2(x,Tx) + \frac{2}{1-\alpha}[(-\alpha)d(x,y) + d(Tx,Ty)]d(x,Tx) + d^2(x,y)$$

Proof. let $x, y \in M$.
(i) Suppose that $0 \le \alpha < 1$. Consider

$$\begin{aligned}
d^2(x,Ty) &\le (d(x,Tx) + d(Tx,Ty))^2 \\
&= d^2(x,Tx) + d^2(Tx,Ty) + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-2\alpha)d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha(d(Tx,x) + d(x,y))^2 + \alpha d^2(x,Ty) + (1-2\alpha)d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,x) + \alpha d^2(x,y) + 2\alpha d(Tx,x)d(x,y) + \alpha d^2(x,Ty) \\
&\quad + (1-2\alpha)d^2(x,y) + 2d(x,Tx)d(Tx,Ty) \\
&= (1+\alpha)d^2(x,Tx) + 2\alpha d(Tx,x)d(x,y) + \alpha d^2(x,Ty) \\
&\quad + (1-\alpha)d^2(x,y) + 2d(x,Tx)d(Tx,Ty).
\end{aligned}$$

We obtain that

$$d^2(x,Ty) \le \frac{(1+\alpha)}{1-\alpha} d^2(x,Tx) + \frac{2}{1-\alpha}(\alpha d(x,y) + d(Tx,Ty))d(Tx,x) + d^2(x,y).$$

(ii) Suppose that $\alpha < 0$. Consider

$$\begin{aligned}
d^2(x,Ty) &\le (d(x,Tx) + d(Tx,Ty))^2 \\
&= d^2(x,Tx) + d^2(Tx,Ty) + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-2\alpha)d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&= d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) - \alpha d^2(x,y) \\
&\quad + 2d(x,Tx)d(Tx,Ty) \\
&\le d^2(x,Tx) + \alpha d^2(Tx,y) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) \\
&\quad - \alpha[d^2(x,Tx) + d^2(Tx,y) + 2d(x,Tx)d(Tx,y)] + 2d(x,Tx)d(Tx,Ty) \\
&= (1-\alpha)d^2(x,Tx) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) \\
&\quad - 2\alpha d(x,Tx)d(Tx,y) + 2d(x,Tx)d(Tx,Ty) \\
&= (1-\alpha)d^2(x,Tx) + \alpha d^2(x,Tx) + \alpha d^2(x,Ty) + (1-\alpha)d^2(x,y) \\
&\quad + 2[(\alpha)d(Tx,y) + d(Tx,Ty)]d(x,Tx),
\end{aligned}$$

this implies that
$d^2(x,Ty) \le d^2(x,Tx) + \frac{2}{1-\alpha}[(-\alpha)d(Tx,y) + d(Tx,Ty)]d(x,Tx) + d^2(x,y)$. ∎

Lemma 3.4. *Let M be a nonempty closed and convex subset of a hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping for some real number $\alpha < 1$. In case $0 \le \alpha < 1$, we have $F(T) \neq \emptyset$ if and only if $\{T^n x\}_{n=1}^{\infty}$ is bounded for some $x \in M$. If M is compact, then $F(T) \neq \emptyset$.*

Proof. Assume that $0 \le \alpha < 1$. The necessity is obvious. We verify the sufficiency. Suppose that $\{T^n x\}_{n=1}^{\infty}$ is bounded for some x in M. Set $x_n := T^n x$ for $n = 1, 2, \ldots$. By the boundedness of $\{x_n\}_{n=1}^{\infty}$, there exists z in X such that $A_M(\{x_n\}) = \{z\}$. It follows from Lemma 2.6 that $z \in M$. Furthermore, we have

$$d^2(x_n, Tz) \le \alpha d^2(x_n, z) + \alpha d^2(x_{n-1}, Tz) + (1 - 2\alpha) d^2(x_n, z), \qquad \forall n = 1, 2, \ldots.$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n, Tz) \le \alpha \limsup_{n\to\infty} d^2(x_n, z) + \alpha \limsup_{n\to\infty} d^2(x_{n-1}, Tz) \\ + (1 - 2\alpha) \limsup_{n\to\infty} d^2(x_n, z).\end{aligned}$$

We obtain

$$\limsup_{n\to\infty} d^2(x_n, Tz) \le \limsup_{n\to\infty} d^2(x_n, z).$$

Consequently, $Tz \in A_M(\{x_n\}) = \{z\}$, we obtain that $F(T) \neq \emptyset$.
Next, we assume that $\alpha < 0$ and M is compact. In particular, T is continuous and the sequence of $x_n := T^n x$ for any $x \in M$ is bounded. We adapt in [Lemmas 3.1 and 3.2][24], we have μ is a Banach limit, i.e., μ is a bounded unital positive linear functional of l_∞ such that $\mu \circ s = \mu$, where s is the left shift operator on l_∞. We write μ_n, a_n for the value of $\mu(a)$ with $a = (a_n)$ in l_∞ as usual. In particular, $\mu_n\ a_{n+1} = \mu(s(a)) = \mu(a) = \mu_n a_n$. We get

$$\mu_n d^2(x_n, Ty) \le \mu_n d^2(x_n, y), \qquad \forall y \in M, \tag{3.2}$$

and

$$g(y) := \mu_n d^2(x_n, y)$$

defines a continuous function from M into $\mathbb{R}$.

By compactness, there exists y in M such that $g(y) = \inf g(M)$. Suppose that there is another z in M such that $g(z) = g(y)$. Let m be the midpoint by definition 2.1, we see that g is convex. Thus, $g(m) = g(y)$ too. Observing the comparison triangles in $\mathbb{E}^2$, we have

$$d^2(x_n, y) + d^2(x_n, z) \ge 2d^2(x_n, m) + \frac{1}{2} d^2(y, z), \qquad \forall n = 1, 2, \ldots.$$

Consequently,

$$\mu_n d^2(x_n, y) + \mu_n d^2(x_n, z) \ge 2\mu_n d^2(x_n, m) + \frac{1}{2}\mu_n d^2(y, z).$$

So,

$$g(y) + g(z) \ge 2g(m) + \frac{1}{2} d^2(y, z).$$

Since $g(y) = g(z) = g(m)$, we have $y = z$. Finally, it follows from (3.2) that $g(Ty) \le g(y) = \inf g(M)$. By uniqueness, we have $Ty = y \in F(T)$. ∎

Lemma 3.5. *Let M be a nonempty closed and convex subset of a hyperbolic space X. Let $T : M \to M$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Proof. Let $\{x_n\} \subset F(T)$ such that $\{x_n\}$ converges to y for some $y \in M$. We will show that $y \in F(T)$. We consider $d^2(x_n, Ty) \leq \alpha d^2(x_n, y) + \alpha d^2(Ty, x_n) + (1-2\alpha)d^2(x_n, y)$. So, we get $(1-\alpha)d^2(x_n, Ty) \leq (1-\alpha)d^2(x_n, y)$ implies that, $d(x_n, Ty) \leq d(x_n, y)$. Since $\lim_{n\to\infty} d(x_n, y) = 0$, then by Sandwish theorem, we obtain that $\lim_{n\to\infty} d(x_n, Ty) = 0$. By uniqueness of limit, we get that $Ty = y$. Hence $y \in F(T)$, and then $F(T)$ is closed.
Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$. By definition of T, we obtain that

$$d^2(x, Tz) \leq \alpha d^2(Tx, z) + \alpha d^2(Tz, x) + (1-2\alpha)d^2(x, z).$$

So, we get $(1-\alpha)d^2(x, Tz) \leq (1-\alpha)d^2(x, z)$,

$$d^2(x, Tz) \leq d^2(x, z) \Longrightarrow d(x, Tz) \leq d(x, z). \tag{3.3}$$

In the other hand, we get

$$d^2(y, Tz) \leq d^2(y, z) \Longrightarrow d(y, Tz) \leq d(y, z) \tag{3.4}$$

Let $z = W(x, y, \eta)$ where $\eta \in [0, 1]$. From (3.3) and (3.4), we obtain

$$\begin{aligned} d(x, y) &\leq d(x, Tz) + d(Tz, y) \\ &\leq d(x, z) + d(z, y) \\ &= d^2(x, W(x, y, \eta)) + d(W(x, y, \eta), y) \\ &\leq (1-\eta)d(x, x) + \eta d(x, y) + (1-\eta)d(x, y) + \eta d(y, y) \\ &= d(x, y). \end{aligned} \tag{3.5}$$

So $d(x, Tz) = d(x, z)$ and $d(y, Tz) = d(y, z)$, because if $d(x, Tz) < d(x, z)$ or $d(y, Tz) < d(y, z)$, which is a contradiction to $d(x, y) < d(x, y)$. Hence $Tz = z$ Therefor $W(x, y, \eta) \in F(T)$, and then $F(T)$ is convex. ∎

Theorem 3.6. *Let M be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping and $\{x_n\}$ be a bounded sequence in M such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Δ-$\lim_{n\to\infty} x_n = x$. Then $x \in F(T)$.*

Proof. Let $\{x_n\}$ be a bounded seqence in X, By Lemma 2.5 we get $\{x_n\}$ has a unique asymptotic center in M. Since, Δ-$\lim_{n\to\infty} x_n = x$, we have that $A(\{x_n\}) = \{x\}$. Using Lemma 3.3 and the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

(i) $d^2(x_n, Tx) \leq \frac{1+\alpha}{1-\alpha} d^2(x_n, Tx_n) + \frac{2}{1-\alpha}(\alpha d(x_n, x) + d(Tx_n, Tx))d(x_n, Tx_n) + d^2(x_n, x)$, where $0 \leq \alpha < 1$,

(ii) $d^2(x_n, Tx) \leq d^2(x_n, Tx_n) + \frac{2}{1-\alpha}[(-\alpha)d(x_n, x) + d(Tx_n, Tx)]d(x_n, Tx_n) + d^2(x_n, x)$, where $\alpha < 0$.

Taking limit superior as $n \to \infty$ with both sides, we obtain that
Case (i) : $0 \leq \alpha < 1$,

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n, Tx) &\leq \frac{1+\alpha}{1-\alpha}\limsup_{n\to\infty} d^2(x_n, Tx_n)\\ &\quad + \frac{2}{1-\alpha}\limsup_{n\to\infty}(\alpha d(x,x) + d(Tx_n, Tx))d(x_n, Tx_n)\\ &\quad + \limsup_{n\to\infty} d^2(x_n, x)\\ &= \limsup_{n\to\infty} d^2(x_n, x).\end{aligned}$$

Case (ii) : $\alpha < 0$,

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n, Tx) &\leq \limsup_{n\to\infty} d^2(x_n, Tx_n)\\ &\quad + \frac{2}{1-\alpha}\limsup_{n\to\infty}[(-\alpha)d(x_n,x) + d(Tx_n, Tx)]d(x_n, Tx_n)\\ &\quad + \limsup_{n\to\infty} d^2(x_n, x)\\ &= \limsup_{n\to\infty} d^2(x_n, x).\end{aligned}$$

So, we get $\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x)$. By the uniqueness of asymptotic center, we obtain that $Tx = x$. Therefore $x \in F(T)$. ∎

Now we recall the quasi nonexpansive mappings as follow: A mapping $T : M \to M$ is said to be quasi-nonexpansive, if

$$d(Tx, p) \leq d(x, p),$$

for each $x \in M$ and $p \in F(T)$.

Lemma 3.7. *Let M be a nonempty subset of a hyperbolic space X. Let $T : M \to M$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

Proof. Let $T : M \to M$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, we let $p \in F(T)$ and $x \in M$. We consider

$$\begin{aligned}d^2(Tx, Tp) &\leq \alpha d^2(Tx, p) + \alpha d^2(x, p) + (1-2\alpha)d^2(x, p)\\ &= \alpha d^2(Tx, p) + (1-\alpha)d^2(x, p),\end{aligned}$$

we obtain that

$$d^2(Tx, Tp) \leq d^2(x, p),$$

implies that

$$d(Tx, p) \leq d(x, p).$$

Hence T is quasi-nonexpansive. ∎

New, we recall Picard normal S_5-iteration process (PNS_5). Let M be a nonempty closed convex subset of a hyperbolic space X and $T : M \to M$ be a mapping which a square α-nonexpansive , for any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Tu_n, 0, 0) \\ u_n = W(v_n, Tv_n, \beta_n) \\ v_n = W(y_n, Ty_n, \gamma_n) \\ y_n = W(z_n, Tz_n, \delta_n) \\ z_n = W(x_n, Tx_n, \zeta_n), \ n \in \mathbb{N}, \end{cases} \tag{3.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$.

Theorem 3.8. *Let M be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (2.1) then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Step1: We prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. Let $p \in F(T)$. Since T is an α-nonexpansive mapping and Lemma 3.7, we get

$$\begin{aligned} d(u_n, p) &= d(W(v_n, Tv_n\beta_n), p) \\ &\leq (1-\beta_n)d(v_n, p) + \beta_n d(Tv_n, p) \\ &= (1-\beta_n)d(v_n, p) + \beta_n d(Tv_n, p) \\ &\leq (1-\beta_n)d(v_n, p) + \beta_n d(v_n, p) \\ &= d(v_n, p), \end{aligned} \tag{3.7}$$

$$\begin{aligned} d(v_n, p) &= d(W(y_n, Ty_n, \gamma_n), p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &= (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(y_n, p) \\ &= d(y_n, p), \end{aligned} \tag{3.8}$$

$$\begin{aligned} d(y_n, p) &= d(W(z_n, Tz_n, \delta_n), p) \\ &\leq (1-\delta_n)d(z_n, p) + \delta_n d(Tz_n, p) \\ &= (1-\delta_n)d(z_n, p) + \delta_n d(Tz_n, Tp) \\ &\leq (1-\delta_n)d(z_n, p) + \delta_n d(z_n, p) \\ &= d(z_n, p), \end{aligned} \tag{3.9}$$

$$\begin{aligned} d(z_n, p) &= d(W(x_n, Tx_n, \zeta_n), p) \\ &\leq (1-\zeta_n)d(x_n, p) + \zeta_n d(Tx_n, p) \\ &= (1-\zeta_n)d(x_n, p) + \zeta_n d(Tx_n, Tp) \\ &\leq (1-\zeta_n)d(x_n, p) + \zeta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.10}$$

By (3.7),(3.8),(3.9), and (3.10), we have

$$\begin{aligned} d(x_{n+1},p) &= d(W(Tu_n,0,0),p) \\ &= d(Tu_n,p) \\ &\le d(u_n,p) \\ &\le d(v_n,p) \\ &\le d(y_n,p) \\ &\le d(z_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{3.11}$$

We obtain $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F$.
Step 2: We will show that $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Suppose that $\lim_{n\to\infty} d(x_n,p)=c\ge 0$. If $c=0$, then

$$\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

Next, we consider $c>0$. By (3.11), we obtain that

$$d(x_{n+1},p)\le d(u_n,p)\le d(v_n,p)\le d(y_n,p)\le d(z_n,p)\le d(x_n,p). \tag{3.12}$$

Taking limsup in (3.12), we get

$$\limsup_{n\to\infty} d(u_n,p)\le \limsup_{n\to\infty} d(v_n,p)\le \limsup_{n\to\infty} d(y_n,p)\le \limsup_{n\to\infty} d(z_n,p)\le c \tag{3.13}$$

Since $d(Tx_n,p)\le d(x_n,p)$, we have

$$\lim_{n\to\infty}\sup d(Tx_n,p)\le c. \tag{3.14}$$

Since $d(x_{n+1},p)\le (z_n,p)$, as $n\to\infty$, we get

$$c=\liminf_{n\to\infty} d(x_{n+1},p)\le \liminf_{n\to\infty} d(z_n,p)\le \limsup_{n\to\infty} d(z_n,p)\le c. \tag{3.15}$$

From (3.14) and (3.15), we have

$$\lim_{n\to\infty} d(z_n,p)=c,$$

it implies that

$$\lim_{n\to\infty} d(W(x_n,Tx_n,\gamma_n),p)=c.$$

By Lemma 2.6, we obtain that

$$\lim_{n\to\infty} d(x_n,Tx_n)=0. \tag{3.16}$$

Step 3: Let $\mathcal{W}_\Delta(x_n):=\bigcup A_M(\{\mu_n\})$, where the union is taken over all subsequence $\{\mu_n\}$ of $\{x_n\}$. Next, we prove that $\mathcal{W}_\Delta(x_n)\subset F(T)$ and contains only one point. Let $u\in\mathcal{W}_\Delta(x_n)$, there exists a subsequence $\{\mu_n\}$ of $\{x_n\}$ such that $A_M(\{\mu_n\})=\{u\}$. By Lemma 2.5 we let subsequence $\{\nu_n\}$ of $\{\mu_n\}$ such that $\Delta-\lim\limits_{n\to\infty}\nu_n=v$, for some $v\in M$. Since, $\lim\limits_{n\to\infty} d(\nu_n,T\nu_n)=0$, we have $v\in F(T)$. Hence, $\{d(u_n,v)\}$ converges and by lemma 2.8, we have that $v=u\in F(T)$. Hence, $\mathcal{W}_\Delta(x_n)\subset F(T)$. Let $A_M(\{x_n\})=x$ and $\{\mu_n\}$ be arbitrary subsequence of $\{x_n\}$ such that $A_M(\{\mu_n\})=\{u\}$. We have that $\{d(x_n,u)\}$ converges, since $u\in F(T)$. Thus, by Lemma 2.8, we have that $u=x\in F(T)$. and $\mathcal{W}_\Delta(x_n)=\{x\}$. Therefore, $\{x_n\}$ Δ-converges to a common fixed point of T. ■

Theorem 3.9. *Let M be a nonempty closed and convex subset of a complete hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by (2.1). Then $\{x_n\}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Proof. First, we show that the fixed point set $F(T)$ is closed, let $\{x_n\}$ be a sequence in $F(T)$ which converges to some point $z \in M$.

$$d(x_n, Tz) = d(Tx_n, Tz) \le d(x_n, z).$$

By taking the limit of both sides we obtain

$$\lim_{n\to\infty} d(x_n, Tz) \le \lim_{n\to\infty} d(x_n, z) = 0.$$

In view of the uniqueness of the limit, we have $z = Tz$, so that $F(T)$ is closed. Suppose that

$$\lim_{n\to\infty} \inf d(x_n, F(T)) = 0.$$

From (3.11),

$$d(x_{n+1}, F(T)) \le d(x_n, F(T)),$$

then $\lim\limits_{n\to\infty} d(x_n, F(T))$ exists. Hence we know $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$.
We have $\lim\limits_{n\to\infty} d(x_n, z) = 0$, and since $0 \le d(x_n, F(T)) \le d(x_n, z)$, it follows that $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely, consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{3^k}$, for all $k \ge 1$ where $\{p_k\}$ is in $F(T)$. By (3.11), we have

$$d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) < \frac{1}{3^k},$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{+1}}, p_k) \\ &< \frac{1}{3^{k+1}} + \frac{1}{3^k} \\ &< \frac{1}{3^{k-1}}. \end{aligned}$$

This show that $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{p_k\}$ is convergent sequence. Let $\lim\limits_{n\to\infty} p_k = p$. In fact, since $d(x_{n_k}, p) \le d(x_{n_k}, p_k) + d(p_k, p) \to 0$ as $k \to \infty$, we have $\lim\limits_{k\to\infty} d(x_{n_k}, p) = 0$. Since $\lim\limits_{n\to\infty} d(x_n, p)$ exists, the sequence $\{x_n\}$ converges to p. ∎

Theorem 3.10. *Let M be a nonempty compact convex subset of a complete hyperbolic space X with monotone modulus of uniformly convexity η. Let $T : M \to M$ be a square α-nonexpansive mapping for some $\alpha < 1$. Let $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \le \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$. In case $\alpha \le 0$, we assume that $\limsup_{k\to\infty} \beta_{n_k} < 1$. Let $\{x_n\}$ be a sequence with x_1 in M defined by (2.1). Then $\{x_n\}$ converges in metric to a fixed point of T.*

Proof. We use Lemma 3.3 and Lemma 3.4, and replacing $\|\cdot,\cdot\|$ with $d(\cdot,\cdot)$ in the proof of [Theroem 3.4][2], we conclude the desired result. ∎

Competing interests

The authors declare that they have no competing interests.

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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An iterative method for solving proximal split feasibility problems and fixed point problems

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Abstract
The purpose of this research is to introduce a regularized algorithm based on the viscosity method for solving the proximal split feasibility problem and the fixed point problem in Hilbert spaces. A strong convergence result of our proposed algorithm for finding a common solution of the proximal split feasibility problem and the fixed point problem for nonexpansive mappings is established. We also apply our main result to the split feasibility problem, and the fixed point problem of nonexpansive semigroups, respectively. Finally, we give numerical examples for supporting our main result.

Keywords Fixed point problems · Proximal split feasibility problems · Nonexpansive mappings

Mathematics Subject Classification 47H09 · 47H10

1 Introduction

Throughout this article, let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator.

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In this paper, we focus our attention on the following proximal split feasibility problem (PSFP): find a minimizer x^* of f, such that Ax^* minimizes g, namely

$$x^* \in \operatorname{argmin} f \text{ such that } Ax^* \in \operatorname{argmin} g, \tag{1.1}$$

where $\operatorname{argmin} f := \{\bar{x} \in H_1 : f(\bar{x}) \leq f(x) \text{ for all } x \in H_1\}$ and $\operatorname{argmin} g := \{\bar{y} \in H_2 : g(\bar{y}) \leq g(y) \text{ for all } y \in H_2\}$. We assume that the problem (1.1) has a nonempty solution set $\Gamma := \operatorname{argmin} f \cap A^{-1}(\operatorname{argmin} g)$.

Censor and Elfving (1994) introduced the split feasibility problem (in short, SFP). The split feasibility problem (SFP) has been used for many applications in various fields of science and technology, such as in signal processing and image reconstruction, and especially applied in medical fields such as intensity-modulated radiation therapy (IMRT) (for details, see Censor et al. (2006) and the references therein). Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and then, the SFP is to find a point:

$$x \in C \text{ such that } Ax \in Q, \tag{1.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator. For solving the problem (1.2), Byrne (2002) introduced a popular algorithm which is called the CQ algorithm as follows:

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n), \quad \forall n \geq 1,$$

where P_C and P_Q denote the metric projection onto the closed convex subsets C and Q, respectively, and A^* is the adjoint operator of A and $\mu_n \in (0, 2/\|A\|^2)$. Many research papers have increasingly investigated split feasibility problem [see, for instance (Lopez et al. 2012; Chang et al. 2014; Qu and Xiu 2005), and the references therein]. If $f = i_C$ [defined as $i_C(x) = 0$ if $x \in C$ and $i_C(x) = +\infty$ if $x \notin C$] and $g = i_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively. Then, the proximal split feasibility problem (1.1) becomes the split feasibility problem (1.2).

Moudafi and Thakur (2014) introduced the split proximal algorithm with a way of selecting the step-sizes, such that its implementation does not need any prior information about the operator norm. Given an initial point $x_1 \in H_1$, assume that x_n has been constructed and $\|A^*(I - \operatorname{prox}_{\lambda g})Ax_n\|^2 + \|(I - \operatorname{prox}_{\lambda f})x_n\|^2 \neq 0$, and then compute x_{n+1} by the following iterative scheme:

$$x_{n+1} = \operatorname{prox}_{\lambda \mu_n f}(x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n), \quad \forall n \geq 1, \tag{1.3}$$

where the stepsize $\mu_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, $h(x) := \frac{1}{2}\|(I - \operatorname{prox}_{\lambda g})Ax\|^2$, $l(x) := \frac{1}{2}\|(I - \operatorname{prox}_{\lambda \mu_n f})x\|^2$ and $\theta^2(x) := \|A^*(I - \operatorname{prox}_{\lambda g})Ax\|^2 + \|(I - \operatorname{prox}_{\lambda \mu_n f})x\|^2$. If $\theta^2(x_n) = 0$, then x_n is a solution of (1.1) and the iterative process stops; otherwise, we set $n := n + 1$ and compute x_{n+1} using (1.3). They also proved the weak convergence of the sequence generated by Algorithm (1.3) to a solution of (1.1) under suitable conditions of parameter ρ_n where $\varepsilon \leq \rho_n \leq \dfrac{4h(x_n)}{h(x_n) + l(x_n)} - \varepsilon$ for some $\varepsilon > 0$.

Yao et al. (2014) gave the regularized algorithm for solving the proximal split feasibility problem (1.1) and proposed a strong convergence theorem under suitable conditions:

$$x_{n+1} = \operatorname{prox}_{\lambda \mu_n f}(\alpha_n u + (1 - \alpha_n)x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n), \quad \forall n \geq 1, \tag{1.4}$$

where the stepsize $\mu_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$.

Shehu et al. (2015) introduced a viscosity-type algorithm for solving proximal split feasibility problems as follows:

$$\begin{cases} y_n = x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n, \\ x_{n+1} = \alpha_n \psi(x_n) + (1-\alpha_n)\,\mathrm{prox}_{\lambda\mu_n f}\, y_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\psi : H_1 \to H_1$ is a contraction mapping. They also proved a strong convergence of the sequences generated by iterative schemes (1.5) in Hilbert spaces.

Recently, Shehu and Iyiola (2015) introduced the following algorithm for solving split proximal algorithms and fixed point problems for k-strictly pseudocontractive mappings in Hilbert spaces:

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = \mathrm{prox}_{\lambda\gamma_n f}(u_n - \gamma_n A^*(I - \mathrm{prox}_{\lambda g})Au_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where the stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. They also showed that, under certain assumptions imposed on the parameters, the sequence $\{x_n\}$ generated by (1.6) converges strongly to $x^* \in F(S) \cap \Gamma$. Many researchers have proposed some methods to solve the proximal split feasibility problem [see, for instance (Shehu et al. 2015; Shehu and Iyiola 2017a, b, 2018; Abbas et al. 2018; Witthayarat et al. 2018), and the references therein].

We note that Algorithm (1.6) is the Halpern-type algorithm with $u \equiv 0$ fixed. However, a viscosity-type algorithm is more general and desirable than a Halpern-type algorithm, because a contraction which is used in the viscosity-type algorithm influences the convergence behavior of the algorithm.

In this paper, inspired and motivated by these studies, we are interested to study the proximal split feasibility problem and the fixed point problem in Hilbert spaces. In Sect. 3, we introduce a regularized algorithm based on the viscosity method for finding a common solution of the proximal split feasibility problem and the fixed point problem of nonexpansive mappings, and prove a strong convergence theorem under some suitable conditions. In Sects. 4 and 5, we apply our main result to the split feasibility problem, and the fixed point problem of nonexpansive semigroups, respectively. In the last section, we first give a numerical result in Euclidean spaces to demonstrate the convergence of our algorithm. We also show the number of iterations of our algorithm by choosing different contractions ψ. In this case, if we take $\psi = 0$ in our algorithm, then we obtain Algorithm (1.6) (Shehu and Iyiola 2015, Algorithm 1). Moreover, we give an example in the infinite-dimensional spaces for supporting our main theorem.

2 Preliminaries

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in C : Tx = x\}$.

Recall that A mapping T of C into itself is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(ii) contraction if there exists a constant $\delta \in [0, 1)$, such that

$$\|Tx - Ty\| \leq \delta\|x - y\|, \quad \forall x, y \in C.$$

Recall that the proximal operator $\operatorname{prox}_{\lambda g} : H \to H$ is defined by:

$$\operatorname{prox}_{\lambda g} x := \operatorname*{argmin}_{u \in H} \left\{ g(u) + \frac{1}{2\lambda}\|u - x\|^2 \right\}. \tag{2.1}$$

Moreover, the proximity operator of f is firmly nonexpansive, namely:

$$\left\langle \operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y), x - y \right\rangle \geq \|\operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y)\|^2; \tag{2.2}$$

for all $x, y \in H$, which is equivalent to

$$\|\operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y)\|^2 \leq \|x - y\|^2 - \|(I - \operatorname{prox}_{\lambda g})(x) - (I - \operatorname{prox}_{\lambda g})(y)\|^2. \tag{2.3}$$

for all $x, y \in H$. For general information on proximal operator, see Combettes and Pesquet (2011a).

In a real Hilbert space H, it is well known that:

(i) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$, for all $x, y \in H$ and $\alpha \in [0, 1]$;
(ii) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$ for all $x, y \in H$;
(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ for all $x, y \in H$.

Recall that the (nearest-point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property:

$$\|x - P_C x\| = \min_{y \in C}\|x - y\|.$$

Lemma 2.1 (Takahashi 2000) *Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality:*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2 (Xu 2003) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying:*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence, such that

1. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
2. $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Definition 2.3 Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $S : C \to C$ is called demi-closed at zero if for any sequence $\{x_n\}$ which converges weakly to x, and if the sequence $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

Lemma 2.4 (Browder 1976) *Let C be a nonempty closed convex subset of a real Hilbert space H. If $S : C \to C$ is a nonexpansive mapping, then $I - S$ is demi-closed at zero.*

Lemma 2.5 (Mainge 2008) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$, such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \longrightarrow \infty$*;*
(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3 Main results

In this section, we introduce an algorithm and prove a strong convergence for solving a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of proximal split feasibility problems (1.1). Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ be a nonexpansive mapping and Let $\psi : H_1 \to H_1$ be a contraction mapping with $\delta \in (0, 1)$.

We introduce the modified split proximal algorithm as follows:

Algorithm 3.1 Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\|A^*(I - \text{prox}_{\lambda g})Ax_n\|^2 + \|(I - \text{prox}_{\lambda f})x_n\|^2 \neq 0$, then compute x_{n+1} by the following iterative scheme:

$$\begin{cases} y_n = \text{prox}_{\lambda\mu_n f}(\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Sy_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where the stepsize $\mu_n := \rho_n \dfrac{\left(\frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2\right) + \left(\frac{1}{2}\|(I - \text{prox}_{\lambda f})x_n\|^2\right)}{\|A^*(I - \text{prox}_{\lambda g})Ax_n\|^2 + \|(I - \text{prox}_{\lambda f})x_n\|^2}$ with $0 < \rho_n < 4$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

We now prove our main theorem.

Theorem 3.2 *Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$be two proper and lower semicontinuous convex functions, and $A : H_1 \to H_2$ be a bounded linear operator. Let $\psi : H_1 \to H_1$ be a contraction mapping with $\delta \in [0, 1)$ and let $S : H_1 \to H_1$ be a nonexpansive mapping, such that $\Omega := F(S) \cap \Gamma \neq 0$. If the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\rho_n\}$ satisfy the following conditions:*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*
(C2) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$*;*
(C3) $\varepsilon \leq \rho_n \leq \dfrac{4(1-\alpha_n)\left(\|(I - \text{prox}_{\lambda g})Ax_n\|^2\right)}{\left(\|(I - \text{prox}_{\lambda g})Ax_n\|^2\right) + \left(\|(I - \text{prox}_{\lambda f})x_n\|^2\right)} - \varepsilon$ *for some* $\varepsilon > 0$.

Then, the sequence $\{x_n\}$ defined by Algorithm 3.1 converges strongly to a point $x^* \in \Omega$ which also solves the variational inequality:

$$\langle(\psi - I)x^*, x - x^*\rangle \leq 0, \quad \forall x \in \Omega.$$

Proof Given any $\lambda > 0$ and $x \in H_1$, we define $h(x) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ax\|^2$, $l(x) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f})x\|^2$, $\theta^2(x) := \|A^*(I - \mathrm{prox}_{\lambda g})Ax\|^2 + \|(I - \mathrm{prox}_{\lambda f})x\|^2$, and hence, $\mu_n = \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ where $0 < \rho_n < 4$.

By Banach fixed point theorem, there exists $x^* \in \Omega$ such that $x^* = P_\Omega \psi(x^*)$. Then, $x^* = \mathrm{prox}_{\lambda\mu_n f}\, x^*$ and $Ax^* = \mathrm{prox}_{\lambda g}\, Ax^*$. Since $\mathrm{prox}_{\lambda g}$ is firmly nonexpansive, we have $I - \mathrm{prox}_{\lambda g}$ is also firmly nonexpansive. Hence

$$\begin{aligned}\langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - x^*\rangle &= \langle (I - \mathrm{prox}_{\lambda g})Ax_n, Ax_n - Ax^*\rangle \\ &= \left\langle (I - \mathrm{prox}_{\lambda g})Ax_n - (I - \mathrm{prox}_{\lambda g})Ax^*, Ax_n - Ax^*\right\rangle \\ &\geq \|(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 = 2h(x_n).\end{aligned} \tag{3.2}$$

From the definition of y_n and the nonexpansivity of $\mathrm{prox}_{\lambda\mu_n f}$, we have:

$$\begin{aligned}\|y_n - x^*\| &= \|\mathrm{prox}_{\lambda\mu_n f}(\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n) - x^*\| \\ &\leq \|\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\| \\ &\leq \alpha_n\|\psi(x_n) - x^*\| + (1-\alpha_n)\left\| x_n - \frac{\mu_n}{(1-\alpha_n)}A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\right\|.\end{aligned} \tag{3.3}$$

From (3.2), we have:

$$\begin{aligned}&\left\| x_n - \frac{\mu_n}{(1-\alpha_n)}A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\right\|^2 \\ &= \|x_n - x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 \\ &\quad - 2\frac{\mu_n}{(1-\alpha_n)}\langle A^*(I - \mathrm{prox}_{\lambda g})Ax_n, x_n - x^*\rangle \\ &\leq \|x_n - x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 - 4\frac{\mu_n}{(1-\alpha_n)}h(x_n) \\ &= \|x_n - x^*\|^2 + \rho_n^2\frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)^2\theta^4(x_n)}\|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\|^2 - 4\rho_n\frac{(h(x_n) + l(x_n))}{(1-\alpha_n)\theta^2(x_n)}h(x_n) \\ &\leq \|x_n - x^*\|^2 + \rho_n^2\frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)^2\theta^2(x_n)} - 4\rho_n\frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\frac{h(x_n)}{(h(x_n) + l(x_n))} \\ &= \|x_n - x^*\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n) + l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\right).\end{aligned} \tag{3.4}$$

By the condition (C3), we have $\dfrac{4h(x_n)}{(h(x_n) + l(x_n))} - \dfrac{\rho_n}{1-\alpha_n} \geq 0$ for all $n \geq 1$. From (3.3) and (3.4), we have:

$$\begin{aligned}\|y_n - x^*\| &\leq \alpha_n\|\psi(x_n) - x^*\| + (1-\alpha_n)\left\| x_n - \frac{\mu_n}{(1-\alpha_n)}A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\right\| \\ &\leq \alpha_n\|\psi(x_n) - \psi(x^*)\| + \alpha_n\|\psi(x^*) - x^*\| + (1-\alpha_n)\left\| x_n - x^*\right\| \\ &\leq \alpha_n\delta\|x_n - x^*\| + \alpha_n\|\psi(x^*) - x^*\| + (1-\alpha_n)\left\| x_n - x^*\right\| \\ &= (1 - \alpha_n(1-\delta))\|x_n - x^*\| + \alpha_n\|\psi(x^*) - x^*\|.\end{aligned} \tag{3.5}$$

Since S is nonexpansive, by (3.1) and (3.5), we obtain:

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\beta_n y_n + (1-\beta_n)Sy_n - x^*\|\\ &\le \beta_n\|y_n - x*\| + (1-\beta_n)\|Sy_n - x^*\|\\ &\le \beta_n\|y_n - x*\| + (1-\beta_n)\|y_n - x^*\|\\ &= \|y_n - x*\|\\ &\le (1-\alpha_n(1-\delta))\|x_n - x^*\| + \alpha_n\|\psi(x^*) - x^*\|\\ &\le \max\left\{\|x_n - x^*\|, \frac{\|\psi(x^*)-x^*\|}{1-\delta}\right\}.\end{aligned}$$

By mathematical induction, we have:

$$\|x_n - x^*\| \le \max\left\{\|x_1 - x^*\|, \frac{\|\psi(x^*)-x^*\|}{1-\delta}\right\}, \quad \forall n \in \mathbb{N}.$$

Hence, $\{x_n\}$ is bounded and so are $\{\psi(x_n)\}, \{Sy_n\}$.

From the definition of y_n and (3.4), we have:

$$\begin{aligned}\|y_n - x^*\|^2 &= \|\operatorname{prox}_{\lambda\mu_n f}(\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I-\operatorname{prox}_{\lambda g})Ax_n) - x^*\|^2\\ &\le \|\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I-\operatorname{prox}_{\lambda g})Ax_n - x^*\|^2,\\ &\le \alpha_n\|\psi(x_n)-x^*\|^2 + (1-\alpha_n)\left\|x_n - \frac{\mu_n}{(1-\alpha_n)}A^*(I-\operatorname{prox}_{\lambda g})Ax_n - x^*\right\|^2\\ &\le \alpha_n\|\psi(x_n)-x^*\|^2 + (1-\alpha_n)\\ &\quad\times\left(\|x_n-x^*\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\right)\right)\\ &= \alpha_n\|\psi(x_n)-x^*\|^2 + (1-\alpha_n)\|x_n-x^*\|^2\\ &\quad - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right).\end{aligned} \tag{3.6}$$

From the definition of x_n and (3.6), we obtain:

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n y_n + (1-\beta_n)Sy_n - x^*\|^2\\ &\le \beta_n\|y_n - x*\|^2 + (1-\beta_n)\|Sy_n - x^*\|^2\\ &\le \|y_n - x*\|^2\\ &\le \alpha_n\|\psi(x_n)-x^*\|^2 + (1-\alpha_n)\|x_n-x^*\|^2\\ &\quad - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right)\\ &\le \alpha_n\|\psi(x_n)-x^*\|^2 + \|x_n-x^*\|^2\\ &\quad - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right).\end{aligned}$$

It implies that

$$\begin{aligned}\rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) &\le \alpha_n\|\psi(x_n)-x^*\|^2 + \|x_n\\ &\quad - x^*\|^2 - \|x_{n+1}-x^*\|^2.\end{aligned} \tag{3.7}$$

It follows from (3.6) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n y_n+(1-\beta_n)Sy_n-x^*\|^2\\ &\le \beta_n\|y_n-x*\|^2+(1-\beta_n)\|Sy_n-x^*\|^2-\beta_n(1-\beta_n)\|y_n-Sy_n\|^2\\ &\le \|y_n-x*\|^2-\beta_n(1-\beta_n)\|y_n-Sy_n\|^2\\ &\le \alpha_n\|\psi(x_n)-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\beta_n(1-\beta_n)\|y_n-Sy_n\|^2\\ &\le \alpha_n\|\psi(x_n)-x^*\|^2+\|x_n-x^*\|^2-\beta_n(1-\beta_n)\|y_n-Sy_n\|^2,\end{aligned}$$

which implies that

$$\beta_n(1-\beta_n)\|y_n-Sy_n\|^2 \le \alpha_n\|\psi(x_n)-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2. \tag{3.8}$$

Now, we divide our proof into two cases.

Case 1 Suppose that there exists $n_0\in\mathbb{N}$, such that $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ is nonincreasing. Then, $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ converges and $\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\to 0$ as $n\to\infty$. From (3.7) and the condition (C1) and (C3), we obtain:

$$\rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))}-\frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right)\to 0 \text{ as } n\to\infty.$$

Hence, we have:

$$\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\to 0 \text{ as } n\to\infty. \tag{3.9}$$

By the linearity and boundedness of A and the nonexpansivity of $\mathrm{prox}_{\lambda g}$, we obtain that $\{\theta^2(x_n)\}$ is bounded.

It follows that

$$\lim_{n\to\infty}\left((h(x_n)+l(x_n))^2\right)=0,$$

which implies that

$$\lim_{n\to\infty}h(x_n)=\lim_{n\to\infty}l(x_n)=0.$$

Next, we show that $\limsup_{n\to\infty}\langle\psi(x^*)-x^*,x_n-x^*\rangle\le 0$, where $x^*=P_\Omega\psi(x^*)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying $x_{n_j}\rightharpoonup\omega$ and

$$\limsup_{n\to\infty}\left\langle\psi(x^*)-x^*,x_n-x^*\right\rangle=\lim_{j\to\infty}\left\langle\psi(x^*)-x^*,x_{n_j}-x^*\right\rangle. \tag{3.10}$$

By the lower semicontinuity of h, we have:

$$0\le h(\omega)\le\liminf_{j\to\infty}h(x_{n_j})=\lim_{n\to\infty}h(x_n)=0.$$

Therefore, $h(\omega)=\frac{1}{2}\|(I-\mathrm{prox}_{\lambda g})A\omega\|^2=0$. Therefore, $A\omega$ is a fixed point of the proximal mapping of g or equivalently, $A\omega$ is a minimizer of g. Similarly, from the lower semicontinuity of l, we obtain:

$$0\le l(\omega)\le\liminf_{j\to\infty}l(x_{n_j})=\lim_{n\to\infty}l(x_n)=0.$$

Therefore, $l(\omega)=\frac{1}{2}\|(I-\mathrm{prox}_{\lambda\mu_n f})\omega\|^2=0$. That is $\omega\in F(\mathrm{prox}_{\lambda\mu_n f})$. Then ω is a minimizer of f. Thus, $\omega\in\Gamma$. We observe that

$$0<\mu_n<4\frac{h(x_n)+l(x_n)}{\theta^2(x_n)}\to 0 \text{ as } n\to\infty,$$

and hence, $\mu_n \to 0$ as $n \to \infty$.

Next, we show that $\omega \in F(S)$. From (3.8) and the condition (C1), (C2), we have:

$$\|y_n - Sy_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

For each $n \geq 1$, let $u_n := \alpha_n \psi(x_n) + (1-\alpha_n)x_n$. Then

$$\begin{aligned}\|u_n - x_n\| &= \|\alpha_n \psi(x_n) + (1-\alpha_n)x_n - x_n\| \\ &= \alpha_n \|\psi(x_n) - x_n\|.\end{aligned}$$

From the condition (C1), we have:

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.12}$$

Observe that

$$\|u_n - \operatorname{prox}_{\lambda\mu_n f} x_n\| \leq \|u_n - x_n\| + \|(I - \operatorname{prox}_{\lambda\mu_n f})x_n\|.$$

From $\lim_{n\to\infty} l(x_n) = \lim_{n\to\infty} \frac{1}{2}\|(I - \operatorname{prox}_{\lambda\mu_n f})x_n\|^2 = 0$ and (3.12), we have:

$$\lim_{n\to\infty} \|u_n - \operatorname{prox}_{\lambda\mu_n f} x_n\| = 0. \tag{3.13}$$

By the nonexpansiveness of $\operatorname{prox}_{\lambda\mu_n f}$, we have:

$$\begin{aligned}\|y_n - \operatorname{prox}_{\lambda\mu_n f} x_n\| &= \left\|\operatorname{prox}_{\lambda\mu_n f}\left(u_n - \mu_n A^*\left(I - \operatorname{prox}_{\lambda g}\right)Ax_n\right) - \operatorname{prox}_{\lambda\mu_n f} x_n\right\| \\ &\leq \|u_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n - x_n\| \\ &\leq \|u_n - x_n\| + \mu_n\|A^*(I - \operatorname{prox}_{\lambda g})Ax_n\|.\end{aligned}$$

From (3.13) and $\mu_n \to 0$ as $n \to \infty$, we have:

$$\lim_{n\to\infty} \|y_n - \operatorname{prox}_{\lambda\mu_n f} x_n\| = 0. \tag{3.14}$$

Since

$$\|y_n - u_n\| \leq \|y_n - \operatorname{prox}_{\lambda\mu_n f} x_n\| + \|u_n - \operatorname{prox}_{\lambda\mu_n f} x_n\|,$$

from (3.13) and (3.14), we obtain:

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.15}$$

From (3.12) and (3.15), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.16}$$

From

$$\|Sy_n - x_n\| \leq \|Sy_n - y_n\| + \|y_n - x_n\|,$$

by (3.11), (3.16), we get:

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0. \tag{3.17}$$

From the definition of x_n, we have:

$$\|x_{n+1} - x_n\| \leq \beta_n\|y_n - x_n\| + (1-\beta_n)\|Sy_n - x_n\|.$$

This implies from (3.16), and (3.17) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.18}$$

Using $x_{n_j} \rightharpoonup \omega \in H_1$ and (3.16), we obtain $y_{n_j} \rightharpoonup \omega \in H_1$. Since $y_{n_j} \rightharpoonup \omega \in H_1$, $\|y_n - Sy_n\| \to 0$ as $n \to \infty$, by Lemma 2.4, we have $\omega \in F(S)$. Hence, $\omega \in \mathcal{F} = F(S) \cap \Gamma$. Since $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$ and $\omega \in \mathcal{F}$, by Lemma 2.1, we have:

$$\begin{aligned}
\limsup_{n\to\infty} \left\langle \psi(x^*) - x^*, x_n - x^* \right\rangle &= \lim_{j\to\infty} \left\langle \psi(x^*) - x^*, x_{n_j} - x^* \right\rangle \\
&= \left\langle (\psi - I)x^*, \omega - x^* \right\rangle \\
&\le 0.
\end{aligned} \tag{3.19}$$

Now, by the nonexpansiveness of S and $\mathrm{prox}_{\lambda\mu_n f}$, and from (3.1) and (3.4), we have:

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n \|y_n - x^*\|^2 + (1-\beta_n)\|Sy_n - x^*\|^2 \le \|y_n - x^*\|^2 \\
&\le \|\alpha_n \psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\|^2 \\
&= (1-\alpha_n)^2 \|x_n - \frac{\mu_n}{(1-\alpha_n)} A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^*\|^2 + \alpha_n^2 \|\psi(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\left\langle \psi(x_n) - x^*, x_n - \frac{\mu_n}{(1-\alpha_n)} A^*(I - \mathrm{prox}_{\lambda g})Ax_n - x^* \right\rangle \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n^2 \|\psi(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \psi(x_n) - x^*, x_n - x^* \rangle \\
&\quad - 2\alpha_n\mu_n \langle \psi(x_n) - x^*, A^*(I - \mathrm{prox}_{\lambda g})Ax_n \rangle \\
&= (1-\alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n^2 \|\psi(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \psi(x_n) - \psi(x^*), x_n - x^* \rangle \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \psi(x^*) - x^*, x_n - x^* \rangle \\
&\quad + 2\alpha_n\mu_n \left\langle x^* - \psi(x_n), A^*(I - \mathrm{prox}_{\lambda g})Ax_n \right\rangle \\
&\le (1 - 2\alpha_n + \alpha_n^2)\|x_n - x^*\|^2 + \alpha_n^2 \|\psi(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\delta \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \psi(x^*) - x^*, x_n - x^* \rangle \\
&\quad + 2\alpha_n\mu_n \|\psi(x_n) - x^*\| \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\| \\
&= (1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n(1-\alpha_n)\delta)\|x_n - x^*\|^2 + \alpha_n^2 \|\psi(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle \psi(x^*) - x^*, x_n - x^* \rangle \\
&\quad + 2\alpha_n\mu_n \|\psi(x_n) - x^*\| \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\| \\
&= (1 - \epsilon_n)\|x_n - x^*\|^2 + \epsilon_n \xi_n,
\end{aligned} \tag{3.20}$$

where $\epsilon_n = \alpha_n(2 - \alpha_n - 2(1-\alpha_n)\delta)$ and

$$\xi_n = \left[\frac{\alpha_n \|\psi(x_n) - x^*\|^2 + 2(1-\alpha_n)\langle \psi(x^*) - x^*, x_n - x^* \rangle + 2\mu_n \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\| \|\psi(x_n) - x^*\|}{2 - \alpha_n - 2(1-\alpha_n)\delta} \right].$$

Note that $\mu_n \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\| = \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)} \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\|$. Thus, $\mu_n \|A^*(I - \mathrm{prox}_{\lambda g})Ax_n\| \to 0$ as $n \to \infty$. From the condition (C1), (3.19), (3.20) and Lemma 2.2, we can conclude that the sequence $\{x_n\}$ converges strongly to x^*.

Case 2 Assume that $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Then, there exists a subsequence n_l of n, such that $\|x_{n_l} - x^*\| < \|x_{n_l+1} - x^*\|$ for all $l \in \mathbb{N}$. Now, we define a positive integer sequence $\tau(n)$ by:

$$\tau(n) := \max\left\{k \in \mathbb{N} : k \leq n, \|x_{n_l} - x^*\| < \|x_{n_l+1} - x^*\|\right\}.$$

for all $n \geq n_0$ (for some n_0 large enough). By Lemma 2.5, we have τ which is a non-decreasing sequence, such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 \leq 0, \quad \forall n \geq n_0.$$

By a similar argument as that of case 1, we can show that

$$\rho_{\tau(n)}\left(\frac{4h(x_{\tau(n)})}{(h(x_{\tau(n)}) + l(x_{\tau(n)}))} - \frac{\rho_{\tau(n)}}{1 - \alpha_{\tau(n)}}\right)\left(\frac{(h(x_{\tau(n)}) + l(x_{\tau(n)}))^2}{\theta^2(x_{\tau(n)})}\right) \to 0 \text{ as } n \to \infty.$$

Then, we have:

$$\frac{(h(x_{\tau(n)}) + l(x_{\tau(n)}))^2}{\theta^2(x_{\tau(n)})} \to 0 \text{ as } n \to \infty. \tag{3.21}$$

It follows that

$$\lim_{n\to\infty}\left((h(x_{\tau(n)}) + l(x_{\tau(n)}))^2\right) = 0,$$

which implies that

$$\lim_{n\to\infty} h(x_{\tau(n)}) = \lim_{n\to\infty} l(x_{\tau(n)}) = 0.$$

Moreover, we have

$$\limsup_{n\to\infty}\left\langle\psi(x^*) - x^*, x_{\tau(n)} - x^*\right\rangle \leq 0.$$

By the same computation as in Case 1, we have:

$$\|x_{\tau(n)+1} - x^*\|^2 \leq (1 - \epsilon_{\tau(n)})\|x_{\tau(n)} - x^*\|^2 + \epsilon_{\tau(n)}\xi_{\tau(n)}, \tag{3.22}$$

where $\epsilon_{\tau(n)} = \alpha_{\tau(n)}(2 - \alpha_{\tau(n)} - 2(1 - \alpha_{\tau(n)})\delta)$ and

$$\xi_{\tau(n)} = \left[\frac{\alpha_{\tau(n)}\|\psi(x_{\tau(n)}) - x^*\|^2 + 2(1 - \alpha_{\tau(n)})\langle\psi(x^*) - x^*, x_{\tau(n)} - x^*\rangle + 2\mu_{\tau(n)}\|A^*(I - \mathrm{prox}_{\lambda g})Ax_{\tau(n)}\|\|\psi(x_{\tau(n)}) - x^*\|}{2 - \alpha_{\tau(n)} - 2(1 - \alpha_{\tau(n)})\delta}\right].$$

Since $\|x_{\tau(n)} - x^*\|^2 \leq \|x_{\tau(n)+1} - x^*\|^2$, then by (3.22), we have:

$$\|x_{\tau(n)} - x^*\|^2 \leq \xi_{\tau(n)}.$$

We note that $\limsup_{n\to\infty}\xi_{\tau(n)} \leq 0$. Thus, it follows from above inequality that

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0.$$

From (3.22), we also have:

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x^*\| = 0.$$

It follows from Lemma 2.5 that

$$0 \leq \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \to 0$$

as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Taking $\psi(x) = u$ in Algorithm 3.1, we have the following Halpern-type algorithm.

Algorithm 3.3 Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\|A^*(I - \text{prox}_{\lambda g})Ax_n\|^2 + \|(I - \text{prox}_{\lambda f})x_n\|^2 \neq 0$, and then compute x_{n+1} by the following iterative scheme:

$$\begin{cases} y_n = \text{prox}_{\lambda \mu_n f}(\alpha_n u + (1 - \alpha_n)x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)Sy_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.23}$$

where the stepsize $\mu_n := \rho_n \dfrac{\left(\frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2\right) + \left(\frac{1}{2}\|(I - \text{prox}_{\lambda f})x_n\|^2\right)}{\|A^*(I - \text{prox}_{\lambda g})Ax_n\|^2 + \|(I - \text{prox}_{\lambda f})x_n\|^2}$ with $0 < \rho_n < 4$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$.

The following result is obtained directly by Theorem 3.2.

Corollary 3.4 *Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_1 \to H_1$ be a nonexpansive mapping, such that $\Omega := F(S) \cap \Gamma \neq 0$. If the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\rho_n\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\varepsilon \leq \rho_n \leq \dfrac{4(1 - \alpha_n)\left(\|(I - \text{prox}_{\lambda g})Ax_n\|^2\right)}{\left(\|(I - \text{prox}_{\lambda g})Ax_n\|^2\right) + \left(\|(I - \text{prox}_{\lambda f})x_n\|^2\right)} - \varepsilon$ *for some* $\varepsilon > 0$.

Then, the sequence $\{x_n\}$ defined by Algorithm 3.3 converges strongly to $z = P_\Omega u$.

4 Convergence theorem for split feasibility problems

In this section, we give an application of Theorem 3.2 to the split feasibility problem.

Algorithm 4.1 Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\|A^*(I - P_Q)Ax_n\|^2 + \|(I - P_C)x_n\|^2 \neq 0$, and then compute x_{n+1} by the following iterative scheme:

$$\begin{cases} y_n = P_C(\alpha_n \psi(x_n) + (1 - \alpha_n)x_n - \mu_n A^*(I - P_Q)Ax_n) \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)Sy_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where the stepsize $\mu_n := \rho_n \dfrac{\left(\frac{1}{2}\|(I - P_Q)Ax_n\|^2\right) + \left(\frac{1}{2}\|(I - P_C)x_n\|^2\right)}{\|A^*(I - P_Q)Ax_n\|^2 + \|(I - P_C)x_n\|^2}$ with $0 < \rho_n < 4$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

We now obtain a strong convergence theorem of Algorithm 4.1 for solving the split feasibility problem and the fixed point problem of nonexpansive mappings as follows:

Theorem 4.2 *Let H_1 and H_2 be two real Hilbert spaces, and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $\psi : H_1 \to H_1$ be a contraction mapping with $\delta \in [0, 1)$ and let $S : H_1 \to H_1$ be a nonexpansive mapping. Assume that $\Omega := F(S) \cap C \cap A^{-1}(Q) \neq \emptyset$. If the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\rho_n\}$ satisfy the following conditions:*

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(C3) $\varepsilon \leq \rho_n \leq \dfrac{4(1-\alpha_n)\left(\|(I-P_Q)Ax_n\|^2\right)}{\left(\|(I-P_Q)Ax_n\|^2\right)+\left(\|(I-P_C)x_n\|^2\right)} - \varepsilon$ *for some* $\varepsilon > 0$.

Then, the sequence $\{x_n\}$ *generated by Algorithm* 4.1 *converges strongly to* $z = P_\Omega \psi(z)$.

Proof Taking $f = i_C$ and $g = i_Q$ in Theorem 3.2 (i_C and i_Q are indicator functions of C and Q, respectively), we have $\mathrm{prox}_{\lambda f} = P_C$ and $\mathrm{prox}_{\lambda g} = P_Q$ for all λ. We also have $\operatorname{argmin} f = C$ and $\operatorname{argmin} g = Q$. Therefore, from Theorem 3.2, we obtain the desired result. □

5 Convergence theorem for nonexpansive semigroups

In this section, we prove a strong convergence theorem for finding a common solution of the proximal split feasibility problem and the fixed point problem of nonexpansive semigroups in Hilbert spaces.

Let C be a nonempty, closed, and convex subset of a real Banach space X. A one-parameter family $S = S(t) : t \geq 0 : C \to C$ is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;
(ii) $S(s+t)x = S(s)S(t)x$ for all $t, s > 0$ and $x \in C$;
(iii) for each $x \in C$ the mapping $t \longmapsto S(t)x$ is continuous;
(iv) $\|S(t)x - S(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t > 0$.

We use $F(S)$ to denote the common fixed point set of the semigroup S, i.e., $F(S) = \bigcap_{t>0} F(S(t)) = \{x \in C : x = S(t)x\}$. It is well known that $F(S)$ is closed and convex (see Browder 1956).

Definition 5.1 (Aleyner and Censor 2005) Let C be a nonempty, closed, and convex subset of a real Hilbert space H, $S = S(t) : t > 0$ be a continuous operator semigroup on C. Then, S is said to be uniformly asymptotically regular (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset K of C, such that

$$\lim_{t\to\infty} \sup_{x\in K} \|S(h)(S(t)x) - S(t)x\| = 0.$$

Lemma 5.2 (Shimizu and Takahashi 1997) *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, and let K be a bounded, closed, and convex subset of C. If we denote $S = S(t) : t > 0$ is a nonexpansive semigroup on C, such that $F(S) = \bigcap_{t>0} F(S(t)) \neq \emptyset$. For all $h > 0$, the set $\sigma_t(x) = \frac{1}{t}\int_0^t S(s)x\,\mathrm{d}s$, then*

$$\lim_{t\to\infty} \sup_{x\in K} \|\sigma_t(x) - S(h)\sigma_t(x)\| = 0.$$

Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator and let $\psi : H_1 \to H_1$ be a contraction mapping with $\delta \in [0, 1)$. Let $S := \{S(t) : t > 0\}$ be a u.a.r nonexpansive semigroup on H_1.

Algorithm 5.3 Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\|A^*(I-\text{prox}_{\lambda g})Ax_n\|^2 + \|(I-\text{prox}_{\lambda f})x_n\|^2 \neq 0$, and then compute x_{n+1} by the following iterative scheme:

$$\begin{cases} y_n = \text{prox}_{\lambda\mu_n f}(\alpha_n\psi(x_n) + (1-\alpha_n)x_n - \mu_n A^*(I-\text{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S(t_n)y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{5.1}$$

where the stepsize $\mu_n := \rho_n \dfrac{\left(\frac{1}{2}\|(I-\text{prox}_{\lambda g})Ax_n\|^2\right) + \left(\frac{1}{2}\|(I-\text{prox}_{\lambda f})x_n\|^2\right)}{\|A^*(I-\text{prox}_{\lambda g})Ax_n\|^2 + \|(I-\text{prox}_{\lambda f})x_n\|^2}$ with $0 < \rho_n < 4$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{t_n\}$ is a positive real divergent sequence.

We now prove a strong convergence result for the problem (1.1) and the fixed point problem of nonexpansive semigroups as follows:

Theorem 5.4 *Suppose that* $\bigcap_{t>0} F(S(t)) \cap \Gamma \neq 0$. *If the control sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\rho_n\}$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\varepsilon \le \rho_n \le \dfrac{4(1-\alpha_n)\left(\|(I-\text{prox}_{\lambda g})Ax_n\|^2\right)}{\left(\|(I-\text{prox}_{\lambda g})Ax_n\|^2\right) + \left(\|(I-\text{prox}_{\lambda f})x_n\|^2\right)} - \varepsilon$ *for some* $\varepsilon > 0$.

Then, the sequence $\{x_n\}$ *generated by Algorithm 5.3 converges strongly to a point* $x^* \in \bigcap_{t>0} F(S(t)) \cap \Gamma$.

Proof By continuing in the same direction as in Theorem 3.2, we have that $\lim_{n\to\infty}\|y_n - S(t_n)y_n\| = 0$. Now, we only show that $\lim_{n\to\infty}\|y_n - S(h)y_n\| = 0$ for all $h \ge 0$. We observe that

$$\begin{aligned} \|y_n - S(h)y_n\| &\le \|y_n - S(t_n)y_n\| + \|S(t_n)y_n - S(h)S(t_n)y_n\| + \|S(h)S(t_n)y_n - S(h)y_n\| \\ &\le 2\|y_n - S(t_n)y_n\| + \sup_{x\in y_n}\|S(t_n)x - S(h)S(t_n)x\|. \end{aligned}$$

Since $\{S(t) : t \ge 0\}$ is a u.a.r. nonexpansive semigroup and $t_n \to \infty$ for all $h \ge 0$, we have:

$$\lim_{n\to\infty}\|y_n - S(h)y_n\| = 0,$$

for all $h \ge 0$. This completes the proof. □

6 Numerical examples

We first give a numerical example in Euclidean spaces to demonstrate the convergence of Algorithm (3.1).

Example 6.1 Let $H_1 = \mathbb{R}^2$ and $H_2 = \mathbb{R}^3$ with the usual norms. Define a mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$S(a,b) := \frac{\sqrt{2}}{2}(a-b, a+b).$$

Table 1 The numerical experiment of Algorithm (6.1) by choosing $\delta = 0.1$

n	a_n	b_n	E_n
1	3.0000000	−2.0000000	–
2	0.1783143	−0.1100519	3.3961470
3	0.0082067	−0.0025830	0.2012117
4	0.0004998	0.0013948	0.0086729
5	0.0001562	0.0010892	0.0004598
6	0.0001076	0.0007884	0.0003047
7	0.0000801	0.0005827	0.0002075
8	0.0000608	0.0004388	0.0001452
9	0.0000467	0.0003353	0.0001045
10	0.0000363	0.0002591	0.0000769
$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	0.0000008	0.0000055	0.0000012
29	0.0000007	0.0000046	0.00000098

One can show that S is nonexpansive. Define two functions $f:\mathbb{R}^2 \to (-\infty,\infty]$ and $g : \mathbb{R}^3 \to (-\infty,\infty]$ by $f := 0$, where 0 is a zero operator and

$$g(a,b,c) := \frac{|-3a+7b-2c|^2}{2}.$$

Then, the explicit forms of the proximity operators of f and g can be written by $\operatorname{prox}_{\lambda f} = I$ and $\operatorname{prox}_{1g} = B^{-1}$, where $B = \begin{pmatrix} 10 & -21 & 6 \\ -21 & 50 & -14 \\ 6 & -14 & 5 \end{pmatrix}$ (see Combettes and Pesquet 2011b). Let $A:\mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$A := \begin{pmatrix} 2 & 1 \\ 7 & -3 \\ -5 & 4 \end{pmatrix},$$

and let $\Omega := F(S) \cap \operatorname{argmin} f \cap A^{-1}(\operatorname{argmin} g)$. Now, we rewrite Algorithm (3.1) in the form:

$$\begin{cases} y_n = \alpha_n \psi(x_n) + (1-\alpha_n)x_n - \mu_n A^T(I-B^{-1})Ax_n \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Sy_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{6.1}$$

where

$$\mu_n = \frac{\rho_n}{2} \frac{\|(I-B^{-1})Ax_n\|^2}{\|A^T(I-B^{-1})Ax_n\|^2}.$$

Take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2}$, $\rho_n = \frac{2n}{n+1}$. Consider a contraction $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\psi(x) = \delta x$ for $0 \le \delta < 1$. We first start with the initial point $x_1 = (3,-2)$ and the stopping criterion for our testing process is set as: $E_n := \|x_n - x_{n-1}\| < 10^{-6}$, where $x_n = (a_n, b_n)$. In Table 1, we show the convergence behavior of Algorithm (6.1) by choosing $\delta = 0.1$. In Table 2, we also show the number of iterations of Algorithm (6.1) by choosing different constants δ. –

Table 2 The number of iterations of Algorithm (6.1) by choosing different constants δ

$\psi : \mathbb{R}^2 \to \mathbb{R}^2, \psi(x) = \delta x$			
Choices of δ	n (no. of iterations)	x_n	E_n
$\delta = 0$ (Shehu and Iyiola 2015, Algorithm 1)	42	$(-0.0000007, -0.0000048)$	0.00000098
$\delta = 0.05$	39	$(-0.0000007, -0.0000046)$	0.00000095
$\delta = 0.1$	29	$(-0.0000007, -0.0000046)$	0.00000098
$\delta = 0.2$	46	$(0.0000007, 0.0000050)$	0.00000099
$\delta = 0.5$	59	$(0.0000007, 0.0000052)$	0.00000097
$\delta = 0.9$	71	$(0.0000007, 0.0000049)$	0.00000088

Remark 6.2 In Example 6.1, by testing the convergence behavior of Algorithm (6.1), we observe that

(i) It converges to a solution, i.e., $x_n \to (0, 0) \in \Omega$.
(ii) The selection of a contraction ψ in our algorithm influences the number of iterations of the algorithm. We also note that if $\psi \equiv 0$ is zero, then our algorithm becomes Algorithm (1.6) (Shehu and Iyiola 2015, Algorithm 1).

Next, we give an example in the infinite-dimensional space L^2 as follows.

Example 6.3 Let $H_1 = L^2([0, 1]) = H_2$. Let $x \in L^2([0, 1])$. Define a bounded linear operator $A : L^2([0, 1]) \to L^2([0, 1])$ by:

$$(Ax)(t) := 3tx(t).$$

Define a mapping $S : L^2([0, 1]) \to L^2([0, 1])$ by:

$$(Sx)(t) := \sin(x(t)).$$

Then, S is nonexpansive. Let

$$C = \left\{x \in L^2([0, 1]) : \langle w, x\rangle \leq 0\right\},$$

where $w \in L^2([0, 1])$, such that $w(t) = 2t^3$, and let

$$Q = \left\{x \in L^2([0, 1]) : x \geq 0\right\}.$$

Define two functions $f, g : L^2([0, 1]) \to (-\infty, \infty]$ by $f := i_C$ and $g := i_Q$, where i_C and i_Q are indicator functions of C and Q, respectively. We can write the explicit forms of the proximity operators of f and g in the following forms:

$$\operatorname{prox}_{\lambda f} x = P_C x = \begin{cases} x - \frac{\langle w,x\rangle}{\|w\|^2} w, & \text{if } x \notin C, \\ x, & \text{if } x \in C, \end{cases}$$

and $\operatorname{prox}_{\lambda g} x = P_Q x = x_+$, where $x_+(t) = \max\{x(t), 0\}$ (see Cegielski 2012). Therefore, Algorithm (3.1) can be rewritten in the form:

$$\begin{cases} y_n = P_C(\alpha_n \psi(x_n) + (1 - \alpha_n)x_n - \mu_n A^*(I - P_Q)Ax_n) \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)Sy_n, \quad \forall n \in \mathbb{N}; \end{cases} \tag{6.2}$$

$\mu_n = \rho_n \frac{\left(\frac{1}{2}\|(I - P_Q)Ax_n\|^2\right) + \left(\frac{1}{2}\|(I - P_C)x_n\|^2\right)}{\|A^*(I - P_Q)Ax_n\|^2 + \|(I - P_C)x_n\|^2}$, for finding a common element in the set $\Omega := F(S) \cap C \cap A^{-1}(Q)$. By choosing the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\rho_n\}$ satisfying the conditions (C1)–(C3) in Theorem 3.2, it can guarantee that the sequence $\{x_n\}$ generated by (6.2) converges strongly to $x^* = 0 \in \Omega$.

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Research Article

The Generalized α-Nonexpansive Mappings and Related Convergence Theorems in Hyperbolic Spaces

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Abstract. In this paper, we propose and analyze a generalized α-nonexpansive mappings on a nonempty subset of a hyperbolic space i.e.,

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \alpha d(y,Tx) + \alpha d(x,Ty) + (1-2\alpha)d(x,y),$$

and prove Δ-convergence theorems and convergence theorems for a generalized α-nonexpansive mappings in a hyperbolic space.

Keywords. Fixed point set; Generalized α-nonexpansive mappings; Δ-convergence theorems and hyperbolic spaces

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1. Introduction

The existence of a fixed point is of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, finances, informatics, engineering and physics. Let M be a nonempty subset of a linear space X, let $F(T) = \{x \in M : Tx = x\}$ denotes the set of fixed points of the mapping T on M. Let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *nonexpansive*, if

$$d(Tx,Ty) \le d(x,y), \tag{1.1}$$

for each $x,y \in M$. Define a mapping T on $[0,1]$ by $Tx = \frac{x}{3}$, it's easy to see that T is nonexpansive. Let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *quasi-nonexpansive*, if

$$d(Tx,p) \le d(x,p)$$

for each $x \in M$ and $p \in F(T)$. Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & x \neq 3, \\ 2, & x = 3. \end{cases}$$

Then $F(T) = \{0\} \neq \emptyset$ and T is quasi-nonexpansive (see [20]). In the last sixty-five years, the numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings.

In 1953, Mann [13] has introduced The Mann iteration process is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is real sequences in $(0,1)$.

In 1974, Ishikawa [6] has introduced The Ishikawa iteration process is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$ and $\{y_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \ \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

In 2007, Agarwal *et al.* [2] introduced a new iteration process whose rate of convergence is similar to Picard iteration and faster than other fixed point iteration processes as follows: For M be a convex subset of a linear space X and $T : M \to M$ a mapping. Then the modified

S-iteration process is a sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = S(x_n, \alpha_n, \beta_n, T^n), \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

In 2008, Suzuki [20] introduced a class of single valued mappings called Suzuki-generalized nonexpansive mappings (or condition C). The condition C is weaker than nonexpansiveness and stronger than quasi-nonexpansive, as follows: Let T be a self-mapping on a subset M of a metric space X. Then T is said to satisfy *condition* C if

$$\frac{1}{2} d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y),$$

for each $x, y \in M$.

It is obvious that every nonexpansive mapping satisfies condition C, but the converse is not true, that is condition C is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The next simple example can show this fact. We see that, if define a mapping $T : [0,3] \to [0,3]$ by

$$Tx = \begin{cases} 0, & x \neq 3, \\ \frac{2}{3}, & x = 3. \end{cases}$$

Then T is condition C, but T is not nonexpansive (see [20]).

In 2011, Aoyama and Kohsaka [3] introduced the class of α-nonexpansive mappings in Banach spaces. This class contains the class of nonexpansive mappings and is related to the class of firmly nonexpansive mappings in Banach spaces as follows: let X be a Banach space and M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *α-nonexpansive* for some real number $\alpha < 1$, if

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|, \tag{1.5}$$

for all $x, y \in C$. Clearly, 0-nonexpansive maps is exactly nonexpansive maps. The next simple example can show this fact. We see that, let $M = [0,4]$ is a subste of $\mathbb{R}$ endowed with the usual normand usual order. Define $T : M \to M$ by

$$Tx = \begin{cases} 0; & x \neq 4, \\ 2; & x = 4. \end{cases}$$

Then, T is a α-nonexpansive mapping with $\alpha \geq \frac{1}{2}$ (see [17]).

In 2011, Sahu [15] has introduced Normal S-iteration Process is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$ and $\{y_n\}$ in M is defined by sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is real sequences in $(0,1)$.

In 2014, Kadioglu [7] defined Picard normal S-iteration process (PNS) is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\},\{y_n\}$ and $\{z_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ is real sequences in $(0,1)$. If $\beta_n = 0$ and $\alpha_n = \beta_n = 0$ in (1.7) then it reduces to Normal S-iteration process and Picard iteration process, respectively.

In 2014, Abbas and Nazir [1] introduced a new iteration process and proved that it is faster than all of Picard, Mann and Agarwal et al. processes as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n T z_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

In 2017, Pant and Shukla [17] introduced a new type of monotone nonexpansive mappings in an ordered Banach space X with partial order $\leq$. This new class of nonlinear mappings properly contains nonexpansive, firmly-nonexpansive and Suzuki-type generalized nonexpansive mappings and partially extends α-nonexpansive mappings as follows: Let X be a Banach space and M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *generalized α-nonexpansive*, if there exists $\alpha \in [0,1)$ such that

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|$$

for all $x, y \in M$. Clearly, generalized 0-nonexpansive maps is exactly Suzuki-generalized nonexpansive maps. The next simple example can show this fact. We see that, let $M = \{(0,0),(2,0),(0,4),(4,0),(4,5),(5,4)\}$ be a subset of $\mathbb{R}$ with dictionary order. Define a norm $\|\cdot\|$ on M by $\|(x_1,x_2)\| = |x_1| + |x_2|$. Then $(X,\|\cdot\|)$ is a Banach space. Define a mapping $T : M \to M$ by $T(0,0) = (0,0)$, $T(2,0) = (0,0)$, $T(0,4) = (0,0)$, $T(4,0) = (2,0)$, $T(4,5) = (4,0)$, $T(5,4) = (0,4)$. Then, T is a generalized α-nonexpansive mapping for $\alpha \geq \frac{1}{5}$, but is neither a Suzuki-generalize nonexpansive nor an a-nonexpansive mapping (see [17]).

In 2018, Mebawondu and Izuchukwu [14] introduced and studied some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces. They further established strong and Δ-convergence theorems for Picard Normal S-iteration scheme generated by a generalized α-nonexpansive mapping in the frame work of uniformly convex hyperbolic spaces. A hyperbolic space is a triple (X,d,W), where (X,d) is a metric space and $W : X^2 \times [0,1] \to X$ is such that

(W1) $d(u,W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$;

(W2) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha - \beta| d(x,y)$;

(W3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(W4) $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y)+\alpha d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$. The setting of hyperbolic spaces introduced by Kohlenbach [10]. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric. All normed spaces and their subsets are the examples of hyperbolic spaces as well convex metric spaces. It is remarked that CAT(0) spaces and Banach spaces are important examples of this type of hyperbolic spaces.

In this paper, we introduce and study some properties of the generalized α-nonexpansive mapping on a nonempty subset of a hyperbolic space and prove fixed point theorems for generalized α-nonexpansive mappings, Δ-convergence theorems and convergence theorems in a hyperbolic space.

2. Preliminaries

Now, we recall definitions on hyperbolic spaces. If $x,y \to X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for the more general setting of convex metric space [21], as follows:

$$d(x,W(x,y,\lambda)) = \lambda d(x,y) \quad \text{and} \quad d(y,W(x,y,\lambda)) = (1-\lambda)d(x,y)$$

for all $x,y \in X$ and $\lambda \in [0,1]$.

A hyperbolic space (X,d,W) is uniformly convex [18] if for any $r>0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$d\left(W\left(x,y,\frac{1}{2}\right),a\right) \leq (1-\delta)r$$

provided $d(x,a) \leq r, d(y,a) \leq r$ and $d(x,y) \geq \varepsilon r$.

A mapping $\eta:(0,1)\times(0,2] \to (0,1]$, which providing such a $\delta=\eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon \in (0,2]$, is called as a modulus of uniform convexity [19]. We call the function η is monotone if it decreases with r (for fixed ε), that is, $\eta(r_2,\varepsilon) \leq \eta(r_1,\varepsilon)$, for all $r_2 \geq r_1 > 0$.

Let M be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X while $\mathrm{diam}(M)$ denote the diameter of M.

Definition 2.1. Let M be a nonempty subset of metric space X and let $\{x_n\}$ be any bounded sequence in M. Let a continuous functional $r_a(\cdot,\{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad \text{for all } x \in X.$$

Then, consider the following:

(i) The infimum of $r_a(\cdot,\{x_n\})$ over M is said to be the *asymptotic radius* of $\{x_n\}$ with respect to M and is denoted by $r_a(M,\{x_n\})$;

(ii) a point $z \in M$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to M if

$$r_a(z,\{x_n\}) = \inf r_a(x,\{x_n\}), x \in M$$

the set of all asymptotic centers of $\{x_n\}$ with respect to M is denoted by $A(M)(M,\{x_n\})$;

(iii) this set may be empty, a singleton, or certain infinitely many points;

(iv) if the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $A(M)(X,\{x_n\}) = A(M)(\{x_n\})$, respectively;

(v) for $x \in X, r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces (see [5]).

Definition 2.2 ([9])**.** A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

Remark 2.3 ([11])**.** We note that Δ-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.4 ([12])**.** *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset M of X.*

Lemma 2.5 ([4])**.** *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_n\}) = \{x_1\}$ and $\{d(x_n,x_1)\}$ converges, then $x = x_1$.*

Lemma 2.6 ([8])**.** *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x) \le c$, $\limsup_{n\to\infty} d(y_n,x) \le c$ and $\lim_{n\to\infty} Wd(x_n,y_n,\alpha_n)$ for some $c \ge 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Definition 2.7. Let M be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to M if for all $x \in M$ and $n \ge 1$,

$$d(x_{n+1},x) \le d(x_n,x).$$

Example 2.8. Let M be a nonempty subset of X, and $T : M \to M$ be a quasi-nonexpansive (in particular, nonexpansive) mapping such that $F(T) \neq \emptyset$ and $x_0 \in M$. Then the sequence $\{x_n\}$ of Picard iterates is Fejér monotone with respect to $F(T)$.

Proposition 2.9 ([5])**.** *Let $\{x_n\}$ be a sequence in X and M be a nonempty subset of X. Suppose that $\{x_n\}$ is Fejér monotone with respect to M, then we have the followings:*

(1) *$\{x_n\}$ is bounded;*

(2) *The sequence $\{d(x_n,p)\}$ is decreasing and converges for all $p \in F(T)$;*

(3) *$\lim_{n\to\infty} d(x_n,F(T))$ exists.*

Definition 2.10 ([16])**.** Let M be a nonempty subset of a metric space X. A self mapping T of M with nonempty fixed point set $F(T)$ in M is said to satisfy Condition I if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$, such that $d(x,Tx) \ge f(D(x,F(T)))$ for all $x \in M$, where $D(x,F(T)) = \inf\{d(x,p) : p \in F(T)\}$.

3. Main Results

In this section, we will prove some property for class of generalized α-nonexpansive mappings in a hyperbolic spaces. First, we introduce generalized α-nonexpansive mappings in a hyperbolic space as follows: Let M be a nonempty subset of hyperbolic space X. Then $T : M \to M$ is said to satisfy *generalized α-nonexpansive*, if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Longrightarrow d(Tx,Ty) \leq \alpha d(y,Tx) + \alpha d(x,Ty) + (1-2\alpha)d(x,y)$$

for all $x, y \in M$.

From [17, Proposition 3.5, Lemma 3.7 and Lemma 3.8], we introduce Lemma 3.1, Lemma 3.2 and Lemma 3.3 in hyperbolic space respectively.

Lemma 3.1. *Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in M$:*

(i) $d(Tx,T^2x) \leq d(x,Tx)$;

(ii) *Either* $\frac{1}{2}d(x,Tx) \leq d(x,y)$ *or* $\frac{1}{2}d(Tx,T^2x) \leq d(Tx,y)$;

(iii) *Either* $d(Tx,Ty) \leq \alpha(Tx,y) + \alpha d(x,Ty) + (1-2\alpha)d(x,y)$ *or* $d(T^2x,Ty) \leq \alpha d(Tx,Ty) + \alpha d(T^2x,y) + (1-2\alpha)d(Tx,y)$.

Proof. (i) Since,

$$\frac{1}{2}d(x,Tx) \leq d(x,Tx)$$

by definition of T, we obtain that

$$\begin{aligned} d(Tx,T^2x) &\leq \alpha d(Tx,Tx) + \alpha d(T^2x,x) + (1-2\alpha)d(x,Tx) \\ &= \alpha d(T^2x,x) + (1-2\alpha)d(x,Tx). \end{aligned}$$

We choose $\alpha = 0 < 1$, then we have $d(Tx,T^2x) \leq d(x,Tx)$.

(ii) We will prove by contradiction, suppose that

$$\frac{1}{2}d(x,Tx) > d(x,y) \quad \text{and} \quad \frac{1}{2}d(Tx,T^2x) > d(Tx,y).$$

So, by (i) we have

$$\begin{aligned} d(x,Tx) &\leq d(x,y) + d(Tx,y) \\ &< \frac{1}{2}d(x,Tx) + \frac{1}{2}d(Tx,T^2x) \\ &\leq d(x,Tx). \end{aligned}$$

This is a contradiction. Hence, we have $\frac{1}{2}d(x,Tx) \leq d(x,y)$ or $\frac{1}{2}d(Tx,T^2x) \leq d(Tx,y)$. (iii) follows from (ii). □

Lemma 3.2. *Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in M$ with $x \leq y$,*

$$d(x,Tx) \leq \frac{(3+\alpha)}{(1-\alpha)}d(x,Tx) + d(x,y).$$

Proof. By Lemma 3.1, we have for all $x,y\in M$ either

$$d(Tx,Ty)\le \alpha(Tx,y)+\alpha d(x,Ty)+(1-2\alpha)d(x,y)$$

or

$$d(T^2x,Ty)\le \alpha d(Tx,Ty)+\alpha d(T^2x,y)+(1-2\alpha)d(Tx,y).$$

In first case, we consider

$$\begin{aligned} d(x,Ty) &\le d(x,Tx)+d(Tx,Ty)\\ &\le d(x,Tx)+\alpha d(Tx,y)+\alpha d(Ty,x)+(1-2\alpha)d(x,y)\\ &\le d(x,Tx)+\alpha d(Tx,x)+\alpha d(x,y)+\alpha d(Ty,x)+(1-2\alpha)d(x,y). \end{aligned}$$

This implies that

$$d(x,Ty)\le \frac{(1+\alpha)}{(1-\alpha)}d(Tx,x)+d(x,y)$$

In other case, we consider

$$\begin{aligned} d(x,Ty) &\le d(x,Tx)+d(Tx,T^2x)+d(T^2x,Ty)\\ &\le 2d(x,Tx)+\alpha d(Tx,Ty)+\alpha d(T^2,y)+(1-2\alpha)d(Tx,y)\\ &\le 2d(x,Tx)+\alpha d(Tx,x)+\alpha d(Ty,x)+\alpha d(T^2x,Tx)+\alpha d(Tx,y)+(1-2\alpha)d(Tx,y)\\ &\le (2+\alpha)d(x,Tx)+\alpha d(Ty,x)+\alpha d(x,Tx)+(1-\alpha)d(Tx,y)\\ &\le (2+\alpha)d(x,Tx)+\alpha d(Ty,x)+\alpha d(x,Tx)+(1-\alpha)d(Tx,x)+(1-\alpha)d(x,y). \end{aligned}$$

This implies that

$$d(x,Ty)\le \frac{(3+\alpha)}{(1-\alpha)}d(x,Tx)+d(x,y).$$ □

Lemma 3.3. *Let M be a nonempty subset of hyperbolic space X and $T:M\to M$ be a generalized α-nonexpansive mapping and $F(T)\neq\emptyset$, then T is a quasi-nonexpansive mapping.*

Proof. Let $p\in F(T)$ and $x\in M$. Since $\frac{1}{2}d(z,Tz)=0\le d(z,x)$, we obtain that

$$\begin{aligned} d(p,Tx) &= d(Tp,Tx)\\ &\le \alpha d(Tp,x)+\alpha d(Tx,p)+(1-2\alpha)d(p,x). \end{aligned}$$

We choose $\alpha=0<1$, then we have

$$d(p,Tx)\le d(p,x).$$

Hence, T is a quasi-nonexpansive mapping. □

Lemma 3.4. *Let X be complete uniformly convex hyperbolic space with monotone modulus of convexity η, M be a nonempty closed convex subset of X and T be a self generalized α-nonexpansive mapping on M. If $\{x_n\}$ is bounded sequence in M such that*

$$\lim_{n\to\infty} d(x_n,Tx_n)=0,$$

then T has a fixed point.

Proof. Since $\{x_n\}$ is bounded sequence in X, and by Lemma 2.4, we have $\{x_n\}$ has unique asymptotic center in M and $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Since T satisfies the generalized α-nonexpansive and from Lemma 3.2 such that

$$d(x_n,Tx) \le \frac{(3+\alpha)}{(1-\alpha)} d(x_n,Tx_n) + d(x_n,x).$$

Now, we take lim sup as $n\to\infty$ both the sides, we have

$$\begin{aligned} r_a(Tx,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,Tx) \\ &\le \limsup_{n\to\infty}\left[\frac{(3+\alpha)}{(1-\alpha)} d(x_n,Tx_n) + d(x_n,x)\right] \\ &\le \limsup_{n\to\infty} d(x_n,x) = r_a(x,\{x_n\}). \end{aligned}$$

By the uniqueness of asymptotic center, $Tx = x$, this implies that x is fixed point of T. Hence, T has a fixed point. □

Lemma 3.5. *Let M be a nonempty and convex subset of a strictly convex hyperbolic space X. Let T be a self generalized α-nonexpansive mapping on M and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Proof. Assume that $\{x_n\} \subseteq F(T)$ such that $\{x_n\}$ converges to some $y \in M$. We will show that $y \in F(T)$. By Lemma 3.2, we get that

$$d(x_n,Ty) \le \frac{(3+\alpha)}{(1-\alpha)} d(x_n,Tx_n) + d(x_n,y),$$

taking lim sup as $n\to\infty$ both the sides, we have

$$\limsup_{n\to\infty} d(x_n,Ty) \le \limsup_{n\to\infty} \frac{(3+\alpha)}{(1-\alpha)} d(x_n,Tx_n) + \limsup_{n\to\infty} d(x_n,y).$$

So, $\limsup_{n\to\infty} d(x_n,Ty) \le \limsup_{n\to\infty} d(x_n,y)$. By the uniqueness of the limit point of M, we obtain that $Ty = y$. Therefore, $F(T)$ is closed.

Next, we show that $F(T)$ is convex, let $x,y \in F(T)$, then for $\beta \in [0,1]$, we have

$$\begin{aligned} d(x,T(W(x,y,\beta))) &\le \frac{(3+\alpha)}{(1-\alpha)} d(x,Tx) + d(x,W(x,y,\beta)) \\ &\le d(x,W(x,y,\beta)) \end{aligned}$$

and

$$\begin{aligned} d(y,T(W(x,y,\beta))) &\le \frac{(3+\alpha)}{(1-\alpha)} d(y,Ty) + d(y,W(x,y,\beta)) \\ &\le d(y,W(x,y,\beta)) \end{aligned}$$

Now, we consider

$$\begin{aligned} d(x,y) &\le d(x,T(W(x,y,\beta))) + d(T(W(x,y,\beta)),y) \\ &\le d(x,W(x,y,\beta)) + d(W(x,y,\beta),y) \\ &\le d(x,y). \end{aligned}$$

Therefore, if $d(x,T(W(x,y,\beta))) < d(x,W(x,y,\beta))$ or $d(T(W(x,y,\beta)),y) < d(W(x,y,\beta),y)$, then which the contradiction to $d(x,y) < d(x,y)$, so $d(x,T(W(x,y,\beta))) = d(x,W(x,y,\beta))$ and

$d(T(W(x,y,\beta)),y) = d(W(x,y,\beta),y)$. Since M is strictly convex, we have $T(W(x,y,\beta) = W(x,y,\beta)$, that is $W(x,y,\beta) \in F(T)$. Hence, $F(T)$ is convex. □

Theorem 3.6. *Let M be a nonempty closed bounded and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and T be a self generalized α-nonexpansive mapping on M. Suppose that $\{x_n\}$ is a sequence in M, with $d(x_n,Tx_n) \to 0$. If $A(M)(M,\{x_n\}) = x$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Proof. Suppose that there exists some approximate fixed point sequence $\{x_n\}$. By Lemma 2.4, the asymptotic center of any bounded sequence is in M has a unique asymptotic center in M. Let $A(M)(M,\{x_n\}) = x$. We will prove that $x = Tx$. From Lemma 3.2, we have

$$d(x_n,Tx) \le \frac{(3+\alpha)}{(1-\alpha)} d(x_n,Tx_n) + d(x_n,x),$$

taking lim sup as $n \to \infty$ both the sides, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n,Tx) &\le \frac{(3+\alpha)}{(1-\alpha)} \limsup_{n\to\infty} d(x_n,Tx_n) + \limsup_{n\to\infty} d(x_n,x)\\ &= \limsup_{n\to\infty} d(x_n,x).\end{aligned}$$

By uniqueness of the asymptotic center implies $Tx = x$. Moreover, $F(T)$ closed and convex, by the prove in Lemma 3.5. □

Corollary 3.7. *Let M be a nonempty closed bounded and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in M, with $d(x_n,Tx_n) \to 0$. If T satisfies generalized α-nonexpansive and $A(M)(M,\{x_n\}) = x$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Now, we expand the result of Abbas and Nazir [1] to generalized α-nonexpansive mappings in hyperbolic spaces, as follows: Let M be a nonempty closed convex subset of a hyperbolic space X and T be a self generalized α-nonexpansive mapping on M. For any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Ty_n, Tz_n, \alpha_n) \\ y_n = W(z_n, Tz_n, \beta_n) \\ z_n = W(x_n, Tx_n, \gamma_n)\ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0,1]$ for all $n \in \mathbb{N}$.

Lemma 3.8. *Let M be a nonempty closed convex subset of a hyperbolic space X and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive. If $\{x_n\}$ is a sequence defined by* (3.1)*, then $\{x_n\}$ is Fejér monotone with respect to $F(T)$.*

Proof. Since T satisfies the generalized α-nonexpansive and $p \in F(T)$, we have

$$\frac{1}{2} d(p,Tp) = 0 \le d(p,x_n),$$

$$\frac{1}{2}d(p,Tp)=0\le d(p,y_n)$$

and

$$\frac{1}{2}d(p,Tp)=0\le d(p,z_n),$$

for all $n\in\mathbb{N}$. We obtain that

$$d(Tp,Tx_n)\le \alpha d(Tp,x_n)+\alpha d(Tx_n,p)+(1-2\alpha)d(p,x_n),$$

$$d(Tp,Ty_n)\le \alpha d(Tp,y_n)+\alpha d(Ty_n,p)+(1-2\alpha)d(p,y_n)$$

and

$$d(Tp,Tz_n)\le \alpha d(Tp,z_n)+\alpha d(Tz_n,p)+(1-2\alpha)d(p,z_n).$$

By (3.1) and Lemma 3.3, we have

$$d(Tp,Tx_n)\le d(p,x_n),$$

$$d(Tp,Ty_n)\le d(p,y_n)$$

and

$$d(Tp,Tz_n)\le d(p,z_n). \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned} d(x_{n+1},p)&=d(W(Ty_n,Tz_n,\alpha_n),p)\\ &\le(1-\alpha_n)d(Ty_n,p)+\alpha_n d(Tz_n,p)\\ &\le(1-\alpha_n)d(y_n,p)+\alpha_n d(z_n,p), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} d(y_n,p)&=d(W(z_n,Tz_n,\beta_n),p)\\ &\le(1-\beta_n)d(z_n,p)+\beta_n d(Tz_n,p)\\ &\le(1-\beta_n)d(z_n,p)+\beta_n d(z_n,p)\\ &=d(z_n,p), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d(z_n,p)&=d(W(x_n,Tx_n,\gamma_n),p)\\ &\le(1-\beta_n)d(x_n,p)+\gamma_n d(Tx_n,p)\\ &\le(1-\beta_n)d(x_n,p)+\gamma_n d(x_n,p)\\ &=d(x_n,p). \end{aligned} \tag{3.5}$$

Now, taking (3.4) in (3.3), we have

$$\begin{aligned} d(x_{n+1},p)&\le(1-\alpha_n)d(z_n,p)+\alpha_n d(z_n,p)\\ &=d(z_n,p). \end{aligned} \tag{3.6}$$

Taking (3.5) in (3.6), we obtain that

$$d(x_{n+1},p)\le d(z_n,p)\le d(x_n,p) \tag{3.7}$$

Hence, $\{x_n\}$ is Fejér monotone with respect to $F(T)$. □

Lemma 3.9. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and T be a self generalized α-nonexpansive mapping on M. If $\{x_n\}$ is a sequence defined by* (3.1)*, then $F(T)$ is nonempty if and only if the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Assume that $F(T)$ is nonempty and let $p \in F(T)$. From Lemma 3.8 and Proposition 2.9, we have $\{x_n\}$ is Fejér monotone with respect to $F(T)$ and bounded such that $\lim_{n\to\infty} d(x_n, p)$ exists, let $\lim_{n\to\infty} d(x_n, p) = k$. We divide into two case

(i) If $k = 0$, we have

$$d(x_n, Tx_n) \le d(x_n, p) + d(p, Tx_n),$$

by Lemma 3.3, we get

$$d(x_n, Tx_n) \le 2d(x_n, p).$$

Taking lim as $n \to \infty$ on both the sides above inequality, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

(ii) If $k > 0$, let $p \in F(T)$ and since T satisfies the generalized α-nonexpansive, by Lemma 3.3, we have

$$d(Tx_n, p) \le d(x_n, p),$$

by taking lim sup as $n \to \infty$ both the sides, we have

$$\limsup_{n\to\infty} d(Tx_n, p) \le k. \tag{3.8}$$

Taking lim sup as $n \to \infty$ both the sides in (3.5), we obtain that

$$\limsup_{n\to\infty} d(z_n, p) \le k. \tag{3.9}$$

From (3.6), we get

$$d(x_{n+1}, p) \le d(z_n, p),$$

so, taking lim inf as $n \to \infty$ both the sides, we obtain that

$$\begin{aligned} \liminf_{n\to\infty} d(x_{n+1}, p) &\le \liminf_{n\to\infty} d(z_n, p) \\ k &\le \liminf_{n\to\infty} d(z_n, p) \end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we have

$$\lim_{n\to\infty} d(z_n, p) = k,$$

which implies that

$$\begin{aligned} k &= \limsup_{n\to\infty} d(z_n, p) \\ &= \limsup_{n\to\infty} d(W(x_n, Tx_n, \gamma_n), p) \\ &\le \limsup_{n\to\infty} [(1-\gamma_n)d(x_n, p) + \gamma_n d(Tx_n, p)] \\ &\le \limsup_{n\to\infty} (1-\gamma_n)d(x_n, p) + \limsup_{n\to\infty} \gamma_n d(Tx_n, p) = k. \end{aligned}$$

Therefore, by Lemma 2.6, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Hence, from Lemma 3.4, we have $Tx = x$, that is $F(T)$ is nonempty. □

Theorem 3.10. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ satisfies the generalized α-nonexpansive, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in* (3.1)*, Δ-converges to a common fixed point of T.*

Proof. By Lemma 3.8, we have $\{x_n\}$ is a bounded sequence then, $\{x_n\}$ has a Δ-convergent subsequence.

Next, we show that every Δ-convergent subsequence of $\{x_n\}$ has unique Δ-limit $F(T)$. Let u and v Δ-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$. By Lemma 2.4, $A(M)(M,\{u_n\}) = \{u\}$ and $A(M)(M,\{v_n\}) = \{v\}$. By Lemma 3.9, we get

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0.$$

From Lemma 3.4, we have u and v are fixed points of T.

Now, we will show that $u = v$. Assume that $u \neq v$, then by uniqueness of asymptotic center we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, u) &= \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v) \\
&< \limsup_{n\to\infty} d(v_n, u) \\
&= \limsup_{n\to\infty} d(x_n, u),
\end{aligned}$$

which is a contradiction, therefore $u = v$. Hence, the sequence $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

Theorem 3.11. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by* (3.1)*, converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} D(x_n, F(T)) = 0$, where $D(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Proof. Assume that $\{x_n\}$ converges to $p \in F(T)$. Thus, $\lim_{n\to\infty} d(x_n, p) = 0$, since $0 \le D(x_n, F(T) \le d(x_n, p) \le 0$. Hence, $\liminf_{n\to\infty} D(x_n, F(T)) = 0$.

Conversely, from Lemma 3.5, we have $F(T)$ is closed. Assume that

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0.$$

From (3.7), we have

$$D(x_{n+1}, F(T)) \le D(x_n, F(T)), \quad n \in \mathbb{N}$$

then by Lemma 3.8 and Proposition 2.9, we obtain that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Then we have $\lim_{n\to\infty} D(x_n, F(T)) = 0$.

Now, we will show that $\{x_n\}$ is convergent to $p \in F(T)$. Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ we get

$$d(x_{n_k}, p_k) < \frac{1}{2^k},$$

for all $k \ge 1$ where $\{p_k\}$ is in $F(T)$. By Lemma 3.8, we have

$$d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) < \frac{1}{2^k},$$

this implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed, $\{p_k\}$ is a convergent sequence. Let $\{p_k\}$ converges to p. Since

$$d(x_{n_k}, p) \le d(x_{n_k}, p_k) + d(p_k, p) \to 0, \text{ as } k \to \infty,$$

such that $\lim_{k\to\infty} d(x_{n_k}, p) = 0$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, the sequence $\{x_n\}$ is convergent to p. This completes the proof. □

Theorem 3.12. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive. Moreover, T satisfies the condition I with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ which is defined by* (3.1)*, converges strongly to some fixed point of T.*

Proof. From Lemma 3.5, we have $F(T)$ is closed. Observe that by Lemma 3.8, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from the condition I that

$$\lim_{n\to\infty} f(D(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Thus, we get

$$\lim_{n\to\infty} f(D(x_n, F(T))) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing mapping satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Rest of the proof follows in lines of Theorem 3.11. Hence the sequence $\{x_n\}$ is convergent to $p \in F(T)$. This completes the proof. □

4. Conclusion

In this paper, we studied some properties of the generalized α-nonexpansive mappings on a nonempty subset of a hyperbolic space, proved fixed point theorems for generalized α-nonexpansive mappings and proved convergence theorems. Moreover, we obtain that corollary for the generalized α-nonexpansive mappings on a nonempty subset of a hyperbolic space as follows:

(1) Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in M$:

   (i) $d(Tx, T^2x) \le d(x, Tx)$;

   (ii) Either $\frac{1}{2}d(x, Tx) \le d(x, y)$ or $\frac{1}{2}d(Tx, T^2x) \le d(Tx, y)$;

   (iii) Either $d(Tx, Ty) \le \alpha(Tx, y) + \alpha d(x, Ty) + (1-2\alpha)d(x, y)$ or $d(T^2x, Ty) \le \alpha d(Tx, Ty) + \alpha d(T^2x, y) + (1-2\alpha)d(Tx, y)$.

(2) Let M be a nonempty closed bounded and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and T be a self generalized α-nonexpansive mapping on M. Suppose that $\{x_n\}$ is a sequence in M, with $d(x_n, Tx_n) \to 0$. If $A(M)(M, \{x_n\}) = x$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

(3) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ satisfies the generalized α-nonexpansive, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (3.1), Δ-converges to a common fixed point of T.

(4) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} D(x_n, F(T)) = 0$, where $D(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.

(5) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive. Moreover, T satisfies the condition I with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of T.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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Research Article

The Finite Family L-Lipschitzian Suzuki-Generalized Nonexpansive Mappings

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Abstract. In this paper, we propose and analyze a L-Lipschitzian Suzuki-generalized nonexpansive mapping on a nonempty subset of a hyperbolic space and prove Δ-convergence theorems and convergence theorems for a L-Lipschitzian Suzuki-generalized nonexpansive mapping in a hyperbolic space.

Keywords. Fixed point set; L-Lipschitzian Suzuki-generalized nonexpansive mappings; Iteration and hyperbolic spaces

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1. Introduction

In mathematics, a fixed point of a function is an element of the function's domain that is mapped to itself by the function. That is to say, x_0 is a fixed point of the function $f(x)$ if $f(x_0) = x_0$. In numerical analysis, fixed-point iteration is a method of computing fixed points of iterated functions. Let M be a nonempty subset of a linear space X, and let $F(T) = \{x \in M : Tx = x\}$ denotes the set of fixed points of the mapping T on M. Many nonlinear equations are naturally formulated as fixed point equations,

$$x = Tx, \tag{1.1}$$

where $T : X \to X$ is a mappings. A solution x of the equation (1.1) is called a *fixed point* of the mapping T. We consider a Picard iteration, which is given by

$$x_{n+1} = Tx_n, \quad \forall\, \mathbb{N}. \tag{1.2}$$

For the Banach contraction mapping theorem, the Picard iteration converges unique fixed point of T, but it fails to approximate fixed point for nonexpansive mappings, even when the existence of a fixed point of T is guaranteed (see [8]). Consider $T : [0,1] \to [0,1]$ defined by $Tx = 1-x$ for $x \in [0,1]$. Then T is nonexpansive with a unique fixed point at $x = \frac{1}{2}$. If we choose a starting value $x = a \neq \frac{1}{2}$, then the successive iteration of T yield the sequence $\{1-a, a, 1-a, \cdots\}$ (see [8]). Next, let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *nonexpansive*, if

$$d(Tx,Ty) \le d(x,y), \tag{1.3}$$

for each $x, y \in M$. Define a mapping T on $[0,1]$ by

$$Tx = x.$$

It is easy to see that T is nonexpansive. In the last fifty years, the numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings (see e.g., [2]-[22]), and compare which one is faster to approximate the fixed point as earliest as possible.

Let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *quasi-nonexpansive*, if

$$d(Tx,p) \le d(x,p)$$

for each $x \in M$ and $p \in F(T)$. Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & x \neq 3, \\ 2, & x = 3. \end{cases}$$

Then $F(T) = \{0\} \neq \emptyset$ and T is quasi-nonexpansive, but T does not satisfy condition C (see [25]).

In 2008, Suzuki [25] introduced a class of single valued mappings called Suzuki-generalized nonexpansive mappings (or condition C), as follows: Let T be a self-mapping on a subset M of a metric space X. Then T is said to satisfy *condition C* if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le d(x,y),$$

for each $x, y \in M$.

It is obvious that every nonexpansive mapping satisfies condition C, but the converse is not true, that is condition C is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The next simple example can show this fact. We see that, if define a mapping

T_1 and T_2 on $[0,3]$ by

$$T_1x = \begin{cases} 0, & x \neq 3, \\ 1, & x = 3 \end{cases}$$

and

$$T_2x = \begin{cases} 0, & x \neq 3, \\ \frac{3}{2}, & x = 3. \end{cases}$$

Then T_1 and T_2 are condition C, but T_1 and T_2 are not nonexpansive (see [25]).

Definition 1.1. Let (X,d) be a metric space and M be its nonempty subset. Then $T : M \to M$ said to be i if there exists a constant $L > 0$ such that

$$d(Tx,Ty) \leq Ld(x,y)$$

for all $x, y \in M$.

Example 1.2. Consider, $T : [0,2] \to [0,2]$, define by

$$Tx = x^2, \quad \forall\ x \in [0,2].$$

It is easy to see that T is L-Lipschitzian, but T is not nonexpansive.

In 2011, Sahu [20] introduced Normal S-iteration Process, whose rate of convergence similar to the Picard iteration process and faster than other fixed point iteration processes, as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, for each $x_1 \in M$, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is real sequences in $(0,1)$.

In 2014, Kadioglu [10] defined Picard normal S-iteration process (PNS) as follows: With C, X and T as in (NS), for each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ is real sequences in $(0,1)$. If $\beta_n = 0$ and $\alpha_n = \beta_n = 0$ in (1.5) then it reduces to Normal S-iteration process and Picard iteration process respectively.

On the other hand, Kohlenbach [13] introduced hyperbolic spaces, as follows: A hyperbolic space is a triple (X,d,W), where (X,d) is a metric space and $W : X^2 \times [0,1] \to X$ is such that

W1: $d(u,W(x,y,\alpha)) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$;

W2: $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha - \beta| d(x,y)$;

W3: $W(x,y,\alpha) = W(y,x,1-\alpha)$;

W4: $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$,

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$.

Moreover, a metric space is said to be a convex metric space in the sense of Takahashi [26], where a triple (X,d,W) satisfy only $W1$. The concept of hyperbolic spaces in [13] is more restrictive than the hyperbolictype introduced by Goebel and Kirk [6] since $W1$-$W2$ together are equivalent to (X,d,W) being a space of hyperbolic type in [6]. But it is slightly more general than the hyperbolic space defined in Reich and Shafrir [19] (see [13]). This class of metric spaces in [13] covers all normed linear spaces, R-trees in the sense of Tits, the Hilbert ball with the hyperbolic metric (see [7]), Cartesian products of Hilbert balls, Hadamard manifolds (see [19]), and CAT(0) spaces in the sense of Gromov (see [4] for a detailed treatment). A thorough discussion of hyperbolic spaces and a detailed treatment of examples can be found in [13] (see also [6], [7], [19]). Define the function $d : X^2 \to [0,\infty)$ by

$$d(x,y) = \|x-y\|$$

as a metric on X, where X is a real Banach space which is equipped with norm $\|\cdot\|$. Then, we have that (X,d,W) is a hyperbolic space with mapping $W : X^2 \times [0,1] \to X$ defined by $W(x,y,\alpha) = (1-\alpha)x + \alpha y$ (see [24]).

In this paper, we prove some properties of a L-Lipschitzian Suzuki-generalized nonexpansive mapping on a nonempty subset of a hyperbolic space and prove Δ-convergence theorems and convergence theorems for a L-Lipschitzian Suzuki-generalized nonexpansive mapping in a hyperbolic space.

Next, we recall the same basic definitions, notations and some results on hyperbolic spaces that will be used in the later section.

2. Preliminaries

Now, we recall definitions on hyperbolic spaces. If $x,y \to X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for the more general setting of convex metric space [26], as follows:

$$d(x,W(x,y,\lambda)) = \lambda d(x,y) \quad \text{and} \quad d(y,W(x,y,\lambda)) = (1-\lambda)d(x,y)$$

for all $x,y \in X$ and $\lambda \in [0,1]$.

A hyperbolic space (X,d,W) is uniformly convex [23] if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$d\left(W\left(x,y,\frac{1}{2}\right),a\right) \le (1-\delta)r$$

provided $d(x,a) \le r, d(y,a) \le r$ and $d(x,y) \ge \varepsilon r$.

A mapping $\eta : (0,1) \times (0,2] \to (0,1]$, which providing such a $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is called as a modulus of uniform convexity [24]. We call the function η is monotone if it decreases with r (for fixed ε), that is, $\eta(r_2,\varepsilon) \le \eta(r_1,\varepsilon)$, $\forall\ r_2 \ge r_1 > 0$.

Let M be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X while diam(M) denote the diameter of M. Consider a continuous functional $r_a(\cdot,\{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad x \in X.$$

The infimum of $r_a(\cdot,\{x_n\})$ over M is said to be the asymptotic radius of $\{x_n\}$ with respect to M and is denoted by $r_a(M,\{x_n\})$. A point $z \in M$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to M if

$$r_a(z,\{x_n\}) = \inf\{r_a(x,\{x_n\}) : x \in M\},$$

the set of all asymptotic centers of $\{x_n\}$ with respect to M is denoted by $AM(M,\{x_n\})$. This set may be empty, a singleton, or certain infinitely many points. If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $AM(X,\{x_n\}) = AM(\{x_n\})$, respectively. We know that for $x \in X, r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$. It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces (see [8]).

Definition 2.1 ([12]). A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write Δ-$\lim_{n\to\infty} x_n = x$.

Remark 2.2. We note that Δ-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.3 ([15]). *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 2.4 ([5]). *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_n\}) = \{x_1\}$ and $\{d(x_n,x_1)\}$ converges, then $x = x_1$.*

Lemma 2.5 ([11]). *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x) \le c, \limsup_{n\to\infty} d(y_n,x) \le c$ and $\lim_{n\to\infty} Wd(x_n,y_n,\alpha_n)$ for some $c \ge 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Lemma 2.6 ([18]). *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that*

$$\delta_{n+1} \le \beta_n\delta_n + \gamma_n$$

for all $n \in \mathbb{N}$. If $\beta_n \ge 1$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Definition 2.7 ([21]). Let (X,d) be a metric space and M be it's nonempty subset of X and T be a self-mapping on M, then a sequence $\{x_n\}$ in M is called approximate fixed point sequence for T (AFPS, in short) if $\lim_{n\to\infty} d(x_n,Tx_n) = 0$.

Theorem 2.8 ([1]). *Let M be a nonempty closed convex subset of a complete CAT(0) space X, $T : M \to M$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{a_n,u_n\}$ such* *that* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty} u_n < \infty$. *Assume that* $F(T)$ *is a closed set. Let* $\{x_n\}$ *be a sequence in* M. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

3. Main Results

In this section, we begin with the definition of L-Lipschitzian Suzuki-generalized nonexpansive mapping.

Definition 3.1. Let T be a self-mapping on a subset M of a metric space X. Then T is said to satisfy *L-Lipschitzian Suzuki-generalized nonexpansive* if there exists a constant $L > 0$ such that

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le Ld(x,y), \quad \forall\ n \ge 1, x,y \in M.$$

Example 3.2. Consider, $T : [0,2] \to [0,2]$, define by

$$Tx = \begin{cases} 0, & x \neq 2, \\ x^2, & x = 2. \end{cases}$$

If $x = 2$ and $y \in (0,1)$, then

$$\frac{1}{2}d(2,T2) = 1 \le d(2,y) \quad \text{and} \quad d(T2,Ty) = 4 \le Ld(2,y).$$

Thus, we see that T is not condition C, because $d(T2,Ty) = 4 > d(2,y)$. In other cases, for any $L \ge \frac{4}{d(2,y)}$, a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive.

Remark 3.3. We consider Example 3.2, for $x = 2$ and $y \in (0,1)$, if we choose $L < \frac{4}{d(2,y)}$, then we have a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive, but T is not L-Lipschitzian.

Example 3.4. Consider, $T : [0,2] \to [0,2]$, define by

$$Tx = \begin{cases} x^2, & x \neq 2, \\ 1, & x = 2. \end{cases}$$

If $x = 2$ and $y \in \left(1, \frac{1}{2}\right)$, then

$$\frac{1}{2}d(2,T2) = \frac{1}{2} \le d(x,y) \ \text{ and } \ d(T2,Ty) \le Ld(2,y) \tag{3.1}$$

hold. Thus, T satisfies L-Lipschitzian Suzuki-generalized nonexpansive mapping for any $L \ge \frac{d(T2,Ty)}{d(2,y)}$, but T is not condition (C) because $d(T2,Ty) > d(2,y)$. In other cases, it's easy to see that, a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive.

Remark 3.5. We will see that in Example 3.4, for $x = 2$ and $y \in \left(1, \frac{1}{2}\right)$, if we choose $L < \frac{d(Tx,Ty)}{d(x,y)}$, then we have a map T satisfies L-Lipschitzian Suzuki-generalized nonexpansive, but T is not L-Lipschitzian.

Proposition 3.6. *Let* $\{T_i\}_{i=1}^k$ *be a self finite family of* L_i*-Lipschitzian Suzuki-generalized* *nonexpansive mappings on M. Then*

$$d(x_n,T_i y) \leq (1+2L_i)d(x_n,T_i x_n)+L_i d(x_n,y)$$

for all $x,y \in M$, $\{x_n\}$ is approximate fixed point sequence in M.

Proof. Let $x,y \in M$, since $\{T_i\}_{i=1}^k$ is a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, we have

$$\frac{1}{2}d(x_n,T_i x_n)=0 \leq d(x_n,y),$$

for all $n \in \mathbb{N}$, then

$$d(T_i x_n,T_i y) \leq L_i d(x_n,y).$$

Now, we consider

$$\begin{aligned} d(x_n,T_i y) &\leq d(x_n,T_i x_n)+d(T_i x_n,T_i^2 x_n)+d(T_i^2 x_n,T_i y)\\ &\leq (1+L_i)d(x_n,T_i x_n)+L_i d(T_i x_n,y)\\ &\leq (1+2L_i)d(x_n,T_i x_n)+L_i d(x_n,y). \end{aligned}$$

Hence, $d(x_n,T_i y) \leq (1+2L_i)d(x_n,T_i x_n)+L_i d(x_n,y)$. □

Let (X,d) be a metric space and let M be a nonempty subset of X. We will denote the fixed point set of mapping $\{T_i\}_{i=1}^k$ by $F(T) := \bigcap_{i=1}^{k} F(T_i)$.

Lemma 3.7. *Let M be a nonempty and convex subset of a strictly convex hyperbolic space X. If $\{T_i\}_{i=1}^k$ be a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, that is there exist a sequence $\{u_n\}$ and $L_i > 0$ such that*

$$\frac{1}{2}d(x,T_i x) \leq d(x,y) \Rightarrow d(T_i x,T_i y) \leq u_n L_i d(x,y),$$

$\forall\ n \geq 1$, $x,y \in M$ with $u_nL_i \to 1$, for all $i=1,2,\dots,k$ and $F(T) \neq \emptyset$. If $\{x_n\},\{y_n\}$ are bounded approximate fixed point sequence in M, then $F(T)$ is closed and convex.

Proof. Assume that $\{x_n\}$ is a sequence in $F(T)$ which converges to some $y \in M$. To show that $y \in F(T)$ by Proposition 3.6, we obtain that

$$\begin{aligned} d(x_n,T_i y) &\leq (1+2L_i)d(x_n,T_i x_n)+u_nL_i d(x_n,y)\\ &\leq (1+2L_i)d(x_n,T_i x_n)+d(x_n,y). \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} d(x_n,T_i y) \leq \limsup_{n\to\infty}(1+2L_i)d(x_n,T_i x_n)+\limsup_{n\to\infty} d(x_n,y).$$

Since $\{x_n\} \subseteq F(T)$, we have $\limsup_{n\to\infty} d(x_n,T_i y) \leq \limsup_{n\to\infty} d(x_n,y)$. By the uniqueness of the limit point we obtain that $T_i y = y$, that is $y \in F(T)$, and then $F(T)$ is closed.

Now, we will to show that $F(T)$ is convex. Let $x,y \in F(T)$ and each $\alpha \in (0,1)$. Then,

$$\begin{aligned} d(x,y) &\leq d(x,T_i(W(x,y,\alpha)))+d(T_i(W(x,y,\alpha)),y)\\ &\leq d(x,W(x,y,\alpha))+d(W(x,y,\alpha),y) \end{aligned}$$

$$\leq d(x,y).$$

Now, we consider

$$\begin{aligned}d(x,T_i(W(x,y,\alpha))) &\leq (1+2L_i)d(x,T_ix)+u_nL_id(x,W(x,y,\alpha))\\ &\leq (1+2L_i)d(x,T_ix)+d(x,W(x,y,\alpha))\\ &\leq d(x,W(x,y,\alpha))\end{aligned}$$

and

$$\begin{aligned}d(y,T_i(W(x,y,\alpha))) &\leq (1+2L_i)d(y,T_iy)+u_nL_id(y,W(x,y,\alpha))\\ &\leq (1+2L_i)d(y,T_iy)+d(y,W(x,y,\alpha))\\ &\leq d(y,W(x,y,\alpha)),\end{aligned}$$

we get that $d(x,T_i(W(x,y,\alpha))) = d(x,(W(x,y,\alpha)))$ and $d(T_i(W(x,y,\alpha)),y) = d(W(x,y,\alpha),y)$, because if $d(x,T_i(W(x,y,\alpha))) \leq d(x,W(x,y,\alpha))$ or $d(T_i(W(x,y,\alpha)),y) \leq d(W(x,y,\alpha),y)$, then which the contradiction to $d(x,y) < d(x,y)$. Since M is strictly convex, we have $T_i(W(x,y,\alpha)) = W(x,y,\alpha)$, so $W(x,y,\alpha) \in F(T)$. Hence, $F(T)$ is convex. □

Lemma 3.8. *Let (X,d) be complete uniformly convex hyperbolic space with monotone modulus of convexity η, M be a nonempty closed convex subset of X and $\{T_i\}_{i=1}^k$ be a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M. Suppose $\{x_n\}$ is bounded sequence in M with $\{x_n\}$ is bounded approximate fixed point sequence for $\{T_i\}_{i=1}^k$, then $\{T_i\}_{i=1}^k$ have a fixed point.*

Proof. Since $\{x_n\}$ is bounded sequence in X, then by Lemma 2.3, has unique asymptotic center in M, that is, $AM(M,\{x_n\}) = \{x\}$ is singleton and $\lim_{n\to\infty} d(x_n,T_ix_n) = 0$. Since $\{T_i\}_{i=1}^k$ satisfies a self finite family of u_nL_i-Lipschitzian Suzuki-generalized nonexpansive on M, there exist a sequence $\{u_n\}$ and $L_i > 0$ such that

$$\begin{aligned}d(x_n,T_ix) &\leq (1+2L_i)d(x_n,T_ix_n)+u_nL_id(x_n,x)\\ &\leq (1+2L_i)d(x_n,T_ix_n)+d(x_n,x).\end{aligned}$$

Taking lim sup as $n \to \infty$ both the sides, we have

$$\begin{aligned}r_a(T_ix,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,T_ix)\\ &\leq \limsup_{n\to\infty}[(1+2L_i)d(x_n,T_ix_n)+d(x_n,x)]\\ &\leq \limsup_{n\to\infty} d(x_n,x) = r_a(x,\{x_n\}).\end{aligned}$$

By the uniqueness of asymptotic center, $T_ix = x$, thus x is fixed point of T. Hence, $F(T)$ is nonempty and then $\{T_i\}_{i=1}^k$ has a fixed point. □

Now, we expand the result of Kadioglu [10] (PNS) to L-Lipschitzian Suzuki-generalized nonexpansive mappings in hyperbolic spaces, as follows: Let M be a nonempty closed convex subset of a hyperbolic space X and $\{T_i\}_{i=1}^k$ be a self finite family of L-Lipschitzian Suzuki-generalized nonexpansive mappings on M. For any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = T_i y \\ y_n = W(z_n, T_i z_n, \alpha_n) \\ z_n = W(x_n, T_i x_n, \beta_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0,1]$ for all $n \in \mathbb{N}$.

Theorem 3.9. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a slef-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in* (3.2)*, Δ-converges to a common fixed point of $\{T_i\}_{i=1}^k$.*

Proof. We divide our proof into three steps.
First, we will show that $\lim\limits_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. Since $\{T_i\}_{i=1}^k$ satisfies the L_i-Lipschitzian Suzuki-generalized and $p \in F(T)$, we have

$$\frac{1}{2}d(p, T_i p) = 0 \le d(p, z_n),$$
$$\frac{1}{2}d(p, T_i p) = 0 \le d(p, y_n)$$
and
$$\frac{1}{2}d(p, T_i p) = 0 \le d(p, x_n),$$

for all $n \in \mathbb{N}$, we get that

$$d(T_i p, T_i z_n) \le L_i d(p, z_n),$$
$$d(T_i p, T_i y_n) \le L_i d(p, y_n)$$
and
$$d(T_i p, T_i x_n) \le L_i d(p, x_n).$$

By (3.2), we have

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T_i x_n, \beta_n), p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(T_i x_n, p) \\ &= (1-\beta_n)d(x_n, p) + \beta_n d(T_i x_n, T_i p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n L_i d(x_n, p) \\ &= (1-\beta_n + \beta_n L_i)d(x_n, p), \quad \forall\ n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3), we have

$$\begin{aligned} d(y_n, p) &= d(W(z_n, T_i z_n, \alpha_n), p) \\ &\le (1-\alpha_n)d(z_n, p) + \alpha_n d(T_i z_n, p) \\ &= (1-\alpha_n)d(z_n, p) + \alpha_n d(T_i z_n, T_i p) \\ &\le (1-\alpha_n)d(z_n, p) + \alpha_n L_i d(z_n, p) \end{aligned}$$

$$\begin{aligned}
&= (1-\alpha_n+\alpha_n L_i)d(z_n,p)\\
&\le (1-\alpha_n+\alpha_n L_i)[(1-\beta_n+\beta_n L_i)d(x_n,p)]\\
&= [(1-\alpha_n+\alpha_n L_i)(1-\beta_n+\beta_n L_i)]d(x_n,p), \quad \forall\ n\in\mathbb{N}. \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
d(x_{n+1},p) &= d(T_i y_n,p)\\
&= d(T_i y_n, T_i p)\\
&\le L_i d(y_n,p)\\
&\le L_i[(1-\alpha_n+\alpha_n L_i)(1-\beta_n+\beta_n L_i)]d(x_n,p)\\
&= [(L_i-\alpha_n L_i-\beta_n L_i)+(\alpha_n L_i^2+\beta_n L_i^2)+(\alpha_n\beta_n L_i^3-2\alpha_n\beta_n L_i^2+\alpha_n\beta_n L_i)]d(x_n,p)\\
&= \mu_n d(x_n,p), \quad \forall\ n\in\mathbb{N} \qquad (3.5)
\end{aligned}$$

where $\mu_n=(L_i-\alpha_n L_i-\beta_n L_i)+(\alpha_n L_i^2+\beta_n L_i^2)+(\alpha_n\beta_n L_i^3-2\alpha_n\beta_n L_i^2+\alpha_n\beta_n L_i)$. Since $\sum_{n=1}^{\infty}\alpha_n<\infty$ and $\sum_{n=1}^{\infty}\beta_n<\infty$, so that $\sum_{n=1}^{\infty}(\mu_n-1)<\infty$. Therefore, by Lemma 2.6, we have that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F(T)$.

Secondary step, we prove that $\lim_{n\to\infty} d(x_n,T_i x_n)=0$. Let $\lim_{n\to\infty} d(x_n,p)=c\ge 0$.

(i) If $c=0$, we obviously have

$$\begin{aligned}
d(x_n,T_i x_n) &\le d(x_n,p)+d(T_i x_n,p)\\
&\le (1+L_i)d(x_n,p),
\end{aligned}$$

taking lim as $n\to\infty$ on both the sides, we have $\lim_{n\to\infty} d(x_n,T_i x_n)=0$.

(ii) If $c>0$, since $\{T_i\}_{i=1}^k$ is a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings and $p\in F(T)$, we have

$$d(T_i x_n,p)\le L_i d(x_n,p),$$

taking lim sup as $n\to\infty$ both the sides, we have

$$\limsup_{n\to\infty} d(T_i x_n,p)\le L_i c,$$

taking lim sup as $n\to\infty$ both the sides in (3.3), we have

$$\limsup_{n\to\infty} d(z_n,p)\le L_i c. \qquad (3.6)$$

Since

$$d(x_{n+1},p)\le L_i(1-\alpha_n+\alpha_n L_i)d(z_n,p),$$

so, we take lim inf as $n\to\infty$ both the sides, we get

$$\begin{aligned}
\liminf_{n\to\infty} d(x_{n+1},p) &\le \liminf_{n\to\infty} d(z_n,p)\\
L_i c &\le \liminf_{n\to\infty} d(z_n,p). \qquad (3.7)
\end{aligned}$$

By (3.6) and (3.7), we have

$$\lim_{n\to\infty} d(z_n,p)=L_i c,$$

it implies that

$$
\begin{aligned}
L_i c &= \limsup_{n\to\infty} d(z_n,p) \\
&= \limsup_{n\to\infty} [d(W(x_n,T_i x_n,\beta_n),p)] \\
&= \limsup_{n\to\infty} [d((1-\beta_n)x_n \oplus \beta_n T_i x_n,p)] \\
&\le \limsup_{n\to\infty} [(1-\beta_n)d(x_n,p)+\beta_n d(T_i x_n,p)] \\
&\le \limsup_{n\to\infty} (1-\beta_n)d(x_n,p)+\limsup_{n\to\infty} \beta_n d(T_i x_n,p) = L_i c.
\end{aligned}
$$

From Lemma 2.5, we have $\lim_{n\to\infty} d(x_n,T_i x_n)=0$.

Finally, we will prove that the sequence $\{x_n\}$ Δ-converges to a fixed point of T_i. Since $\{d(x_n,p)\}$ is bounded, by Lemma 2.3, it follows that $\{x_n\}$ has a unique asyptotic center. Let u,v Δ-limits of the subsequence of $\{u_n\},\{v_n\}\subset\{x_n\}$. Since $F(T)\neq\emptyset$, we have u and v are fixed points of $\{T_i\}_{i=1}^k$. Now, we claim that $u=v$. Let $u\neq v$, then by uniqueness of asymptotic center

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n,u) &= \limsup_{n\to\infty} d(u_n,u) \\
&< \limsup_{n\to\infty} d(u_n,v) \\
&= \limsup_{n\to\infty} d(x_n,v) \\
&= \limsup_{n\to\infty} d(v_n,v) \\
&< \limsup_{n\to\infty} d(v_n,u) \\
&= \limsup_{n\to\infty} d(x_n,u),
\end{aligned}
$$

which is a contradiction. Therefore $u=v$, the sequence $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

Theorem 3.10. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a self-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T)\neq\emptyset$ and $F(T)$ is closed. Then the sequence $\{x_n\}$ defined in* (3.2) *converges strongly to $p\in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$, where $d(x_n,F(T))=\inf_{p\in F(T)} d(x_n,p)$.*

Proof. Necessity is obvious, we only prove the sufficiency. Assume that

$$\liminf_{n\to\infty} d(x_n,F(T))=0.$$

From (3.5)

$$d(x_{n+1},F(T))\le \mu_n d(x_n,F(T)),\quad n\in\mathbb{N}$$

then $\lim_{n\to\infty} d(x_n,F(T))$ exists. Hence by the hypothesis, $\liminf_{n\to\infty} d(x_n,F(T))=0$, then we have $\lim_{n\to\infty} d(x_n,F(T))=0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. By Theorem 2.8, we obtained the following inequality

$$d(x_{n+m},p) \le K d(x_n,p)$$

for each $p \in F(T)$ and for all $m,n \in \mathbb{N}$, where $K = e^{\left(\sum_{j=n}^{n+m-1} \mu_j\right)} > 0$. As, $\sum_{n=1}^{\infty} \mu_n < \infty$ thus $K^* = e^{\left(\sum_{n=1}^{\infty} \mu_n\right)} \ge K = e^{\left(\sum_{j=n}^{n+m-1} \mu_j\right)} > 0$. Let $\epsilon > 0$ be arbitrarily. Since $\lim_{n\to\infty} d(x_n,F(T)) = 0$, there exists a positive integer n_0 such that

$$d(x_n,F(T)) < \frac{\epsilon}{4K^*}, \quad \forall\, n \ge n_0.$$

In particular, $\inf\{d(x_{n_0},p) : p \in F(T)\} < \frac{\epsilon}{4K^*}$. So there exist $p^* \in F(T)$ such that

$$d(x_{n_0},p^*) < \frac{\epsilon}{2K^*}.$$

Thus, for $n \ge n_0$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\le d(x_{n+m},p^*) + d(p^*,x_n) \\ &\le 2K^* d(x_{n_0},p^*) \\ &< 2K^* \left(\frac{\epsilon}{2K^*}\right) = \epsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in M. Since M is a closed subset of a complete uniformly convex hyperbolic space, so it must converge strongly to a point p in M. Since $F(T)$ is closed, $\lim_{n\to\infty} d(x_n,F(T)) = 0$, that is, $p \in F(T)$. This completes the proof. □

4. Conclusion

In this paper, we introduce an algorithm by the iteration process of Kadioglu (PNS) to approximating a fixed point for L-Lipschitzian Suzuki-generalized nonexpansive mappings in hyperbolic spaces and introduce a L-Lipschitzian Suzuki-generalized nonexpansive mapping, i.e.,

$$\frac{1}{2} d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le L d(x,y).$$

We obtain fixed point theorems, Δ-convergence theorems, and convergence theorems for L-Lipschitzian Suzuki-generalized nonexpansive mappings in a hyperbolic space. Moreover, we obtain that examples, lemmas and theorems for L-Lipschitzian Suzuki-generalized nonexpansive mappings on a nonempty subset of a hyperbolic spaces in the following way:

(1) Let $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M. Then

$$d(x_n,T_i y) \le (1+2L_i) d(x_n,T_i x_n) + L_i d(x_n,y)$$

for all $x,y \in M$, $\{x_n\}$ is approximate fixed point sequence in M.

(2) Let M be a nonempty and convex subset of a strictly convex hyperbolic space X. If $\{T_i\}_{i=1}^k$ be a self finite family of $u_n L_i$-Lipschitzian Suzuki-generalized nonexpansive mappings

on M, that is there exist a sequence $\{u_n\}$ and $L_i > 0$ such that

$$\frac{1}{2}d(x,T_ix) \le d(x,y) \Rightarrow d(T_ix,T_iy) \le u_nL_id(x,y), \quad \forall\ n \ge 1, x,y \in M.$$

with $u_nL_i \to 1$, for all $i = 1,2,\dots,k$ and $F(T) \neq \emptyset$. If $\{x_n\}$, $\{y_n\}$ are bounded approximate fixed point sequence in M, then $F(T)$ is closed and convex.

(3) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a slef-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (3.2), Δ-converges to a common fixed point of $\{T_i\}_{i=1}^k$.

(4) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let a slef-map $\{T_i\}_{i=1}^k$ be a self finite family of L_i-Lipschitzian Suzuki-generalized nonexpansive mappings on M, such that $F(T) \neq \emptyset$ and $F(T)$ is closed. Then the sequence $\{x_n\}$ defined in (3.2) converges strongly to $p \in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n,F(T)) = 0$, where $d(x_n,F(T)) = \inf_{p\in F(T)} d(x_n,p)$.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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The generalized viscosity explicit rules for a family of strictly pseudo-contractive mappings in a q-uniformly smooth Banach space

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Abstract

In this paper, we construct an iterative method by a generalized viscosity explicit rule for a countable family of strictly pseudo-contractive mappings in a q-uniformly smooth Banach space. We prove strong convergence theorems of proposed algorithm under some mild assumption on control conditions. We apply our results to the common fixed point problem of convex combination of family of mappings and zeros of accretive operator in Banach spaces. Furthermore, we also give some numerical examples to support our main results.

Keywords: Strict pseudo-contractions; Banach space; Strong convergence; Fixed point problem; Iterative method

1 Introduction

In this paper, we assume that E is a real Banach space with dual space E^* and C is a nonempty subset of E. Let $q > 1$ be a real number. The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^q, \|\bar{x}\| = \|x\|^{q-1}\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between elements of E and E^*. In particular, $J_q = J_2$ is called the *normalized duality mapping*. If E is smooth, then J_q is single-valued and denoted by j_q (see [1]). If $E := H$ is a real Hilbert space, then $J = I$, where I is the identity mapping. Further, we have the following properties of the generalized duality mapping J_q:

- $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$.
- $J(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \geq 0$.
- $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let T be a self-mapping of C. We denote the fixed point set of the mapping T by $F(T) = \{x \in C : x = Tx\}$. A mapping $f : C \to C$ is said to be a *contraction* if there exists a constant $\rho \in (0, 1)$ satisfying

$$\|f(x) - f(y)\| \leq \rho\|x - y\|, \quad \forall x, y \in C.$$

We use Π_C to denote the collection of all contractions from C into itself. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $T : C \to C$ is said to be λ-*strict pseudo-contraction* if for all $x, y \in C$, there exist $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^q - \lambda\|(I - T)x - (I - T)y\|^q, \quad \forall x, y \in C. \tag{1}$$

It is not hard to show that (1) equivalent to the following inequality:

$$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq \lambda\|(I - T)x - (I - T)y\|^q, \quad \forall x, y \in C. \tag{2}$$

If $E := H$ is a Hilbert space, then (1) (and so (2)) is equivalent to the following inequality:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{3}$$

where $k = 1 - 2\lambda < 1$. We assume that $k \geq 0$, so that $k \in [0, 1)$. Note that the class of strictly pseudo-contractive mappings include the class of nonexpansive mappings as a particular case in Hilbert spaces. Clearly, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. Strict pseudo-contractions were first introduced by Browder and Petryshyn [2] in 1967. They have more powerful applications than nonexpansive mappings do in solving inverse problems (see, e.g., [3]). Therefore it is more interesting to study the theory of iterative methods for strictly pseudo-contractive mappings. Several researchers studied the class of strictly pseudo-contractive mappings in Hilbert and Banach spaces (see, e.g., [4–9] and the references therein).

Now, we give some examples of λ-strictly pseudo-contractive mappings.

Example 1.1 ([8]) Let $E = \mathbb{R}$ with the usual norm, and let $C = (0, \infty)$. Let $T : C \to C$ be defined by

$$Tx = \frac{x^2}{1 + x}, \quad x \in C.$$

Then, T is a 1-strict pseudo-contraction.

Example 1.2 ([8]) Let $E = \mathbb{R}$ with the usual norm, and let $C = [-1, 1]$. Let $T : C \to C$ be defined by

$$Tx = \begin{cases} x, & x \in [-1, 0], \\ x - x^2, & x \in [0, 1]. \end{cases}$$

Then, T is a λ-strict pseudo-contraction with constant $\lambda > 0$.

Over the last several years, the implicit midpoint rule (IMR) has become a powerful numerical method for numerically solving time-dependent differential equations (in particular, stiff equations) (see [10–15]) and differential algebraic equations (see [16]). Consider the following initial value problem:

$$x'(t) = f\big(x(t)\big), \quad x(t_0) = x_0, \tag{4}$$

where $f : \mathbb{R}^M \to \mathbb{R}^M$ is a continuous function. The IMR is an implicit method given by the following finite difference scheme [17]:

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + hf(\frac{y_n+y_{n+1}}{2}), \quad n \geq 0, \end{cases} \tag{5}$$

where $h > 0$ is a time step. It is known that if $f : \mathbb{R}^M \to \mathbb{R}^M$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{y_n\}$ converges to the exact solution of (4) as $h \to 0$ uniformly over $t \in [t_0, t^*]$ for any fixed $t^* > 0$. If the function f is written as $f(x) = x - g(x)$, then (5) becomes

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + h[\frac{y_n+y_{n+1}}{2} - g(\frac{y_n+y_{n+1}}{2})], \quad n \geq 0, \end{cases} \tag{6}$$

and the critical points of (4) are the fixed points of the problem $x = g(x)$.

Based on IMR (5), Alghamdi et al. [18] introduced the following two algorithms for the solution of the fixed point problem $x = Tx$, where T is a nonexpansive mapping in a Hilbert space H:

$$x_{n+1} = x_n - t_n\left[\frac{x_n + x_{n+1}}{2} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right], \quad n \geq 0, \tag{7}$$

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{8}$$

for $x_0 \in H$, with $\{t_n\}_{n=1}^{\infty} \subset (0,1)$. They proved that these two schemes converge weakly to a point in $F(T)$.

To obtain strong convergence, Xu et al. [19] applied the viscosity approximation method introduced by Moudafi [20] to the IMR for a nonexpansive mapping T and proposed the following *viscosity implicit midpoint rule* in Hilbert spaces H as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 1, \tag{9}$$

where $\{\alpha_n\}$ is a real control condition in $(0,1)$. They also proved that the sequence $\{x_n\}$ generated by (9) converges strongly to a point $x^* \in F(T)$, which solves the variational inequality

$$\big\langle (f - I)x^*, z - x^*\big\rangle \leq 0, \quad z \in F(T). \tag{10}$$

Later, Ke and Ma [21] improved the viscosity implicit midpoint rule by replacing the midpoint by any point of the interval $[x_n, x_{n+1}]$. They introduced the so-called *generalized viscosity implicit rules* to approximating the fixed point of a nonexpansive mapping T in

Hilbert spaces H as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\big(s_n x_n + (1-s_n)x_{n+1}\big), \quad n \geq 1. \tag{11}$$

They also proved that the sequence $\{x_n\}$ generated by (11) converges strongly to a point $x^* \in F(T)$ that solves the variational inequality (10).

In numerical analysis, it is clear that the computation by the IMR is not an easy work in practice. Because the IMR need to compute at every time steps, it can be much harder to implement. To overcome this difficulty, for solving (4), we consider the helpful method, the so-called *explicit midpoint method* (EMR), given by the following finite difference scheme [22, 23]:

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + hf(y_n), \\ y_{n+1} = y_n + hf(\frac{y_n+\bar{y}_{n+1}}{2}), \quad n \geq 0. \end{cases} \tag{12}$$

Note that the EMR (12) calculates the system status at a future time from the currently known system status, whereas IMR (5) calculates the system status involving both the current state of the system and the later one (see [23, 24]).

In 2017, Marino et al. [25] combined the generalized viscosity implicit midpoint rules (11) with the EMR (12) for a quasi-nonexpansive mapping T and introduced the following so-called *generalized viscosity explicit midpoint rule* in Hilbert spaces H as follows:

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 1. \end{cases} \tag{13}$$

They also showed that, under certain assumptions imposed on the parameters, the sequence $\{x_n\}$ generated by (13) converges strongly to a point $x^* \in F(T)$, which solves the variational inequality (10).

The above results naturally bring us to the following questions.

Question 1 Can we extend the generalized viscosity explicit midpoint rule (13) to higher spaces other than Hilbert spaces? Such as a 2-uniformly smooth Banach space or, more generally, in a q-uniformly smooth Banach space.

Question 2 Can we obtain a strong convergence result of generalized viscosity explicit midpoint rule (13) for finding the set of common fixed points of a family of mappings? Such as a countable family of strict pseudo-contractions.

The purpose of this paper is to give some affirmative answers to the questions raised. We introduce an iterative algorithm for finding the set of common fixed points of a countable family of strict pseudo-contractions by a generalized viscosity explicit rule in a q-uniformly smooth Banach space. We prove the strong convergence of the proposed algorithm under some mild assumption on control conditions. We apply our results to the common fixed point problem of a convex combination of a family of mappings and zeros of an accretive operator in Banach spaces. Furthermore, we also give some numerical examples to support our main results.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and dual space E^* of E. The symbol $\langle x, x^*\rangle$ denotes the pairing between E and E^*, that is, $\langle x, x^*\rangle = x^*(x)$, the value of x^* at x. The *modulus of convexity* of E is the function $\delta : (0,2] \to [0,1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}.$$

A Banach space E is said to be *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. For $p > 1$, we say that E is said to be *p-uniformly convex* if there is $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for all $\epsilon \in (0,2]$.

The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\|, \|y\| \leq 1\right\}.$$

A Banach space E is said to be *uniformly smooth* if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. For $q > 1$, a Banach space E is said to be *q-uniformly smooth* if there exists $c_q > 0$ such that $\rho_E(\tau) \leq c_q \tau^q$ for all $\tau > 0$. If E is q-uniformly smooth, then $q \leq 2$, and E is also uniformly smooth. Further, E is p-uniformly convex (q-uniformly smooth) if and only if E^* is q-uniformly smooth (p-uniformly convex), where $p \geq 2$ and $1 < q \leq 2$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. It is well known that Hilbert spaces L_p and l_p ($p > 1$) are uniformly smooth (see [26]). More precisely, the spaces L_p and l_p are $\min\{p,2\}$-uniformly smooth for every $p > 1$.

Definition 2.1 Let C a be nonempty closed convex subsets of E, and let Q be a mapping of E onto C. Then Q is said to be:

- *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$.
- *retraction* if $Qx = x$ for all $x \in C$.
- a sunny nonexpansive retraction if Q is sunny, nonexpansive, and a retraction from E onto C.

It is known that if $E := H$ is a real Hilbert space, then a sunny nonexpansive retraction Q coincides with the metric projection from E onto C. Moreover, if E is uniformly smooth and T is a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$, then $F(T)$ is a sunny nonexpansive retraction from E onto C (see [27]). We know that in a uniformly smooth Banach space, a retraction $Q : C \to E$ is sunny and nonexpansive if and only if $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$ for all $x \in E$ and $y \in C$ (see [28]).

Lemma 2.2 ([29]) *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $S : C \to C$ be a nonexpansive self-mapping such that $F(S) \neq \emptyset$ and $f \in \Pi_C$. Let $\{z_t\}$ be the net sequence defined by*

$$z_t = tf(z_t) + (1-t)Sz_t, \quad t \in (0,1).$$

Then:

(i) $\{x_t\}$ *converges strongly as $t \to 0$ to a point $Q(f) \in F(S)$, which solves the variational inequality*

$$\big\langle (I-f)Q(f), j_q\big(Q(f) - z\big)\big\rangle \leq 0, \quad z \in F(S).$$

(ii) *Suppose that* $\{x_n\}$ *is a bounded sequence such that* $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. *If* $Q(f) := \lim_{t\to 0} x_t$ *exists, then*

$$\limsup_{n\to\infty}\langle (f-I)Q(f), j_q(x_n - Q(f))\rangle \leq 0.$$

Lemma 2.3 ([30]) *Let* C *be a nonempty closed convex subset of a real* q*-uniformly smooth Banach space* E. *Let* $T : C \to C$ *be a* λ*-strict pseudo-contraction. For all* $x \in C$, *we define* $T_\theta x := (1-\theta)x + \theta Tx$. *Then, as* $\theta \in (0,\delta]$, $\delta = \min\{1, (\frac{q\lambda}{\kappa_q})^{\frac{1}{q-1}}\}$, *where* κ_q *is the* q*-uniform smoothness constant, and* $T_\theta : C \to C$ *is nonexpansive such that* $F(T_\theta) = F(T)$.

Using the concept of subdifferentials, we have the following inequality.

Lemma 2.4 ([31]) *Let* $q > 1$, *and let* E *be a real normed space with the generalized duality mapping* J_q. *Then, for any* $x, y \in E$, *we have*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle, \tag{14}$$

where $j_q(x+y) \in J_q(x+y)$.

Lemma 2.5 ([32]) *Let* $p > 1$ *and* $r > 0$ *be two fixed real numbers, and let* E *be a uniformly convex Banach space. Then, for all* $x, y \in B_r$ *and* $t \in [0,1]$,

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - t(1-t)c\|x-y\|^p,$$

where $c > 0$.

Lemma 2.6 ([33]) *Suppose that* $q > 1$. *Then*

$$ab \leq \frac{1}{q}a^q + \left(\frac{q-1}{q}\right)b^{\frac{q}{q-1}}$$

for positive real numbers a, b.

Lemma 2.7 ([34]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers,* $\{\gamma_n\}$ *be a sequence of* $(0,1)$ *with* $\sum_{n=1}^{\infty}\gamma_n = \infty$, $\{c_n\}$ *be a sequence of nonnegative real number with* $\sum_{n=1}^{\infty} c_n < \infty$, *and let* $\{b_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty} b_n \leq 0$. *Suppose that*

$$a_{n+1} = (1-\gamma_n)a_n + \gamma_n b_n + c_n$$

for all $n \in \mathbb{N}$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 ([35]) *Let* $\{s_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $s_{n_i} < s_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists an increasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $\lim_{k\to\infty} m_k = \infty$ *and the following properties are satisfied by all sufficiently large numbers* $k \in \mathbb{N}$:

$$s_{m_k} \leq s_{m_k+1} \quad \textit{and} \quad s_k \leq s_{m_k+1}.$$

In fact, $m_k := \max\{j \leq k : s_j \leq s_{j+1}\}$.

Definition 2.9 ([34]) Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of C into itself. We say that $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition if

$$\sum_{n=1}^{\infty} \sup_{w\in C} \|T_{n+1}w - T_n w\| < \infty. \tag{15}$$

Lemma 2.10 ([34]) *Let C be a nonempty closed convex subset of a real Banach space E. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, let T be the mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Then, $\lim_{n\to\infty} \sup_{w\in C} \|Tw - T_n w\| = 0$.*

In the following, we will write that $(\{T_n\}, T)$ satisfies the $AKTT$-condition if $\{T_n\}$ satisfies the $AKTT$-condition and T is defined by Lemma 2.10 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E. Let $f \in \Pi_C$ with coefficient $\rho \in (0,1)$, and let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudo-contractions such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For all $x \in C$, define the mapping $S_n x = (1-\theta_n)x + \theta_n T_n x$, where $0 < \theta_n \leq \delta$, $\delta = \min\{1, (\frac{q\lambda}{\kappa_q})^{\frac{1}{q-1}}\}$, and $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) S_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n(t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad n \geq 1, \end{cases} \tag{16}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n)(1-t_n) > 0$.

Suppose in addition that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the $AKTT$-condition. Then, $\{x_n\}$ defined by (16) converges strongly to $x^* = Q(f) \in \Omega$, which solves the variational inequality

$$\langle (I-f)Q(f), j_q(Q(f) - z)\rangle \leq 0, \quad z \in \Omega, \tag{17}$$

where Q is a sunny nonexpansive retraction of C onto Ω.

Proof First, we show that $\{x_n\}$ is bounded. From Lemma 2.3 we have that S_n is nonexpansive such that $F(S_n) = F(T_n)$ for all $n \geq 1$. Put $z_n := t_n x_n + (1-t_n)\bar{x}_{n+1}$. For each $z \in \Omega := \bigcap_{n=1}^{\infty} F(T_n)$, we have

$$\begin{aligned} \|z_n - z\| &= \|t_n(x_n - z) + (1-t_n)(\bar{x}_{n+1} - z)\| \\ &\leq t_n\|x_n - z\| + (1-t_n)\|\bar{x}_{n+1} - z\| \\ &\leq t_n\|x_n - z\| + (1-t_n)\big(\beta_n\|x_n - z\| + (1-\beta_n)\|S_n x_n - z\|\big) \\ &\leq t_n\|x_n - z\| + (1-t_n)\beta_n\|x_n - z\| + (1-t_n)(1-\beta_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{18}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\alpha_n f(x_n)+(1-\alpha_n)S_nz_n-z\| \\
&= \|\alpha_n(f(x_n)-f(z))+\alpha_n(f(z)-z)+(1-\alpha_n)(S_nz_n-z)\| \\
&\le \alpha_n\|f(x_n)-f(z)\|+\alpha_n\|f(z)-z\|+(1-\alpha_n)\|S_nz_n-z\| \\
&\le (1-(1-\rho)\alpha_n)\|x_n-z\|+(1-\rho)\alpha_n\frac{\|f(z)-z\|}{1-\rho} \\
&\le \max\left\{\|x_n-z\|,\frac{\|f(z)-z\|}{1-\rho}\right\}.
\end{aligned}$$

By induction we have

$$\|x_n-z\|\le\max\left\{\|x_1-z\|,\frac{\|f(z)-z\|}{1-\rho}\right\},\quad n\ge 1.$$

Hence $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{f(x_n)\}$ and $\{S_n(t_nx_n+(1-t_n)\bar{x}_{n+1})\}$ are bonded. Let $x^*=Q(f)$. By the convexity of $\|\cdot\|^q$ and Lemma 2.5 we have

$$\begin{aligned}
\|S_nz_n-x^*\|^q &\le \|z_n-x^*\|^q \\
&= \|t_n(x_n-x^*)+(1-t_n)(\bar{x}_{n+1}-x^*)\|^q \\
&\le t_n\|x_n-x^*\|^q+(1-t_n)\|\bar{x}_{n+1}-x^*\|^q \\
&= t_n\|x_n-x^*\|^q+(1-t_n)\|\beta_n(x_n-x^*)+(1-\beta_n)(S_nx_n-x^*)\|^q \\
&\le t_n\|x_n-x^*\|^q+(1-t_n)\big[\beta_n\|x_n-x^*\|^q+(1-\beta_n)\|S_nx_n-x^*\|^q \\
&\quad -\beta_n(1-\beta_n)c\|x_n-S_nx_n\|^q\big] \\
&\le \|x_n-x^*\|^q-\beta_n(1-\beta_n)(1-t_n)c\|x_n-S_nx_n\|^q.
\end{aligned} \tag{19}$$

It follows from Lemma 2.4 and (19) that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^q \\
&\quad= \|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(S_nz_n-x^*)\|^q \\
&\quad= \|\alpha_n(f(x_n)-f(x^*))+\alpha_n(f(x^*)-x^*)+(1-\alpha_n)(S_nz_n-x^*)\|^q \\
&\quad\le \|\alpha_n(f(x_n)-f(x^*))+(1-\alpha_n)(S_nz_n-x^*)\|^q+q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle \\
&\quad\le \alpha_n\|f(x_n)-f(x^*)\|^q+(1-\alpha_n)\|S_nz_n-x^*\|^q+q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle \\
&\quad\le \alpha_n\|f(x_n)-f(x^*)\|^q+(1-\alpha_n)\big[\|x_n-x^*\|^q-\beta_n(1-\beta_n)(1-t_n)c\|x_n-S_nx_n\|^q\big] \\
&\quad\quad+q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle \\
&\quad\le (1-(1-\rho)\alpha_n)\|x_n-x^*\|^q-(1-\alpha_n)\beta_n(1-\beta_n)(1-t_n)c\|x_n-S_nx_n\|^q \\
&\quad\quad+q\alpha_n\langle f(x^*)-x^*,j_q(x_{n+1}-x^*)\rangle.
\end{aligned} \tag{20}$$

The rest of the proof will be divided into two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=n_0}^{\infty}$ is nonincreasing. This implies that $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is convergent. From (20) we see that

$$(1-\alpha_n)\beta_n(1-\beta_n)(1-s_n)c\|x_n - S_n x_n\|^q \le \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + \alpha_n M,$$

where $c > 0$ and $M = \sup_{n\ge 1}\{q\|f(x^*) - x^*\|\|x_{n+1} - x^*\|^{q-1}, (1-\rho)\|x_n - x^*\|^q\} < \infty$. From $(C1)$ and $(C2)$ we get that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{21}$$

We observe that

$$\begin{aligned}
&\sup_{x\in\{x_n\}} \|S_{n+1}x - S_n x\| \\
&\quad = \sup_{x\in\{x_n\}} \big\|(1-\theta_{n+1})x + \theta_{n+1}T_{n+1}x - (1-\theta_n)x - \theta_n T_n x\big\| \\
&\quad \le |\theta_{n+1} - \theta_n| \sup_{x\in\{x_n\}} \|x\| + \theta_{n+1} \sup_{x\in\{x_n\}} \|T_{n+1}x - T_n x\| + |\theta_{n+1} - \theta_n| \sup_{x\in\{x_n\}} \|T_n x\| \\
&\quad \le |\theta_{n+1} - \theta_n| \Big(\sup_{x\in\{x_n\}} \|x\| + \sup_{x\in\{x_n\}} \|T_n x\| \Big) + \sup_{x\in\{x_n\}} \|T_{n+1}x - T_n x\|.
\end{aligned}$$

Since $\{T_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition and $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$, we have

$$\sum_{n=1}^{\infty} \sup_{x\in\{x_n\}} \|S_{n+1}x - S_n x\| < \infty,$$

that is, $\{S_n\}_{n=1}^{\infty}$ satisfies the $AKTT$-condition. From this we can define the nonexpansive mapping $S : C \to C$ by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$. Since $\{\theta_n\}$ is bounded, there exists a subsequence $\{\theta_{n_i}\}$ of $\{\theta_n\}$ such that $\theta_{n_i} \to \theta$ as $i \to \infty$. It follows that

$$Sx = \lim_{i\to\infty} S_{n_i}x = \lim_{i\to\infty} \big[(1-\theta_{n_i})x + \theta_{n_i}T_{n_i}x\big] = (1-\theta)x + \theta Tx, \quad x \in C.$$

This shows that $F(S) = F(T) = \bigcap_{n=1}^{\infty} F(T_n) := \Omega$. By (21) and Lemma 2.10 we have

$$\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\
&\le \|x_n - S_n x_n\| + \sup_{x\in\{x_n\}} \|S_n x - Sx\| \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{22}$$

Let $\{z_t\}$ be a sequence defined by

$$z_t = f(z_t) + (1-t)Sz_t, \quad t \in (0,1).$$

From Lemma 2.2(i) we know that $\{x_t\}$ converges strongly to $x^* = Q(f)$, which solves the variational inequalities

$$\big\langle (I-f)Q(f), j_q\big(Q(f) - z\big)\big\rangle \le 0, \quad z \in \Omega.$$

Moreover, we obtain that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, j_q(x_n-x^*)\rangle \le 0. \tag{23}$$

Note that

$$\begin{aligned}\|S_n z_n - x_n\| &\le \|S_n z_n - S_n x_n\| + \|S_n x_n - x_n\|\\ &\le \|z_n - x_n\| + \|S_n x_n - x_n\|\\ &= (1-s_n)(1-\beta_n)\|S_n x_n - x_n\| + \|S_n x_n - x_n\|\\ &\le 2\|x_n - S_n x_n\|.\end{aligned}$$

From (21), we get that

$$\lim_{n\to\infty}\|S_n z_n - x_n\| = 0. \tag{24}$$

It follows that

$$\begin{aligned}&\|x_{n+1}-x_n\|\\ &\quad\le \big\|\alpha_n\big(f(x_n)-x_n\big)+(1-\alpha_n)(S_n z_n - x_n)\big\|\\ &\quad\le \alpha_n\big\|f(x_n)-x_n\big\| + (1-\alpha_n)\|S_n z_n - x_n\| \to 0 \quad \text{as } n\to\infty. \end{aligned}\tag{25}$$

We also have

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, j_q(x_{n+1}-x^*)\rangle \le 0. \tag{26}$$

Again from (20), we have

$$\big\|x_{n+1}-x^*\big\|^q \tag{27}$$
$$\quad\le \big(1-(1-\rho)\alpha_n\big)\big\|x_n - x^*\big\|^q + q\alpha_n\langle f(x^*)-x^*, j_q(x_{n+1}-x^*)\rangle. \tag{28}$$

Apply Lemma 2.7 and (26) to (27), we obtain that $x_n \to x^*$ as $n\to\infty$.

Case 2. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\big\|x_{n_i}-x^*\big\| \le \big\|x_{n_{i+1}}-x^*\big\|$$

for all $i\in\mathbb{N}$. By Lemma 2.8, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$ as $k\to\infty$ and

$$\big\|x_{m_k}-x^*\big\| \le \big\|x_{m_k+1}-x^*\big\| \quad\text{and}\quad \big\|x_k-x^*\big\| \le \big\|x_{m_k+1}-x^*\big\| \tag{29}$$

for all $k\in\mathbb{N}$. From (20) we have

$$\begin{aligned}&(1-\alpha_{m_k})\beta_{m_k}(1-\beta_{m_k})(1-s_{m_k})c\|x_{m_k}-S_{m_k}x_{m_k}\|^q\\ &\quad\le \big\|x_{m_k}-x^*\big\|^q - \big\|x_{m_k+1}-x^*\big\|^q + \alpha_{m_k}M\\ &\quad\le \alpha_{m_k}M,\end{aligned}$$

where $c > 0$ and $M < \infty$. This implies by $(C1)$ and $(C2)$ that

$$\|x_{m_k} - S_{m_k}x_{m_k}\| \to 0 \quad \text{as } k \to \infty. \tag{30}$$

Since

$$\begin{aligned}
&\sup_{x\in\{x_{m_k}\}} \|S_{m_k+1}x - S_{m_k}x\| \\
&\quad= \sup_{x\in\{x_{m_k}\}} \big\|(1-\theta_{m_k+1})x + \theta_{m_k+1}T_{m_k+1}x - (1-\theta_{m_k})x - \theta_{m_k}T_{m_k}x\big\| \\
&\quad\le |\theta_{m_k+1} - \theta_{m_k}| \sup_{x\in\{x_{m_k}\}} \|x\| + \theta_{m_k+1} \sup_{x\in\{x_{m_k}\}} \|T_{m_k+1}x - T_{m_k}x\| \\
&\qquad + |\theta_{m_k+1} - \theta_{m_k}| \sup_{x\in\{x_{m_k}\}} \|T_{m_k}x\| \\
&\quad\le |\theta_{m_k+1} - \theta_{m_k}| \Big(\sup_{x\in\{x_{m_k}\}} \|x\| + \sup_{x\in\{x_{m_k}\}} \|T_{m_k}x\|\Big) + \sup_{x\in\{x_{m_k}\}} \|T_{m_k+1}x - T_{m_k}x\| < \infty,
\end{aligned}$$

that is, $\{S_{m_k}\}_{k=1}^{\infty}$ satisfies the $AKTT$-condition. Then, by (30) and Lemma 2.10, we get that

$$\begin{aligned}
&\|x_{m_k} - Sx_{m_k}\| \\
&\quad\le \|x_{m_k} - S_{m_k}x_{m_k}\| + \|S_{m_k}x_{m_k} - Sx_{m_k}\| \\
&\quad\le \|x_{m_k} - S_{m_k}x_{m_k}\| + \sup_{x\in\{x_{m_k}\}} \|S_{m_k}x - Sx\| \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{31}$$

By the same argument as in Case 1, we can show that

$$\limsup_{k\to\infty} \big\langle f(x^*) - x^*, j(x_{m_k} - x^*)\big\rangle \le 0. \tag{32}$$

It follows from (31) that

$$\begin{aligned}
\|S_{m_k}z_{m_k} - x_{m_k}\| &\le \|S_{m_k}z_{m_k} - S_{m_k}x_{m_k}\| + \|S_{m_k}x_{m_k} - x_{m_k}\| \\
&\le \|z_{m_k} - x_{m_k}\| + \|S_{m_k}x_{m_k} - x_{m_k}\| \\
&= (1-s_{m_k})(1-\beta_{m_k})\|S_{m_k}x_{m_k} - x_{m_k}\| + \|S_{m_k}x_{m_k} - x_{m_k}\| \\
&\le 2\|x_{m_k} - S_{m_k}x_{m_k}\| \to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{m_k+1} - x_{m_k}\| &\le \big\|\alpha_{m_k}\big(f(x_{m_k}) - x_{m_k}\big) + (1-\alpha_{m_k})(S_{m_k}z_{m_k} - x_{m_k})\big\| \\
&\le \alpha_{m_k}\big\|f(x_{m_k}) - x_{m_k}\big\| + (1-\alpha_{m_k})\|S_{m_k}z_{m_k} - x_{m_k}\| \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Then, we also have

$$\limsup_{k\to\infty} \big\langle f(x^*) - x^*, j_q(x_{m_k+1} - x^*)\big\rangle \le 0. \tag{33}$$

Again from (27) we have

$$\begin{aligned}&\|x_{m_k+1}-x^*\|^q\\&\quad\le\bigl(1-(1-\rho)\alpha_{m_k}\bigr)\|x_{m_k}-x^*\|^q+q\alpha_{m_k}\bigl\langle f(x^*)-x^*,j_q\bigl(x_{m_k+1}-x^*\bigr)\bigr\rangle,\end{aligned}\tag{34}$$

which implies that

$$\begin{aligned}(1-\rho)\alpha_{m_k}\|x_{m_k}-x^*\|^q&\le\|x_{m_k}-x^*\|^q-\|x_{m_k+1}-x^*\|^q\\&\quad+q\alpha_{m_k}\bigl\langle f(x^*)-x^*,j_q\bigl(x_{m_k+1}-x^*\bigr)\bigr\rangle\\&\le q\alpha_{m_k}\bigl\langle f(x^*)-x^*,j_q\bigl(x_{m_k+1}-x^*\bigr)\bigr\rangle.\end{aligned}\tag{35}$$

Since $\alpha_{m_k}>0$, we get $\lim_{k\to\infty}\|x_{m_k}-x^*\|=0$. So, we have

$$\begin{aligned}\|x_k-x^*\|&\le\|x_{m_k+1}-x^*\|\\&=\|x_{m_k}-x^*\|+\|x_{m_k+1}-x^*\|-\|x_{m_k}-x^*\|\\&\le\|x_{m_k}-x^*\|+\|x_{m_k+1}-x_{m_k}\|\to0\quad\text{as }k\to\infty,\end{aligned}$$

which implies that $x_k\to x^*$ as $k\to\infty$. This completes the proof. □

Applying Theorem 3.1 to a 2-uniformly smooth Banach space, we obtain the following result.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real uniformly convex and 2-uniformly smooth Banach space E. Let $f\in\Pi_C$ with coefficient $\rho\in(0,1)$, and let $\{T_n\}_{n=1}^{\infty}:C\to C$ be a family of λ-strict pseudo-contractions such that $\Omega:=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. For all $x\in C$, define the mapping $S_nx=(1-\theta)x+\theta T_nx$, where $0<\theta\le\delta$, $\delta=\min\{1,\frac{\lambda}{K^2}\}$, and $\sum_{n=1}^{\infty}|\theta_{n+1}-\theta_n|<\infty$. For given $x_1\in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}\bar{x}_{n+1}=\beta_nx_n+(1-\beta_n)S_nx_n,\\x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)S_n(t_nx_n+(1-t_n)\bar{x}_{n+1}),\quad n\ge1,\end{cases}\tag{36}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$ are sequences in $(0,1)$ satisfying the conditions (C1) and (C2) of Theorem 3.1. Suppose in addition that $(\{T_n\}_{n=1}^{\infty},T)$ satisfies the AKTT-condition. Then $\{x_n\}$ converges strongly to $x^*=Q(f)\in\Omega$, which solves the variational inequality

$$\bigl\langle(I-f)Q(f),j\bigl(Q(f)-z\bigr)\bigr\rangle\le0,\quad\forall z\in\Omega,\tag{37}$$

where Q is a sunny nonexpansive retraction of C onto Ω.

Utilizing the fact that a Hilbert space H is uniformly convex and 2-uniformly smooth with the best smooth constant $\kappa_2=1$, we obtain the following result.

Corollary 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $f\in\Pi_C$ with coefficient $\rho\in(0,1)$, and let $\{T_n\}_{n=1}^{\infty}:C\to C$ be a family of λ-strict pseudo-contractions with $\lambda\in[0,1)$ such that $\Omega:=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. For all $x\in C$, define the mapping*

$S_n x = (1-\theta_n)x + \theta_n T_n x$, where $0 < \theta_n \leq \delta$, $\delta = \min\{1, 2\lambda\}$, and $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) S_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n(t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad n \geq 1, \end{cases} \tag{38}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$ are sequences in $(0,1)$ satisfying conditions (C1) and (C2) of Theorem 3.1. Suppose, in addition, that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition. Then $\{x_n\}$ converges strongly to $x^* = P(f) \in \Omega$, which solves the variational inequality

$$\langle (I-f)P(f), P(f) - z \rangle \leq 0, \quad z \in \Omega, \tag{39}$$

where P is a metric projection of C onto Ω.

4 Application

4.1 The generalized viscosity explicit rules for convex combination of family of mappings

In this subsection, we apply our main result to convex combination of a countable family of strict pseudo-contractions. The following lemmas can be found in [36, 37].

Lemma 4.1 ([36, 37]) *Let C be a closed convex subset of a smooth Banach space E. Suppose that $\{T_n\}_{n=1}^{\infty} : C \to C$ is a family of λ-strictly pseudo-contractive mappings with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\mu_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \mu_n = 1$. Then the following conclusions hold*:

(i) *A mapping $G : C \to E$ defined by $G := \sum_{n=1}^{\infty} \mu_n T_n$ is a λ-strictly pseudocontractive mapping.*

(ii) *$F(G) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 4.2 ([37]) *Let C be a closed convex subset of a smooth Banach space E. Suppose that $\{T_k\}_{k=1}^{\infty} : C \to C$ is a countable family of λ-strictly pseudocontractive mappings with $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$. For all $n \in \mathbb{N}$, define $S_n : C \to C$ by $S_n x := \sum_{k=1}^{n} \mu_n^k T_k x$ for all $x \in C$, where $\{\mu_n^k\}$ is a family of nonnegative numbers satisfying the following conditions*:

(i) *$\sum_{k=1}^{n} \mu_n^k = 1$ for all $n \in \mathbb{N}$;*

(ii) *$\mu^k := \lim_{n\to\infty} \mu_n^k > 0$ for all $k \in \mathbb{N}$;*

(iii) *$\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\mu_{n+1}^k - \mu_n^k| < \infty$.*

Then:

(1) *Each T_n is a λ-strictly pseudocontractive mapping.*

(2) *$\{T_n\}$ satisfies the AKTT-condition.*

(3) *If $T : C \to C$ is defined by $Tx = \sum_{k=1}^{\infty} \mu^k S_k x$ for all $x \in C$,*

then, $Tx = \lim_{n\to\infty} T_n x$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

Using Theorem 3.1 and Lemmas 4.1 and 4.2, we obtain the following result.

Theorem 4.3 *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E. Let $f \in \Pi_C$ with coefficient $\rho \in (0,1)$, and let $\{T_k\}_{k=1}^{\infty} : C \to C$ be a countable family of λ_k-strict pseudo-contractions with $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$.*

For all $x \in C$, define a mapping $S_n x := (1-\theta_n)x + \theta_n \sum_{k=1}^{n} \mu_n^k T_k x$ such that $\Omega := \bigcap_{k=1}^{\infty} F(T_k) \neq \emptyset$, where $0 < \theta_n \leq \delta$, $\delta = \min\{1, (\frac{q\lambda}{\kappa_q})^{\frac{1}{q-1}}\}$, and $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) S_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S_n (t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad n \geq 1, \end{cases} \tag{40}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$ are sequences in $(0,1)$ satisfy conditions $(C1)$ and $(C2)$ of Theorem 3.1, and $\{\mu_n^k\}$ is a real sequence satisfying (i)–(iii) *of Lemma 4.2. Then $\{x_n\}$ converges strongly to a $x^* \in \Omega$.*

4.2 The generalized viscosity explicit rules for zeros of accretive operators

In this subsection, we apply our main result to problem of finding a zero of an accretive operator. An operator $A \subset E \times E$ is said to be accretive if for all (x_1, y_1) and $(x_2, y_2) \in A$, there exists $j_q \in J_q(x_1 - x_2)$ such that $\langle y_1 - y_2, j_q \rangle \geq 0$. An operator A is said to satisfy the range condition if $\overline{D(A)} = R(I + \lambda A)$ for all $\lambda > 0$, where $D(A)$ is the domain of A, $R(I + \lambda A)$ is the range of $I + \lambda A$, and $\overline{D(A)}$ is the closure of $D(A)$. If A is an accretive operator that satisfies the range condition, then we can defined a single-valued mapping $J_\lambda^A : R(I + \lambda A) \to D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$, which is called the *resolvent* of A. We denote $A^{-1}0$ by the set of zeros of A, that is, $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. It is well known that J_λ is nonexpansive and $F(J_\lambda) = A^{-1}0$ (see [38]). We also know the following [39]: For all $\lambda, \mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, we have

$$\|J_\lambda x - J_\mu x\| \leq \frac{|\lambda - \mu|}{\lambda} \|x - J_\lambda x\|.$$

Lemma 4.4 ([34]) *Let C be a nonempty closed convex subset of a Banach space E. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$, which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$. Suppose that $\{\lambda_n\} \subset (0, \infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$. Then, $\{J_{\lambda_n}\}$ satisfies the AKTT-condition. Consequently, for each $x \in C$, $\{J_{\lambda_n} x\}$ converges strongly to some point of C. Moreover, let $J_\lambda : C \to C$ be defined by $J_\lambda x = \lim_{n\to\infty} J_{\lambda_n} x$ for all $x \in C$ and $F(J_\lambda) = \bigcap_{n=1}^{\infty} F(J_{\lambda_n})$, where $\lambda_n \to \lambda$ as $n \to \infty$. Then,* $\lim_{n\to\infty} \sup_{x\in C} \|J_\lambda x - J_{\lambda_n} x\| = 0$.

Utilizing Theorem 3.1 and and Lemma 4.4, we obtain the following result.

Theorem 4.5 *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E. Let $f \in \Pi_C$ with coefficient $\rho \in (0,1)$ and let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$. Suppose that $\{\lambda_n\} \subset (0, \infty)$ is such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. For given $x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) J_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{\lambda_n} (t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad n \geq 1, \end{cases} \tag{41}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$ are sequences in $(0,1)$ satisfying conditions $(C1)$ and $(C2)$ of Theorem 3.1. Then $\{x_n\}$ converges strongly to $x^* \in A^{-1}0$.

4.3 The generalized viscosity explicit rules with weak contraction

In this subsection, we apply our main result to the viscosity approximation method with weak contraction.

Definition 4.6 ([40–42]) Let C be a closed and convex subset of a real Banach space E. A mapping $g : C \to C$ is said to be *weakly contractive* if there exists a continuous strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$ such that

$$\bigl\|g(x) - g(y)\bigr\| \le \|x - y\| - \psi\bigl(\|x - y\|\bigr), \quad x, y \in C.$$

As a particular case, if $\psi(t) = (1 - \rho)t$ for all $t \ge 0$, where $\rho \in (0,1)$, then the weakly contractive mapping is contraction with coefficient ρ.

In 2001, Rhoades [42] first proved Banach's contraction principle for the weakly contractive mapping in complete metric space.

Lemma 4.7 ([42]) *Let (E,d) be a complete metric space, and let g be a weakly contractive mapping on E. Then g has a unique fixed point in E.*

Lemma 4.8 ([43]) *Assume that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real number, and $\{\lambda_n\}$ is a sequence of a positive real number satisfying the conditions $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{\lambda_n} = 0$. Suppose that*

$$a_{n+1} \le a_n - \lambda_n \psi(a_n) + b_n, \quad n \ge 1,$$

where $\psi(t)$ is a continuous strictly increasing function on $\mathbb{R}$ with $\psi(0) = 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Utilizing Theorem 3.1, we obtain the following result.

Theorem 4.9 *Let C be a nonempty closed convex subset of a real uniformly convex and q-uniformly smooth Banach space E. Let $g : C \to C$ be a weak contraction, and let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudo-contractions such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For all $x \in C$, define the mapping $S_n x = (1 - \theta_n)x + \theta_n T_n x$, where $0 < \theta_n \le \delta$, $\delta = \min\{1, (\frac{q\lambda}{\kappa_q})^{\frac{1}{q-1}}\}$, and $\sum_{n=1}^{\infty} |\theta_{n+1} - \theta_n| < \infty$. For given $x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) S_n(t_n x_n + (1 - t_n)\bar{x}_{n+1}), \quad n \ge 1, \end{cases} \tag{42}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{t_n\}$ are sequences in $(0,1)$ satisfy conditions $(C1)$ and $(C2)$ of Theorem 3.1. Suppose in addition that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Proof By the smoothness of E there exists a sunny nonexpansive retraction Q from C onto Ω. Moreover, $Q(g)$ is a weakly contractive mapping of C into itself. For all $x, y \in C$, we have

$$\bigl\|Q\bigl(g(x)\bigr) - Q\bigl(g(y)\bigr)\bigr\| \le \bigl\|g(x) - g(y)\bigr\| \le \|x - y\| - \psi\bigl(\|x - y\|\bigr).$$

Lemma 4.7 guarantees that $Q(g)$ has a unique fixed point $x^* \in C$ such that $x^* = Q(g)$. Now, we define a sequence $\{y_n\}$ and $y_1 \in C$ as follows:

$$\begin{cases} \bar{y}_{n+1} = \beta_n y_n + (1-\beta_n)S_n y_n, \\ y_{n+1} = \alpha_n g(y_n) + (1-\alpha_n)S_n(t_n y_n + (1-t_n)\bar{y}_{n+1}), \quad n \geq 1. \end{cases}$$

Then, by Theorem 3.1 with a constant $f = g(x^*)$, we have that $\{y_n\}$ converges strongly to $x^* = Q(g)) \in \Omega$. Next, we show that $x_n \to x^*$ as $n \to \infty$. Since

$$\|\bar{x}_{n+1} - \bar{y}_{n+1}\| \leq \beta_n\|x_n - y_n\| + (1-\beta_n)\|S_n x_n - S_n y_n\| \leq \|x_n - y_n\|,$$

it follows that

$$\begin{aligned} &\|x_{n+1} - y_{n+1}\| \\ &\quad = \bigl\|\alpha_n\bigl(g(x_n) - g\bigl(x^*\bigr)\bigr) + (1-\alpha_n)\bigl(S_n\bigl(t_n x_n + (1-t_n)\bar{x}_{n+1}\bigr) - S_n\bigl(t_n y_n + (1-t_n)\bar{y}_{n+1}\bigr)\bigr)\bigr\| \\ &\quad \leq \alpha_n\bigl\|g(x_n) - g\bigl(x^*\bigr)\bigr\| + (1-\alpha_n)\bigl\|S_n\bigl(t_n x_n + (1-t_n)\bar{x}_{n+1}\bigr) - S_n\bigl(t_n y_n + (1-t_n)\bar{y}_{n+1}\bigr)\bigr\| \\ &\quad \leq \alpha_n\bigl\|g(x_n) - g(y_n)\bigr\| + \alpha_n\bigl\|g(y_n) - g\bigl(x^*\bigr)\bigr\| \\ &\qquad + (1-\alpha_n)\bigl(t_n\|x_n - y_n\| + (1-t_n)\|\bar{x}_{n+1} - \bar{y}_{n+1}\|\bigr) \\ &\quad \leq \alpha_n\|x_n - y_n\| - \alpha_n\psi\bigl(\|x_n - y_n\|\bigr) + \alpha_n\bigl\|y_n - x^*\bigr\| \\ &\qquad - \alpha_n\psi\bigl(\bigl\|y_n - x^*\bigr\|\bigr) + (1-\alpha_n)\|x_n - y_n\| \\ &\quad \leq \|x_n - y_n\| - \alpha_n\psi\bigl(\|x_n - y_n\|\bigr) + \alpha_n\bigl\|y_n - x^*\bigr\|. \end{aligned} \tag{43}$$

Since $\{y_n\}$ converges strongly to x^*, applying Lemma 4.8 to (43), we obtain that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Therefore $x_n \to x^*$. This completes the proof. □

5 Numerical examples

In this section, we present a numerical example of our main result.

Example 5.1 Let $E = \ell_4$ and $C = \{\mathbf{x} = (x_1, x_2, x_3, x_4, \ldots) \in \ell_4 : x_i \in \mathbb{R} \text{ for } i = 1, 2, 3, \ldots\}$ with norm $\|\mathbf{x}\|_{\ell_4} = (\sum_{i=1}^{\infty}|x_i|^4)^{1/4}$. Let $f : C \to C$ be the contraction defined by $f(\mathbf{x}) = \frac{1}{3}\mathbf{x}$. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be the strictly pseudo-contractive mapping defined by

$$T_n\mathbf{x} = \begin{cases} \frac{1}{n}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \ldots) - 2\mathbf{x} & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0}, \end{cases}$$

where $\mathbf{0} = (0, 0, 0, 0, 0, 0, 0, \ldots)$ is the null vector on ℓ_4.

- We show that T_n is strictly pseudo-contractive. For each $n \geq 1$, if $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, then

$$\begin{aligned} \bigl\langle (I - T_n)\mathbf{x} - (I - T_n)\mathbf{y}, j_2(\mathbf{x} - \mathbf{y})\bigr\rangle &= \bigl\langle 3\mathbf{x} - 3\mathbf{y}, j_2(\mathbf{x} - \mathbf{y})\bigr\rangle \\ &= 3\|\mathbf{x} - \mathbf{y}\|_{\ell_4}^2 \\ &= \frac{1}{3}\|3\mathbf{x} - 3\mathbf{y}\|_{\ell_4}^2 \\ &\geq \lambda\bigl\|(I - T_n)\mathbf{x} - (I - T_n)\mathbf{y}\bigr\|_{\ell_4}^2 \end{aligned}$$

for $\lambda \leq \frac{1}{3}$. Then, we can choose $\lambda = \frac{1}{3}$. Thus, T_n is $\frac{1}{3}$-strictly pseudo-contractive with $\bigcap_{n=1}^{\infty} F(T_n) = \{\mathbf{0}\}$. Further, we observe that T_n is not nonexpansive.

- We show that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition. Since

$$\begin{aligned}
&\sup_{\mathbf{x}\in\ell_4} \|T_{n+1}\mathbf{x} - T_n\mathbf{x}\|_{\ell_4} \\
&\quad = \sup_{\mathbf{x}\in\ell_4} \left\| \frac{1}{n+1}\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots\right) - 2\mathbf{x} - \frac{1}{n}\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots\right) + 2\mathbf{x} \right\|_{\ell_4} \\
&\quad = \left\| \frac{1}{n+1}\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots\right) - \frac{1}{n}\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots\right) \right\|_{\ell_4} \\
&\quad = \left(\frac{1}{n} - \frac{1}{n+1}\right) \left\| \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots\right) \right\|_{\ell_4}.
\end{aligned}$$

So we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \sup_{\mathbf{x}\in\ell_4} \|T_{n+1}\mathbf{x} - T_n\mathbf{x}\|_{\ell_4} &= \lim_{n\to\infty} \sum_{k=1}^{n} \sup_{\mathbf{x}\in\ell_4} \|T_{k+1}\mathbf{x} - T_k\mathbf{x}\|_{\ell_4} \\
&= \left\| \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots\right) \right\|_{\ell_4} < \infty,
\end{aligned}$$

that is, $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, where $T : C \to C$ is defined by

$$T\mathbf{x} = \lim_{n\to\infty} T_n\mathbf{x} = -2\mathbf{x}, \quad \mathbf{x} \in C.$$

Since in ℓ_4, $q = 2$ and $\kappa_2 = 3$, we can choose $\theta_n = \frac{1}{9n} + \frac{1}{9}$. Define the mapping $\{S_n\}_{n=1}^{\infty} : C \to C$ by

$$S_n\mathbf{x} = \begin{cases} (\frac{2}{3} - \frac{1}{3n})\mathbf{x} + (\frac{1}{9n^2} + \frac{1}{9n})(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, \dots) & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT condition, we also have that $(\{S_n\}_{n=1}^{\infty}, S)$ satisfies the AKTT condition, where $S : C \to C$ is defined by

$$S\mathbf{x} = \lim_{n\to\infty} S_n\mathbf{x} = \frac{2}{3}\mathbf{x}, \quad \mathbf{x} \in C.$$

Then, we have $F(S) = F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \{\mathbf{0}\}$. Let $\alpha_n = \frac{1}{32n+1}$, $\beta_n = \frac{1}{100n+3} + 0.32$, and $t_n = \frac{n}{2n+1}$. So our algorithm (16) has the following form:

$$\begin{cases} \bar{\mathbf{x}}_{n+1} = (\frac{1}{100n+3} + 0.32)\mathbf{x}_n + (0.68 - \frac{1}{100n+3})S_n\mathbf{x}_n, \\ \mathbf{x}_{n+1} = \frac{1}{32n+2}f(\mathbf{x}_n) + \frac{32n}{32n+1}S_n(\frac{n}{2n+1}\mathbf{x}_n + \frac{n+1}{2n+1}\bar{\mathbf{x}}_{n+1}), \quad n \geq 1. \end{cases} \tag{44}$$

Let $\mathbf{x_1} = (1, -0.25, 1.46, 1.85, 0, 0, 0, \dots)$ be the initial point. Then, we obtain numerical results in Table 1 and Fig. 1.

Table 1 The values of the sequences $\{\mathbf{x}_n\}$

n	$\mathbf{x}_n$	$\Vert\mathbf{x}_{n+1}-\mathbf{x}_n\Vert_{\ell_4}$
1	(1.000000, –0.250000, 1.460000, 1.850000, 0, 0, 0,…)	1.459e+00
50	(0.007006, 0.003503, 0.002335, 0.001751, 0, 0, 0,…)	1.471e–04
100	(0.003416, 0.001708, 0.001139, 0.000854, 0, 0, 0,…)	3.531e–05
150	(0.002258, 0.001129, 0.000753, 0.000565, 0, 0, 0,…)	1.549e–05
200	(0.001687, 0.000843, 0.000562, 0.000422, 0, 0, 0,…)	8.657e–06
⋮	⋮	⋮
400	(0.000838, 0.000419, 0.000279, 0.000210, 0, 0, 0,…)	2.143e–06
450	(0.000745, 0.000372, 0.000248, 0.000186, 0, 0, 0,…)	1.692e–06
500	(0.000670, 0.000335, 0.000223, 0.000167, 0, 0, 0,…)	1.369e–06

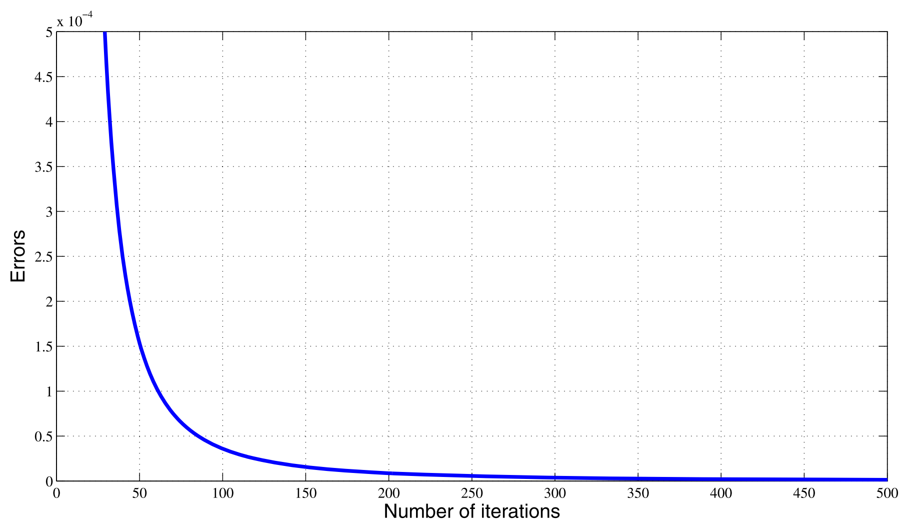

Figure 1 The behavior of errors

6 Conclusion

In this work, we introduce an algorithm by a generalized viscosity explicit rule for finding a common fixed point of a countable family of strictly pseudo-contractive mappings in a q-uniformly smooth Banach space. We obtain some strong convergence theorem for the sequence generated by the proposed algorithm under suitable conditions. However, we should like remark the following:

(1) We extend the results of Ke and Ma [21] and Marino et al. [25] from a one nonexpansive mapping in Hilbert spaces to a countable family of strictly pseudo-contractive mappings in a q-uniformly smooth Banach space.

(2) Our result is proved with a new assumption on the control conditions $\{\beta_n\}$ and $\{t_n\}$.

(3) The method of proof of our result is simpler in comparison with the results of [19, 21, 44, 45]). Moreover, we remove the conditions $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $0<\epsilon\leq s_n\leq s_{n+1}<1$ in Theorem 3.1 of [21].

(4) We give a numerical example that shows the efficiency and implementation of our main result in the space ℓ_4, which is a uniformly convex and 2-uniformly smooth Banach space but not a Hilbert space.

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CONVERGENCE THEOREMS FOR A BIVARIATE NONEXPANSIVE OPERATOR

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Abstract. In this paper, we prove some fixed point theorems for coupled-nonexpansive mapping and prove strong convergence and weakly convergence theorems for a double Mann-type iterative process to approximating a fixed point for coupled-nonexpansive operator in Hilbert spaces. Moreover, we prove some properties of the coupled fixed point set for coupled-nonexpansive mapping and prove fixed point theorem for such mapping on Banach spaces.

Keywords: fixed point; coupled-nonexpansive; strong convergence; weakly convergence; mann iterative.

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1. Introduction

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Let $(X, ||\cdot||)$ be a real Banach space and let K be a nonempty subset of X. A mapping $T: K \to K$ is said to be nonexpansive, if $||Tx - Ty|| \leq ||x - y||$, for each $x, y \in K$. (see [1]). During last four decades many authors have investigated nonexpansive mappings and the set of its fixed points. We now review the needed definitions and results. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{R}$ the set of all real numbers, respectively. A nonempty subset $K \subseteq X$ is said to be *convex*, if $\alpha x + (1-\alpha)y \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$. A Banach space K be *strictly convex* if $\| \frac{x+y}{2} \| < 1$ for each $x, y \in K$ with $||x|| = ||y|| = 1$ and $x \neq y$. A Banach space K be *uniformly convex* if for any $\varepsilon \in (0,2]$ there exists $\delta = \delta(\varepsilon) > 0$, whenever $x, y \in K$, $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \varepsilon$ then $||\frac{x+y}{2}|| < 1 - \delta$. It is clear that uniform convexity implies strict convexity (see [2]). A mapping $T: K \to K$ have a coupled fixed point, if there exist $x, y \in X$ such that $T(x,y) = x$ and $T(y,x) = y$. Let $\{x_n\}$ be a bounded sequence in a Banach space $(X, ||\cdot||)$. For $x \in X$, we define a continuous functional $r(\cdot, x_n)$: $X \to [0, \infty)$ by $r(x, x_n) = \limsup_{n\to\infty} ||x - x_n||$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\}$. The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subseteq X$ is the set $A_K(\{x_n\}) = \{x \in X : r(x, x_n) \leq r(y, x_n), \ \forall y \in K\}$. This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot, x_n)$ in K. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$ (see [3]).

Lemma 1. [3] *Let $(X, ||\cdot||)$ be a uniformly convex Banach space with modulus of convexity of δ. Then every bounded sequence $\{x_n\}$ in K has a unique asymptotic center in K.*

The Banach fixed point theorem concerns certain mappings of a complete metric space itself. It states conditions sufficient for the existence and uniqueness of a fixed point and it's also given a constructive procedure for obtaining better and better approximations to the fixed point, this is a method such that we choose x_0 in a given set and calculate recursively a sequence $x_0, x_1, x_2, \ldots$ from a relation of the form $x_n = Tx_{n-1} = T^n x_0, \quad \forall n \geq 1$. It is also know as the *Picard iteration* starting at x_o. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings have been studied many mathematicians as follows:

Definition 2. Let $(X, ||\cdot||)$ be a normed space and $K \subseteq X$ be a closed and convex. Three classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $T : K \to K$.

Krasnoselskij's iteration

The first one is introduced by Schaefer [4] which is defined as follows: $x_0 \in K$

$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n, \;\; n \geq 0,$$

where $\lambda \in (0,1)$.

Halpern's iteration

The second one is introduced by Halpern [5] which is defined as follows: $x_0 \in K$

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \;\; n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$.

Mann's iteration

The third one is introduced by Mann [6] which is defined as follows: $x_0 \in K$

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \;\; n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$.

Next, Takahashi [2] proved fixed point theorems for nonexpansive mappings in Hilbert spaces as follows:

Theorem 3. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $F : K \to K$ be a nonexpansive. Then the following are equivalent:*
(i) The set $Fix(F)$ of fixed points of T is nonempty;
(ii) $\{F^n x\}$ is bounded for some $x \in K$.

In 2006, Bhaskar and Lakshmikantham [7] established a fixed point theorem for mixed monotone mappings in partially ordered metric spaces. Moreover, the double sequence $\{(x_n, y_n)\}n \geq 0$, defined by the Picard-type iteration $x_{n+1} = T(x_n, y_n), \quad y_{n+1} = T(y_n, x_n), \quad n \geq 0$, with $x_0, y_0 \in X$, is convergent and its limit is always a coupled fixed point of F. In 2013, Olaoluwa et al.[8] introduced the definitions of nonexpansive condituon for coupled maps in product spaces and proved the existence of coupled fixed points of such mappings when X is a uniformly convex as follows:

Definition 4. Let X be a Banach spaces and K be a nonempty subset of X. A mapping $T: K \times K \to K$ is said to be *coupled-nonexpansive* if

$$||T(x,y) - T(u,v))|| \leq \frac{1}{2}(||x-u|| + ||y-v||), \tag{1}$$

for all $x, y, u, v \in K$.

Throughout this paper, a mapping $T: K \times K \to K$ is call *bivariate nonexpansive* or *coupled-nonexpansive* if T satisfies 1.

Example 5. Let $X = \mathbb{R}$. Defined

$$||x|| = |x|,$$

for every $x \in \mathbb{R}$ and $T: X \times X \to X$ be defined by $T(x,y) = \frac{x-y}{2}$, for all $x, y \in X$. Indeed for all $x, y \in X$, we consider

$$\begin{aligned}
||T(x,y) - T(u,v)|| &= |\frac{x-y}{2} - \frac{u-v}{2}| \\
&= \frac{1}{2}|(x-y) - (u-v)| \\
&\leq \frac{1}{2}(|x-y| + |u-v|) \\
&= \frac{||x-u|| + ||y-v||}{2}.
\end{aligned}$$

Hence T is coupled-nonexpansive mapping.

Next, Berinde et al.[9] proved weak and strong convergence theorems for a double Krasnoselskij-type iterative method to approximate coupled solutions of a bivariate nonexpansive operator $T: K \times K \to K$, where K is a nonempty closed and convex subset of a Hilbert space as follows:

Definition 6. A mapping $T: K \times K \to K$ is called *demicompact* if it has the property that whenever $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K with the property that $\{T(u_n, v_n) - u_n\}$ and $\{T(v_n, u_n, \ ) - v_n\}$ converge strongly to 0, then there exists a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ such that $u_{n_k} \to u$ and $v_{n_k} \to v$ strongly.

Theorem 7. *Let K be a bounded, closed and convex subset of a Hilbert space H and let T : $K \times K \to K$ be weakly nonexpansive and demicompact operator. Then the set of coupled fixed points of T is nonempty and the double iterative algorithm $\{(x_n, x_n)\}_{n=0}^{\infty}$ given by x_0 in K and*

$$x_{n+1} = \lambda x_n + (1-\lambda)T(x_n, x_n), \ \ n \geq 0,$$

where $\{\lambda\}_{n=0}^{\infty} \in (0,1)$, converges (strongly) to a coupled fixed point of T.

In this paper, we prove some fixed point theorems for coupled-nonexpansive mapping and prove strong convergence and weakly convergence theorems for a double Mann-type iterative process to approximating a fixed point for coupled-nonexpansive operator in Hilbert spaces. Moreover, we prove some properties of the coupled fixed point set for coupled-nonexpansive mapping and prove fixed point theorem for such mapping on Banach spaces.

2. Fixed Point Theorems

In this section, we prove fixed point theorems for coupled-nonexpansive mapping and prove strong convergence theorems in Banach spaces.

Theorem 8. *Let K be a nonempty closed and convex subset of a Hilbert space H and let T : $K \times K \to K$ be a coupled-nonexpansive. Then the following are equivalent:*
(i) The set $F_c(T)$ of fixed points of T is nonempty;
(ii) $\{T^n(x,x)\}$ is bounded for some $(x,x) \in K \times K$.

Proof. Let $F : K \to K$ be given by $Fx = T(x,x)$, for all $x \in K$. By the coupled-nonexpansiveness property of T, we obtain the nonexpansiveness of F and hence, by Theorem 3, it follows that $F_c(T) \neq \emptyset$. □

3. The Properties of Coupled-Fixed Point Set

In this section, we will prove some properties of coupled-fixed point set for coupled-nonexpansive mapping in a Banach space. Let $(X, ||\cdot||)$ be a Banach space. and let K be a nonempty subset of X. We will denote *the coupled fixed point set* of a mapping T by $F_c(T) = \{(x,y) \in K \times K : T(x,y) = x \text{ and } T(y,x) = y\}$.

Lemma 9. *Let K be a nonempty bounded closed convex subset of strictly Banach spaces X, with $||(x,y)||_{X^2} = ||x|| + ||y||$ for all $x,y \in X$. Let $T : K \times K \to K$ be coupled-nonexpansive and $F_c(T) \neq \emptyset$, then $F_c(T)$ are closed and convex.*

Proof. Suppose that $\{x_n\}$ is a sequence in $F_c(T)$ which converges to some $x \in K$, where $x_n = (y_n, z_n)$ and $x = (y,z)$. Then $||y_n - y|| + ||z_n - z|| = ||x_n - x||_{X^2} \to 0$ as $n \to \infty$. Then $y_n \to y$ and $z_n \to z$. We will to show that $x \in F_c(T)$. We consider

$$||y_n - Tx|| = ||Tx_n - Tx|| = ||T(y_n,z_n) - T(y,z)|| \leq \frac{1}{2}||y_n - y|| + \frac{1}{2}||z_n - z|| \tag{2}$$

and

$$||z_n - T(z,y)|| = ||T(z_n,y_n) - T(z,y)|| \leq \frac{1}{2}||z_n - z|| + \frac{1}{2}||y_n - y||. \tag{3}$$

So

$$\lim_{n\to\infty} ||y_n - Tx|| \leq \lim_{n\to\infty} \frac{1}{2}||y_n - y|| + \lim_{n\to\infty} \frac{1}{2}||z_n - z|| = 0$$

and

$$\lim_{n\to\infty} ||z_n - T(z,y)|| \leq \lim_{n\to\infty} \frac{1}{2}||z_n - z|| + \lim_{n\to\infty} \frac{1}{2}||y_n - y|| = 0.$$

Thus $T(y,z) = y$ and $T(z,y) = z$ by the uniqueness of limit point. Hence $F(T)$ is closed. Next, we will to show that $F_c(T)$ is convex, let $u,v \in F_c(T)$ and each $\alpha \in [0,1]$, where $u = (u_1,u_2)$, $v = (v_1,v_2)$. Now, to show that $w = \alpha u + (1-\alpha)v \in F(T)$. Let $w = (w_1,w_2)$. Then $w_1 = \alpha u_1 + (1-\alpha)v_1$ and $w_2 = \alpha u_2 + (1-\alpha)v_2$. Since

$$||u_1 - v_1|| = ||Tu - Tv|| \leq \frac{1}{2}(||u_1 - y_1|| + ||u_2 - v_2||)$$

and

$$||u_2 - v_2|| = ||T(u_2,v_1) - T(y_2,v_1)|| \leq \frac{1}{2}(||u_2 - v_2|| + ||u_1 - v_1||),$$

we have $||u_1 - v_1|| \leq ||u_2 - v_2||$ and $||u_1 - v_1|| \geq ||u_2 - v_2||$. Thus,

$$||u_1 - v_1|| = ||u_2 - v_2||. \tag{4}$$

Since

$$||u_1 - w_1|| = ||u_1 - (\alpha u_1 + (1-\alpha)v_1)|| = (1-\alpha)||u_1 - v_1||$$

and

$$||u_2 - w_2|| = ||u_2 - (\alpha u_2 + (1-\alpha)v_2)|| = (1-\alpha)||u_2 - v_2||,$$

we get

$$||u_1 - w_1|| = ||u_2 - w_2||. \tag{5}$$

Since

$$||v_1 - w_1|| = ||v_1 - (\alpha u_1 + (1-\alpha)v_1)|| = \alpha||u_1 - v_1||$$

and

$$||v_2 - w_2|| = ||v_2 - (\alpha u_2 + (1-\alpha)v_2)|| = \alpha||u_2 - v_2||,$$

it follows that

$$||v_1 - w_1|| = ||v_2 - w_2||. \tag{6}$$

Similar to the above proof, it follows that

$$||u_1 - Tw|| = ||Tu - Tw|| \leq \frac{1}{2}(||u_1 - w_1|| + ||u_2 - w_2||) = ||u_1 - w_1||,$$

$$||u_2 - T(w_2, w_1)|| \leq \frac{1}{2}(||u_2 - w_2|| + ||u_1 - w_1||) = ||u_2 - w_2||,$$

$$||v_1 - Tw|| = ||Tv - T(z_1, z_2)|| \leq \frac{1}{2}(||v_1 - w_1|| + ||v_2 - w_2||) = ||v_1 - w_1||,$$

and

$$||v_2 - T(w_2, w_1)|| = ||T(v_2, v_1) - T(w_2, w_1)|| \leq \frac{1}{2}(||v_2 - w_2|| + ||v_1 - w_1||) = ||v_2 - w_2||.$$

For $w = \alpha u + (1-\alpha)v$ where $w = (w_1, w_2)$, we consider

$$\begin{aligned} ||u_1 - v_1|| &\leq ||u_1 - Tw|| + ||Tw - v_1|| \leq ||u_1 - w_1|| + ||w_1 - v_1|| \\ &= ||u_1 - (\alpha u_1 + (1-\alpha)v_1)|| + ||(\alpha u_1 + (1-\alpha)v_1) - v_1|| \\ &= ||u_1 - v_1||, \qquad (7) \end{aligned}$$

and

$$
\begin{aligned}
||u_2 - v_2|| &\le ||u_2 - T(w_2,w_1)|| + ||T(w_2,w_1) - v_2|| \\
&\le ||u_2 - w_2|| + ||w_2 - v_2|| \\
&= ||u_2 - (\alpha u_2 + (1-\alpha)v_2|| + ||(\alpha u_2 + (1-\alpha)v_2) - v_2|| = ||u_2 - v_2||. \qquad (8)
\end{aligned}
$$

Thus $||u_1 - Tw|| = ||u_1 - w_1||$ and $||Tw - v_1|| = ||w_1 - v_1||$, because if $||u_1 - Tw|| < ||u_1 - w_1||$ or $||Tw - v_1|| < ||w_1 - v_1||$, then which the contradiction to $||u_1 - v_1|| < ||u_1 - v_1||$. Since X is strictly convex, we have $Tw = \alpha u_1 + (1-\alpha)v_1 = w_1$. Similarly, $T(w_2,w_1) = \alpha w_2 + (1-\alpha)v_2 = w_2$, and then w is a coupled fixed point of T, that is $\alpha u + (1-\alpha)v \in F_c(T)$. Hence $F_c(T)$ is convex.

□

Theorem 10. *Let K be a nonempty bounded closed convex subset of uniformly convex Banach space X such that $||(x,y)||_{X^2} = ||x|| + ||y||$ with uniformly convex Banach space with modulus of convexity of δ. Suppose that a map $T : K \times K \to K$ be coupled-nonexpansive and $\{x_n\}$ and $\{y_n\}$ are a sequences in K defined by $x_{n+1} = T(x_n,y_n)$, $y_{n+1} = T(y_n,x_n)$ and $||x_n - z||$, $||y_n - z||$ are increase sequences in $\mathbb{R}$ for all $z \in K$ with $||x_{n-1} - x|| = ||y_{n-1} - y||$ for all $n \in \mathbb{N}$. Then $F_c(T)$ are nonempty, closed and convex.*

Proof. By Lemma 1, the asymptotic center of any bounded sequence in K, particularly, the asymptotic center of approximate coupled fixed point sequence for T is in K. Let $A(\{x_n\}) = \{x\}$ and $A(\{y_n\}) = \{y\}$. We consider

$$
\begin{aligned}
||x_n - T(x,y)|| &\le ||x_{n+1} - T(x,y)|| = ||T(x_n,y_n) - T(x,y)|| \\
&\le \frac{1}{2}(||x_n - x|| + ||y_n - y||) = ||x_n - x||, \qquad (9)
\end{aligned}
$$

thus $\limsup_{n\to\infty} ||x_n - T(x,y)|| \le \limsup_{n\to\infty} ||x_n - x||$. Similarly,

$$
\begin{aligned}
||y_n - T(y,x)|| &\le ||y_{n+1} - T(y,x)|| = ||T(y_n,x_n) - T(y,x)|| \\
&\le \frac{1}{2}(||y_n - x|| + ||x_n - y||) = ||y_n - y||, \qquad (10)
\end{aligned}
$$

hence $\limsup_{n\to\infty}||y_n - T(y,x)|| \leq \limsup_{n\to\infty}||y_n - y||$. By the uniqueness of the asymptotic center, $T(x,y) = x$ and $T(y,x) = y$. Hence $F_c(T)$ is nonempty. By Lemmas 9,we conclude that $F_c(T)$ are nonempty, closed and convex. $\square$

4. Iterative Approximation of Fixed Points

Theorem 11. *Let K be a nonempty, bounded, closed and convex subset of a Hilbert space H and let $T : K \times K \to K$ be coupled-nonexpansive operator. Then the Mann iterative $\{x_n\}_{n=0}^{\infty}$ given by x_0 in K and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n,x_n), \ \ n \geq 0,$$

(11)

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq (0,1)$, weakly converges to coupled fixed point of T.

Proof. As in proof of Theorem 8, for each $w \in Fix(T)$ and each n, we have,

$$||x_{n+1} - w|| \leq ||x_n - w||.$$

Defined $g : Fix(T) \to [0,\infty)$ by $g(w) = \lim_{n\to\infty}||x_n - w||$. we see that g is well defined and is a lower semi-continuous convex function on the nonempty convex set $Fix(F)$. Let $r = \inf\{g(w) : w \in Fix(F)\}$. For each $\varepsilon > 0$, the set $M_\varepsilon = \{z : g(z) \leq r + \varepsilon\}$ is closed and convex and then weakly compact. Therefore $\cap_{\varepsilon>0} M_\varepsilon$ in fact $\varepsilon > 0$, $\ M_\varepsilon = z : g(z) = r \equiv L$. Moreover, L contains exactly one point. Indeed, since L is convex and closed, for $w_1, w_2 \in L$, and $w_{\alpha_n} = \alpha_n w_1 + (1-$

$\alpha_n)w_2$, we get

$$
\begin{aligned}
g^2(w_{\alpha_n}) &= \lim_{n\to\infty} ||w_{\alpha_n} - x_n||^2 \\
&= \lim_{n\to\infty} ||\alpha_n w_1 + (1-\alpha_n)w_2 - x_n||^2 \\
&= \lim_{n\to\infty} (\alpha_n^2 ||w_1 - x_n||^2 + (1-\alpha_n)^2 ||w_2 - x_n||^2 \\
&\quad + \alpha_n(1-\alpha_n)\langle w_1 - x_n, w_2 - x_n\rangle) \\
&\leq \lim_{n\to\infty} (\alpha_n^2 ||w_1 - x_n||^2 + (1-\alpha_n)^2 ||w_2 - x_n||^2 \\
&\quad + \alpha_n(1-\alpha_n)||w_1 - x_n||\, ||w_2 - x_n||) \\
&\quad + \lim_{n\to\infty} \alpha_n(1-\alpha_n)[\langle w_1 - x_n, w_2 - x_n\rangle - ||w_1 - x_n||\, ||w_2 - x_n||]. \qquad (12)
\end{aligned}
$$

So $\lim_{n\to\infty} \alpha_n(1-\alpha_n)[\langle w_1 - x_n, w_2 - x_n\rangle - ||w_1 - x_n||\, ||w_2 - x_n||] = 0$. Since

$$\lim_{n\to\infty} ||w_1 - x_n|| = r = r \lim_{n\to\infty} ||w_2 - x_n||,$$

we have

$$
\begin{aligned}
||w_1 - w_2||^2 &= ||w_1 - x_n - (w_2 - x_n)||^2 \\
&= ||w_1 - x_n||^2 + ||w_2 - x_n||^2 - 2\langle w_1 - x_n, w_2 - x_n\rangle \to r^2 + r^2 - 2r^2 = 0, \qquad (13)
\end{aligned}
$$

a contradiction. Now, we will show that $x_n = F^n(x_0, x_0) \rightharpoonup w_1$, it suffices to assume that $x_{n_j} \rightharpoonup w$ for an infinite subsequence and then prove that $w = w_1$. By the arguments in the proof of Theorem 8, $w \in Fix(F)$. Considering the definition of g and the fact that $x_{n_j} \to p$, we obtain that

$$
\begin{aligned}
||x_{n_j} - w_1||^2 &= ||x_{n_j} - w - (w_1 - w)||^2 \\
&= ||x_{n_j} - w||^2 + ||w_1 - w||^2 - 2\langle x_{n_j} - p, w_1 - p\rangle \to g^2(w) + ||w_1 - w||^2 \\
&= g^2(w_1) = r^2. \qquad (14)
\end{aligned}
$$

Since $g^2(w_{\alpha_n}) \geq r^2$, we conclude that $||w - w_1|| \leq 0$. Therefore $w = w_1$. □

Theorem 12. *Let K be a nonempty, bounded, closed, and convex subset of a Hilbert space H and let $T : K \times K \to K$ be coupled-nonexpansive and demicompact operator. Then the set of* *coupled fixed points of* T *is nonempty and the double iterative algorithm* $\{(x_n,x_n)\}_{n=0}^{\infty}$ *given by* x_0 *in* K *and*

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)T(x_n,x_n),\ \ n\geq 0, \tag{15}$$

where $\{\alpha_n\}_{n=0}^{\infty}\subseteq[0,1]$, *strongly converges to a coupled fixed point of* T.

Proof. By Theorem 8, T has at least one coupled fixed point with equal components, say $(x',x')\in C\times C$. We will to show that the sequence $\{x_n-T(x_n,x_n)\}$ converges strongly to 0. We consider

$$\begin{aligned}||x_{n+1}-x'||^2&=||\alpha_n x_n+(1-\alpha_n)T(x_n,x_n)-x'||^2\\&=\alpha_n^2||x_n-x'||^2+(1-\alpha_n)^2||T(x_n,x_n)-x'||^2\\&\quad+\alpha_n(1-\alpha_n)\langle x_n-x',T(x_n,x_n)-x'\rangle,\end{aligned} \tag{16}$$

and

$$\begin{aligned}||x_n-T(x_n,x_n)||^2&=||x_n-x'||^2+||T(x_n,x_n)-x'||^2\\&\quad+\langle x_n-x',T(x_n,x_n)-x'\rangle.\end{aligned} \tag{17}$$

Since T coupled-nonexpansive and $T(x',x')=x'$, we obtain

$$||T(x_n,x_n)-x'||=||T(x_n,x_n)-T(x',x')||\leq||x_n-x'||. \tag{18}$$

Now, by (16),(17) and (18), it follows that for any $\{\beta_n\}$ we get

$$\begin{aligned}||x_{n+1}-x'||^2+\beta_n^2||x_n-T(x_n,x_n)||^2&\leq(2\beta_n^2+\alpha_n^2+(1-\alpha_n)^2)||x_n-x'||^2\\&\quad+2(\alpha_n(1-\alpha_n)-\beta_n)\langle T(x_n,x_n)-x',x_n-x'\rangle.\end{aligned} \tag{19}$$

If we choose now a sequence such that $0\leq\beta_n^2\leq\alpha_n(1-\alpha_n),\ \ \forall n\geq 1$ then from the inequality (16), we obtain

$$\begin{aligned}||x_{n+1}-x'||^2+\beta_n^2||x_n-T(x_n,x_n)||^2&\leq(2\beta_n^2+\alpha_n^2+(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)-2\beta_n^2)||x_n-x'||\\&=||x_n-x'||\end{aligned} \tag{20}$$

By the Cauchy-Schwarz inequality,

$$\langle T(x_n,x_n)-x',x_n-x'\rangle \leq ||T(x_n,x_n)-x'||\,||x_n-x'|| \leq ||x_n-x'||.$$

By (20), we get

$$\beta_n^2||x_n-T(x_n,x_n)||^2 \leq ||x_n-x'||^2-||x_{n+1}-x'||^2,\ \ \forall n\geq 1. \tag{21}$$

Thus $\{||x_n-x'||\}$ is a decreasing sequence, hence it is convergent. By the inequality (20), we get

$$\begin{aligned} 0 &\leq ||x_n-T(x_n,x_n)||^2 \\ &\leq \frac{1}{\beta_n^2}(||x_n-x'||^2-||x_{n+1}-x'||^2)||x_n-x'||^2-||x_{n+1}-x'||^2,\ \ \forall n\geq 1, \end{aligned} \tag{22}$$

Take $n\to\infty$, we have $||x_n-T(x_n,x_n)||\to 0$. By demicompactness of T that there exist a converges subsequence $\{x_{n_k}\}$ of x_n in K, say $x_{n_k}\to w$. Since T is coupled-nonexpansive, we get T is continuous, and then $T(x_{n_k},x_{n_k})\to T(w,w)$. Since $||x_n-T(x_n,x_n)||\to 0$, we have $x_{n_k}-T(x_{n_k},x_{n_k})\to w-Tw,w)$, which shows that (w,w) is a coupled fixed point of T. Using (20), with $x'=w$, we deduce that the sequence of nonnegative real numbers $\{x_n-w\}$ is non-increasing, so convergent. Since its subsequence $\{x_{n_k}-w\}$ converges to 0, it follows that the sequence $\{x_n-w\}$ itself converges to 0, Therefore $\{(x_n,x_n)\}$ converges strongly to (w,w) as $n\to\infty$. □

Example 13. Let $\mathbb{R}$ be a real number. Defined $\langle x,y\rangle = xy$, and $|x|^2=\langle x,x\rangle$, for every $x,y\in\mathbb{R}$ and $T:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by

$$T(x,y)=\frac{x-y}{2},$$

for all $x,y\in\mathbb{R}$. Indeed, we see that consider, T satisfies (1) and is demicompact. Hence, all the assumptions of Theorem 8 are satisfied. It is easy to see that T possesses a unique coupled fixed point, $(0,0)$, and the Mann-type iteration (15) yields the sequence $x_n=(1-\alpha_n)^n x_0$, $n\geq 0$, where $\alpha_n=\frac{2n+2}{2n}\subseteq[0,1]$. Since $\lim_{n\to\infty}\alpha_n=\frac{1}{2}$, it follows that $(x_n,x_n)\to(0,0)$ as $n\to\infty$, for any $x_0\in K$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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Approximation of Common Solutions for Proximal Split Feasibility Problems and Fixed Point Problems in Hilbert Spaces

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Abstract : In this paper, a new iterative algorithm is proposed for finding the minimum-norm solution of a proximal split minimization problem and fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. Under suitable conditions, it is proved that the sequence generated by the proposed algorithm converges strongly to a common solution of the two above described problems. The iterative algorithm are proposed in such a way that the selection of the step-sizes does not need any prior information about the operator norm.

Keywords : Fixed point problem, proximal split feasibility problems, quasi-nonexpansive mapping.

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1 Introduction

Throughout this article, let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$be two proper and lower semicontinuous convex functions and $A : H_1 \to H_2$ be a bounded linear operator. Now, we will introduce one of the famous problems in many fields of pure and applied sciences, that is the split feasibility problem (SFP) was first introduced by Censor and Elfving [18] in 1994: Find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ be a bounded linear operator. Split feasibility problem can be applied to medical image reconstruction, especially intensity-modulated therapy (see, [2]). In the past decade, many researchers have increasingly stuided the split feasibility problem, see, for instance [3, 4, 5, 6, 7, 8, 9, 10], and the references therein.

In this paper, we study more general problem which is the following: find a solution $z \in H_1$ such that

$$\min_{z \in H_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.2}$$

where $g_\lambda(y) := \min_{u \in H_2}\{g(u) + \frac{1}{2\lambda}\|u-y\|^2\}$ is the Moreau-Yosida approximate of the function f of parameter λ, also called proximal operator of f of order λ and below denoted by $\text{prox}_{\lambda g}(x)$. If $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ ortherwise] and $g = \delta_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively. Then problem (1.2) reduces to

$$\min_{x \in H_1} \{\delta_C(x) + (\delta_Q)_\lambda(Ax)\} \Leftrightarrow \min_{x \in H_1} \{\frac{1}{2\lambda}\|(I - P_Q)(Ax)\|^2\}$$

which is equivalent to **SFP** when $C \cap A^{-1}(Q)$.

In the case $\operatorname{argmin} f \cap A^{-1}(\operatorname{argmin} g) \neq \emptyset$, the split minimization problem (**SMP**) is to find a minimizer z of f such that Az minimizes g ; that is,

$$z \in \operatorname{argmin} f \text{ such that } Az \in \operatorname{argmin} g, \tag{1.3}$$

where $\operatorname{argmin} f := \{\bar{x} \in H_1 : f(\bar{x}) \leq f(x) \text{ for all } x \in H_1\}$ and $\operatorname{argmin} g := \{\bar{y} \in H_2 : g(\bar{y}) \leq g(y) \text{ for all } y \in H_2\}$. The solution set of the problem (1.3) is denote by Γ.

Recall that the proximal operator $\text{prox}_{\lambda g} : H \to H$ is defined by

$$\text{prox}_{\lambda g}(x) := \operatorname*{argmin}_{u \in H}\{g(u) + \frac{1}{2\lambda}\|u - x\|^2\}. \tag{1.4}$$

Moreover, the proximity operator of f is firmly nonexpansive, namely,

$$\langle \text{prox}_{\lambda g}(x) - \text{prox}_{\lambda g}(y), x - y\rangle \geq \|\text{prox}_{\lambda g}(x) - \text{prox}_{\lambda g}(y)\|^2. \tag{1.5}$$

for all $x, y \in H$, which is equivalent to

$$\|\operatorname{prox}_{\lambda g}(x) - \operatorname{prox}_{\lambda g}(y)\|^2 \leq \|x-y\|^2 - \|(I-\operatorname{prox}_{\lambda g})(x) - (I-\operatorname{prox}_{\lambda g})(y)\|^2. \quad (1.6)$$

for all $x, y \in H$. For general information on proximal operator, see the research paper by Combettes and Pesquet [23].

In 2014, Moudafi and Thakur [24] introduced the split proximal algorithm for estimating the stepsizes which do not need prior knowledge of the operator norms for solving **SMP** (1.3) as follows.

$$x_{n+1} = \operatorname{prox}_{\lambda\gamma_n f}(x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n)\forall n \geq 1, \quad (1.7)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, $h(x) := \frac{1}{2}\|(I-\operatorname{prox}_{\lambda g})Ax\|^2$, $l(x) := \frac{1}{2}\|(I - \operatorname{prox}_{\lambda\gamma_n f})x\|^2$ and $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$. Thay also proved the weak convergence theorem of the sequence generated by algorithm (1.7) to a solution of **SMP** (1.3).

In 2014, Yao *et al.* [25] introduced the regularized algorithm for solving the split proximal algorithm as follows:

$$x_{n+1} = \operatorname{prox}_{\lambda\gamma_n f}(\alpha_n u + (1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n), \forall n \geq 1, \quad (1.8)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. Then, they proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter α_n and γ_n.

Recently, Shehu and Ogbuisi [12]introduced the following algorithm for solving split proximal algorithms and fixed point problems for k-strictly pseudocontractive mappings in Hilbert spaces:

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = \operatorname{prox}_{\lambda\gamma_n f}(u_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Au_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \forall n \in \mathbb{N}, \end{cases} \quad (1.9)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. They also showed that, under certain assumptions imposed on the parameters, the sequence $\{x_n\}$ generated by (1.9) converges strongly to $x^* \in Fix(S) \cap \Gamma$.

Very recently, Abbas *et al.* [16] studied the following algorithm for finding the minimum-norm solution of split proximal algorithm, that is,

$$x_{n+1} = \operatorname{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n)\forall n \geq 1, \quad (1.10)$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. Using the split proximal algorithm 1.10, they also proved a strong convergence theorem of the sequences generated by the proposed algorithms under some appropriate conditions.

After we have studied research related to split proximal algorithm and fixed point problem, we obtain the following question.

Question Is it possible to obtain a strong convergence theorem for finding the minimum-norm solution of a proximal split minimization problem and the set of common fixed points of a family of mappings in Hilbert spaces ? Such as a countable family of quasi-nonexpansive mappings.

In this paper, we give the answer for the mentioned questions and introduce a new iterative algorithm for finding the minimum-norm solution of a proximal split minimization problem and fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. Under suitable conditions, it is proved that the sequence generated by the proposed algorithm converges strongly to a common solution of the two above described problems. The iterative algorithm are proposed in such a way that the selection of the step-sizes does not need any prior information about the operator norm.

2 Preliminaries

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$. A point $z \in H$ is called a *mimimum norm* fixed point of T if and only if $z \in Fix(T)$ and $\|z\| = \min\{\|x\| : x \in Fix(T)\}$.

Definition 2.1. *Let $T : C \to C$ be a nonlinear mapping, then*

(i) T is said to be nonexpansive *if*

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C,$$

(ii) T is said to be quasi-nonexpansive *if*

$$\|Tx - p\| \leq \|x - p\|, \forall x \in C \text{ and } \forall p \in Fix(T),$$

Lemma 2.2. *[28] Let C be a nonempty closed convex subset of a real Hilbert space H.For every $i = 1, 2, 3, .., N$, let $T_i : H_1 \to H_1$ be a finte fammily of quasi-nonexpansive mapping such that $\bigcap_{i=1}^N Fix(T_i) \neq 0$ and $I - T_i$ are demiclosed at zero. Put $T = \sum_{i=1}^N a_i T_i$, where $0 < a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\sum_{i=1}^{N} a_i = 1$. Then the following hold:*

1. *$Fix(T) = \bigcap_{i=1}^N Fix(T_i)$;*
2. *T is a quasi-nonexpansive mapping;*

3. T is demiclosed at zero.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.3 ([21])**.** *Given $x \in H_1$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Lemma 2.4 ([19])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n \to \infty} s_n = 0$.

Lemma 2.5. ([22]) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq ...$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0$.

3 Main Theorem

In this section, we prove a strong convergence theorem for for finding the minimum-norm solution of a proximal split minimization problem and fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$be two proper and lower semicontinuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, 3, .., N$, let $T_i : H_1 \to H_1$ be a finite family of quasi-nonexpansive mapping such that $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ and $I - T_i$ are demiclosed at zero.

Now, we introduce the following algorithm for finding the solution set of $\Gamma \cap \bigcap_{i=1}^{N} Fix(T_i)$.

Algorithm 3.1

Step 1: Choose an initial point $x_1 \in H_1$.

Step 2: Assume that x_n has been constructed.
Set $\theta(x_n) := \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$ where $h(x_n) := \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax_n\|^2$ and $l(x_n) := \frac{1}{2}\|(I - \text{prox}_{\lambda f})x_n\|^2$ with $\theta(x_n) \neq 0$.
We compute x_{n+1} in the following iterative scheme:

$$\begin{cases} y_n = \text{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \text{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\sum_{i=1}^{N} a_i T_i y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\displaystyle\sum_{i=1}^{N} a_i = 1$.

Using algorithm (3.1), we prove a strong convergence theorem for approximation of solutions of problem (1.3) and the set of fixed points of quasi-nonexpansive mappings as follows:

Theorem 3.1. *Suppose that $\Omega := \Gamma \cap \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. If the parameters satisfy the following conditions:*

(C1) $\displaystyle\lim_{n\to\infty} \alpha_n = 0$ *and* $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\varepsilon \leq \rho_n \leq \dfrac{4(1-\alpha_n)h(x_n)}{h(x_n) + l(x_n)} - \varepsilon$ *for some* $\varepsilon > 0$ *and for any* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to a solution z which is also a minimum norm solution of Ω. In other words, $z = P_\Omega(0)$.

Proof. Let $z = P_\Omega(0)$. Then $z = \text{prox}_{\lambda\gamma_n f} z$ and $Az = \text{prox}_{\lambda g} z$. Note that $\nabla h(x_n) = A^*(I - \text{prox}_{\lambda g})Ax_n$, $\nabla l(x_n) = (I - \text{prox}_{\lambda\gamma_n f})x_n$

Since $\text{prox}_{\lambda g}$ is firmly nonexpansive, we have that $I - \text{prox}_{\lambda g}$ is also firmly nonexpansive. Hence

$$\begin{aligned} \langle A^*(I - \text{prox}_{\lambda g})Ax_n, x_n - z\rangle &= \langle (I - \text{prox}_{\lambda g})Ax_n, Ax_n - Az\rangle \\ &= \langle (I - \text{prox}_{\lambda g})Ax_n, Ax_n - Az\rangle \\ &= \langle (I - \text{prox}_{\lambda g})Ax_n - (I - \text{prox}_{\lambda g})Az, Ax_n - Az\rangle \\ &\geq \|(I - \text{prox}_{\lambda g})Ax_n\|^2 = 2h(x_n). \end{aligned} \tag{3.2}$$

From the deffinition of y_n and the nonexpansivity of $\operatorname{prox}_{\lambda\gamma_n f}$, we have

$$\begin{aligned}
\|y_n - z\| &= \| \operatorname{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n) - z\| \\
&\leq \|(1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\| \\
&= \|\alpha_n(z) + (1-\alpha_n)\left(x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right)\| \\
&\leq \alpha_n\|z\| + (1-\alpha_n)\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right\|. \quad (3.3)
\end{aligned}$$

Since $\nabla h(x_n) = A^*(I - \operatorname{prox}_{\lambda g})Ax_n$, $\nabla l(x_n) = (I - \operatorname{prox}_{\lambda\gamma_n f})x_n$ and (3.2), we have

$$\begin{aligned}
&\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right\|^2 \\
&= \|x_n - z\|^2 + \frac{\gamma_n^2}{(1-\alpha_n)^2}\|A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\|^2 \\
&\quad - 2\frac{\gamma_n}{(1-\alpha_n)}\langle A^*(I - \operatorname{prox}_{\lambda g})Ax_n, x_n - z\rangle \\
&= \|x_n - z\|^2 + \frac{\gamma_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 - 2\frac{\gamma_n}{(1-\alpha_n)}\langle \nabla h(x_n), x_n - z\rangle \\
&\leq \|x_n - z\|^2 + \frac{\gamma_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 - 4\frac{\gamma_n}{(1-\alpha_n)}h(x_n) \\
&= \|x_n - z\|^2 + \rho_n^2\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)^2\theta^4(x_n)}\|\nabla h(x_n)\|^2 - 4\rho_n\frac{(h(x_n)+l(x_n))}{(1-\alpha_n)\theta^2(x_n)}h(x_n) \\
&\leq \|x_n - z\|^2 + \rho_n^2\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)^2\theta^4(x_n)} - 4\rho_n\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\frac{h(x_n)}{(h(x_n)+l(x_n))} \\
&= \|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\right). \quad (3.4)
\end{aligned}$$

Without loss of generality, by condition (C3), we can assume that $\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n} \geq 0$ for all $n \geq 1$. From (3.3), (3.4), we have

$$\begin{aligned}
\|y_n - z\| &\leq \alpha_n\|z\| + (1-\alpha_n)\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right\| \\
&\leq \alpha_n\|z\| + (1-\alpha_n)\|x_n - z\|. \quad (3.5)
\end{aligned}$$

Put $T = \sum_{i=1}^N a_i T_i$, where $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\sum_{i=1}^N a_i = 1$.

From Lemma 2.2, we have T is a quasi-nonexpansive mapping. It follows that

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\| \\
&\leq \beta_n\|y_n - z\| + (1-\beta_n)\|Ty_n - z\| \\
&\leq \beta_n\|y_n - z\| + (1-\beta_n)\|y_n - z\| \\
&= \|y_n - z\| \\
&\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\|z\| \\
&\leq \max\{\|x_n - z\|, \|z\|\}.
\end{aligned}$$

By mathematical induction, we have

$$\|x_n - z\| \leq \max\{\|x_1 - z\|, \|z\|\}, \forall n \in \mathbb{N}.$$

It implies that $\{x_n\}$ is bounded and so are , $\{T(y_n)\}$.

From the definition of y_n, we have

$$\begin{aligned}
\|y_n - z\|^2 &= \|\operatorname{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I-\operatorname{prox}_{\lambda g})Ax_n) - z\|^2 \\
&\leq \|(1-\alpha_n)x_n - \gamma_n A^*(I-\operatorname{prox}_{\lambda g})Ax_n - z\|^2, \\
&= \|\alpha_n(z) + (1-\alpha_n)\left(x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I-\operatorname{prox}_{\lambda g})Ax_n - z\right)\|^2 \\
&\leq \alpha_n\|z\|^2 + (1-\alpha_n)\left\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I-\operatorname{prox}_{\lambda g})Ax_n - z\right\|^2 \\
&\leq \alpha_n\|z\|^2 + (1-\alpha_n)\left(\|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}\right)\right) \\
&= \alpha_n\|z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right).
\end{aligned} \tag{3.6}$$

It follows from (3.6), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\|^2 \\
&\leq \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\
&\leq \|y_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\
&\leq \alpha_n\|z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\
&\leq \alpha_n\|z\|^2 + \|x_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2.
\end{aligned}$$

It implies that

$$\beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \leq \alpha_n\|z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2. \tag{3.7}$$

From the definition of x_n and (3.6), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n y_n + (1-\beta_n)Ty_n - z\|^2 \\
&\le \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 \\
&\le \|y_n - z\|^2 \\
&\le \alpha_n\|z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) \\
&\le \alpha_n\|z\|^2 + \|x_n - z\|^2 - \rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right).
\end{aligned}$$

It implies that

$$\rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) \le \alpha_n\|z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2. \tag{3.8}$$

Now we divide the rest of the proof into two cases.

CASE 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=1}^{\infty}$ is non-increasing. Then $\{\|x_n - z\|\}_{n=1}^{\infty}$ coverges and $\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0$ as $n \to \infty$. From (3.8), the condition (C1) and (C3), we obtain

$$\rho_n\left(\frac{4h(x_n)}{(h(x_n)+l(x_n))} - \frac{\rho_n}{1-\alpha_n}\right)\left(\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)}\right) \to 0 \text{ as } n \to \infty.$$

Then, we have

$$\frac{(h(x_n)+l(x_n))^2}{\theta^2(x_n)} \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Observe that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded (see [16]). It follows that

$$\lim_{n\to\infty}((h(x_n)+l(x_n))^2) = 0.$$

It implies that

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0.$$

Next, we will show that $\limsup_{n\to\infty}\langle -z, x_n - z\rangle \le 0$, where $z = P_\omega(0)$. To show this,since $\{x_n\}$ is bounded, there exits a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle -z, x_n - z\rangle = \lim_{j\to\infty}\left\langle -z, x_{n_j} - z\right\rangle.$$

By the lower semicontinuity of h, we have

$$0 \le h(q) \le \liminf_{j\to\infty} h(x_{n_j}) = \lim_{n\to\infty} h(x_n) = 0.$$

So, $h(q) = \frac{1}{2}\|(I - \operatorname{prox}_{\lambda g})Aq\|^2 = 0$. Therefore, Aq is a fixed point of the proximal mapping of g or equivalently $0 \in \partial f(Aq)$. In other words, Aq is a minimizer of g . Similarly, from the lower semicontinuity of l, we obtain

$$0 \leq l(q) \leq \liminf_{j\to\infty} l(x_{n_j}) = \lim_{n\to\infty} l(x_n) = 0.$$

So, $l(q) = \frac{1}{2}\|(I - \operatorname{prox}_{\lambda\gamma_n f})q\|^2 = 0$. Therefore, q is a fixed point of the proximal mapping of f or equivalently $0 \in \partial g(q)$. In other words, q is a minimizer of f. Hence $q \in \Gamma$.
From the definition of γ_n, we have

$$0 < \gamma_n < 4\frac{h(x_n) + l(x_n)}{\theta^2(x_n)} \to 0 \text{ as } n \to \infty$$

implies that $\gamma_n \to 0$ as $n \to \infty$.
Next, we will show that $q \in Fix(T) = \bigcap_{i=1}^N Fix(T_i)$. From (3.7) and the condition (C1) (C2), we have

$$\|y_n - Ty_n\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

For each $n \geq 1$, let $u_n := (1 - \alpha_n)x_n$. Then,

$$\begin{aligned}\|u_n - x_n\| &= \|(1 - \alpha_n)x_n - x_n\| \\ &= \alpha_n\|x_n\|.\end{aligned}$$

From the condition (C1), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.11}$$

Observe that

$$\|u_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\| \leq \|u_n - x_n\| + \|(I - \operatorname{prox}_{\lambda\gamma_n f})x_n\|.$$

From $\lim_{n\to\infty} l(x_n) = \lim_{n\to\infty} \frac{1}{2}\|(I - \operatorname{prox}_{\lambda\gamma_n f})x_n\|^2 = 0$ and (3.11), we have

$$\lim_{n\to\infty} \|u_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\| = 0. \tag{3.12}$$

By the nonexpansiveness of $\operatorname{prox}_{\lambda\gamma_n f}$, we have

$$\begin{aligned}\|y_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\| &= \|\operatorname{prox}_{\lambda\gamma_n f}(u_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n) - \operatorname{prox}_{\lambda\gamma_n f} x_n\| \\ &\leq \|u_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n - x_n\| \\ &\leq \|u_n - x_n\| + \gamma_n\|A^*(I - \operatorname{prox}_{\lambda g})Ax_n\|.\end{aligned}$$

From (3.12) and $\gamma_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|y_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\| = 0. \tag{3.13}$$

We observe that

$$\|y_n - u_n\| \leq \|y_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\| + \|u_n - \operatorname{prox}_{\lambda\gamma_n f} x_n\|.$$

From (3.12) and (3.13), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.14}$$

Also, observe that $\|y_n - x_n\| \leq \|y_n - u_n\| + \|u_n - x_n\|$ and from (3.12) and (3.13), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.15}$$

Using $x_{n_j} \rightharpoonup q \in H_1$ and (3.15), we obtain $y_{n_j} \rightharpoonup q \in H_1$. Since $y_{n_j} \rightharpoonup q \in H_1$, $\|y_n - Ty_n\| \to 0$ as $n \to \infty$ and Lemma 2.2, we have $q \in Fix(T) = \bigcap_{i=1}^N Fix(T_i)$. Hence $q \in \Omega = \bigcap_{i=1}^N Fix(T_i) \cap \Gamma$. Since $x_{n_j} \rightharpoonup q$ as $j \to \infty$ and $q \in \Omega$. Lemma 2.3, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle -z, x_n - z\rangle &= \lim_{j\to\infty} \left\langle -z, x_{n_j} - z\right\rangle \\
&= \langle -z, q - z\rangle \\
&\leq 0.
\end{aligned} \tag{3.16}$$

Now, from (3.1)and (3.4) , we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 \\
&\leq \beta_n\|y_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\
&\leq \|y_n - z\|^2 \\
&\leq \|(1-\alpha_n)x_n - \gamma_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\|^2 \\
&= \|(1-\alpha_n)\left(x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\right) + \alpha_n z\|^2 \\
&= (1-\alpha_n)^2\|x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z\|^2 + \alpha_n^2\|z\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle x_n - \frac{\gamma_n}{(1-\alpha_n)}A^*(I - \operatorname{prox}_{\lambda g})Ax_n - z, -z\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n^2\|z\|^2 + 2\alpha_n(1-\alpha_n)\langle x_n - z, -z\rangle \\
&\quad - 2\alpha_n\gamma_n\langle A^*(I - \operatorname{prox}_{\lambda g})Ax, -z\rangle \\
&= (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n^2\|z\|^2 + 2\alpha_n(1-\alpha_n)\langle x_n - z, -z\rangle \\
&\quad + 2\alpha_n\gamma_n\langle \nabla h(x_n), z\rangle \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\big(\alpha_n\|z\|^2 + 2(1-\alpha_n)\langle x_n - z, -z\rangle \\
&\quad + 2\gamma_n\|\nabla h(x_n)\|\|z\|\big).
\end{aligned} \tag{3.17}$$

Since $\nabla h(x_n)$ is Lipschitz continuous with Lipschitzian constant $\|A\|^2$and $\nabla l(x_n)$ is nonexpansive, $\nabla h(x_n)$, $\nabla l(x_n)$, and $\theta^2(x_n)$ are bounded. From the condition

(C1), (3.16), (3.17) and Lemma 2.4, we can conclude that the sequence $\{x_n\}$ converges strongly to z.

CASE 2. Assume that $\{\|x_n - z\|\}$ is not monotonically decreasing sequence. Then there exists a subsequence n_k of n such that $\|x_{n_k} - \bar{x}\| < \|x_{n_k+1} - \bar{x}\|$ for all $k \in \mathbb{N}$. Now we define a positive interger sequence $\tau(n)$ by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \|x_{n_k} - \bar{x}\| < \|x_{n_k+1} - \bar{x}\|\}.$$

for all $n \geq n_0$ (for some n_0 large enough). By lemma 2.5, we have τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\tau(n)} - \bar{x}\|^2 - \|x_{\tau(n)+1} - \bar{x}\|^2 \leq 0, \forall n \geq n_0.$$

By continuing in the same direction as in CASE 1, we can show that

$$\rho_{\tau(n)} \left(\frac{4h(x_{\tau(n)})}{(h(x_{\tau(n)}) + l(x_{\tau(n)}))} - \frac{\rho_{\tau(n)}}{1 - \alpha_{\tau(n)}} \right) \left(\frac{(h(x_{\tau(n)}) + l(x_{\tau(n)}))^2}{\theta^2(x_{\tau(n)})} \right) \to 0 \text{ as } n \to \infty.$$

Hence, we have

$$\frac{(h(x_{\tau(n)}) + l(x_{\tau(n)}))^2}{\theta^2(x_{\tau(n)})} \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Consequently, we have

$$\lim_{n\to\infty} ((h(x_{\tau(n)}) + l(x_{\tau(n)}))^2) = 0.$$

It implies that

$$\lim_{n\to\infty} h(x_{\tau(n)}) = \lim_{n\to\infty} l(x_{\tau(n)}) = 0.$$

Moreover, By continuing in the same direction as in Case 1, we can prove that

$$\limsup_{n\to\infty} \left\langle -z, x_{\tau(n)} - z \right\rangle \leq 0.$$

From (3.17), we have

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - z\|^2 - \|x_{\tau(n)} - z\|^2 \\ &\leq (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \alpha_{\tau(n)}\rho_{\tau(n)} - \|x_{\tau(n)} - z\|^2 \\ &= \alpha_{\tau(n)}(\rho_{\tau(n)} - \|x_{\tau(n)} - z\|^2). \end{aligned}$$

It follows that

$$\|x_{\tau(n)} - z\|^2 \leq \rho_{\tau(n)},$$

where $\rho_{\tau(n)} = \alpha_{\tau(n)}\|z\|^2 + 2(1 - \alpha_{\tau(n)})\langle x_{\tau(n)} - z, -z\rangle + 2\gamma_{\tau(n)}\|\nabla h(x_{\tau(n)})\|\|z\|$. By using Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0.$$

It follows from Lemma 2.5 that

$$0 \leq \|x_{\tau(n)} - \bar{x}\| \leq \|x_{\tau(n)+1} - \bar{x}\| \to 0$$

as $n \to \infty$. Hence $\{x_n\}$ converges strongly to z. This completes the proof.

□

As a direct proof of Theorem 3.1, we obtain the following results.

When $f = \delta_C$ and $g = \delta_Q$ are indicator functions of nonempty, closed, and convex sets C and Q of H_1 and H_2, respectively, then SMP (1.3) reduces to the split feasibility problem (1.1). In this case, we obtain the following results.

Algorithm 3.2

Step 1: Choose an initial point $x_1 \in H_1$.

Step 2: Assume that x_n has been constructed.
Set $h(x_n) := \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ with $\|\nabla h(x_n)\| \neq 0$. We compute x_{n+1} in the following iterative scheme:

$$\begin{cases} y_n = P_C((1-\alpha_n)x_n - \gamma_n A^*(I - P_Q)Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)\displaystyle\sum_{i=1}^{N} a_i T_i y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.19}$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n)}{\|\nabla h(x_n)\|^2}$ with $0 < \rho_n < 4$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$, and $0 \leq a_i \leq 1$, for every $i = 1, 2, ..., N$ with $\displaystyle\sum_{i=1}^{N} a_i = 1$.

Using algorithm 3.2, we prove a strong convergence theorem for approximation of solutions of problem (1.1) and the set of fixed points of quasi-nonexpansive mappings as follows:

Corollary 3.1. *Suppose that $\Omega := \Psi \cap \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$. If the parameters satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\varepsilon \leq \rho_n \leq 4(1-\alpha_n) - \varepsilon$ *for some* $\varepsilon > 0$*.*

Then the sequence $\{x_n\}$ converges strongly to a solution z which is also a minimum norm solution of Ω. In other words, $z = P_\Omega(0)$.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to \mathbb{R}\cup\{+\infty\}$ and $g : H_2 \to \mathbb{R}\cup\{+\infty\}$ be two proper and lower semicontinuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $T : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $Fix(T) \neq \emptyset$ and $I - T$ are demiclosed at zero. Suppose that $\Omega := \Gamma \cap Fix(T) \neq \emptyset$. Set $\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$ where $h(x) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g})Ax\|^2$ and $l(x) := \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f})x\|^2$ with $\theta(x) \neq 0$ for each $n \geq 1$. For given $x_1 \in H_1$ and let $\{x_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{cases} y_n = \mathrm{prox}_{\lambda\gamma_n f}((1-\alpha_n)x_n - \gamma_n A^*(I - \mathrm{prox}_{\lambda g})Ax_n) \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.20}$$

where stepsize $\gamma_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$, and $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$. If the parameters satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(C3) $\varepsilon \leq \rho_n \leq \dfrac{4(1-\alpha_n)h(x_n)}{h(x_n) + l(x_n)} - \varepsilon$ *for some* $\varepsilon > 0$.

Then the sequence $\{x_n\}$ converges strongly to a solution z which is also a minimum norm solution of Ω. In other words, $z = P_\Omega(0)$.

Proof. Take $T = T_i$ for all $i = 1, 2, 3, ..., N$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. □

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Existence and Convergence Theorem for Fixed Point Problem of Various Nonlinear Mappings and Variational Inequality Problems without Some Assumptions

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Abstract. The purpose of this article, we give a necessary and sufficient condition for the modified Mann iterative process in order to obtain a strong convergence theorem for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and variational inequality problem in Hilbert space without the conditions $\bigcap_{i=1}^{N} Fix(T_i) \cap VI(C,A) \neq \emptyset$. Moreover, we utilize our main result to fixed point problems of strictly pseudocontractive mappings and the set of solutions of variational inequality problem.

1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$.
A mapping T of C into itself is called *nonexpansive* if

$$\left\|Tx - Ty\right\| \leq \left\|x - y\right\|, \forall x, y \in C.$$

Mann's iteration process [8] is often used to approximate a fixed point of a nonexpansive mapping. But Mann's iteration process has only weak convergence. To obtain strong convergence theorems, the Mann's iteration is modified by many researchers; see for instance [7], [12], and the references therein.

Let $A : C \to H$. *The variational inequality problem* is to find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \tag{1}$$

for all $y \in C$. The set of solution of (1) is denoted by $VI(C, A)$. In 1964, Stampacchia [13] introduced and investigated the variational inequality problem. It is well known that the application of the variational inequality problem has been expanded to problems from economics, finance, optimization and game theory; see [15]. Several authors have studied the variational inequality problem; see [16], [3], [4], and references cited therein.

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In 2003, Takahashi and Toyoda [5] introduce an iterative scheme of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an inverse strongly-monotone mapping as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n A x_n), \forall n \geq 1,$$

where $T : C \to C$ is a nonexpansive mapping and A is an a inverse strongly-monotone mapping of C into H. Then, they proved a weak convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2013, Kangtunyakarn [6] proved a strong convergence theorem for finding a common element of the set of fixed point problem of a nonexpansive mapping and the set of solution of (1) without assumption $Fix(T) \cap VI(C, A) \neq \emptyset$. He defined the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha T x_n + (1-\alpha)P_C(I - \rho A)x_n, \forall n \geq 1, \tag{2}$$

where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and positive real numbers α, ρ. In the last few decades many authors have studied strong convergence theorems for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and the set of variational inequality problem by using condition $\bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A) \neq \emptyset$; see for instance [11] and references therein.

In this paper, motivated and inspired by [5] and [6], we give a necessary and sufficient condition for the modified Mann iterative process in order to obtain a strong convergence theorem for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and the set of solutions of variational inequality problem in Hilbert space without the conditions $\bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A) \neq \emptyset$. Moreover, we utilize our main result to fixed point problems of strictly pseudocontractive mappings and the set of solutions of variational inequality problem.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.1 ([9]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.2 (See [14]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.3 ([1]). *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and S: $C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself. Let the sequence $\{x_n\}$ be generated by $x_0 \in H$ and*

$$\begin{cases} y_n^i = \beta T_i x_n + (1-\beta)x_n, \\ x_{n+1} = \alpha x_n + (1-\alpha)P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i, \forall n \geq 0, \end{cases} \tag{3}$$

where $0 < \alpha, \beta < 1$, $0 < \rho < \|A\|^{-1}$, *and* $\sum_{i=1}^{N} a^i = 1$. *Then the following are equivalent.*

(i) *The sequence* $\{x_n\}$ *defined by (3) converges strongly to* $x^* \in \bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A)$;

(ii) $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, *for all* $i = 1, 2, \ldots, N$.

Proof. $(i) \Rightarrow (ii)$. *Let condition* (i) *hold. Since* $x^* \in \bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A)$, *we have*

$$\|T_i x_n - x_n\| \leq \|T_i x_n - T_i x^*\| + \|x^* - x_n\| \leq 2\|x^* - x_n\|,$$

which implies that $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$.
Next we claim that $(ii) \Rightarrow (i)$, *let condition* (ii) *hold. Let* $x, y \in C$. *Since* A *is a strongly positive linear bounded operator and Lemma 2.1 , we have*

$$\begin{aligned}\|(I - \rho A)x - (I - \rho A)y\| &= \|(I - \rho A)(x - y)\| \\ &\leq (1 - \rho\bar{\gamma})\|x - y\|.\end{aligned}$$

We have $I - \rho A$ *is a contractive mapping with coefficient* $1 - \rho\bar{\gamma}$.
For every $n \in \mathbb{N}$, $i = 1, 2, ..., N$, *and the definition of* $\{y_n^i\}$, *we have*

$$\begin{aligned}\|y_{n+1}^i - y_n^i\| &= \|\beta T_i x_n + (1 - \beta)x_n - \beta T_i x_{n-1} - (1 - \beta)x_{n-1}\| \\ &= \|\beta(T_i x_n - T_i x_{n-1}) + (1 - \beta)(x_n - x_{n-1})\| \\ &\leq \beta\|T_i x_n - T_i x_{n-1}\| + (1 - \beta)\|x_n - x_{n-1}\| \\ &\leq \beta\|x_n - x_{n-1}\| + (1 - \beta)\|x_n - x_{n-1}\| \\ &= \|x_n - x_{n-1}\|. \end{aligned} \tag{4}$$

From the definition of $\{x_n\}$ *and (4), we have*

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha x_n + (1 - \alpha)P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - \alpha x_{n-1} - (1 - \alpha)P_C(I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\| \\ &= \|\alpha(x_n - x_{n-1}) + (1 - \alpha)\left(P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - P_C(I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\right)\| \\ &\leq \alpha\|x_n - x_{n-1}\| + (1 - \alpha)\|P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - P_C(I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\| \\ &\leq \alpha\|x_n - x_{n-1}\| + (1 - \alpha)\|(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - (I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\| \\ &\leq \alpha\|x_n - x_{n-1}\| + (1 - \alpha)(1 - \rho\bar{\gamma})\sum_{i=1}^{N} a^i\|y_n^i - y_{n-1}^i\| \\ &\leq \alpha\|x_n - x_{n-1}\| + (1 - \alpha)(1 - \rho\bar{\gamma})\|x_n - x_{n-1}\| \\ &= (1 - \rho\bar{\gamma}(1 - \alpha))\|x_n - x_{n-1}\| \\ &= a\|x_n - x_{n-1}\| \\ &\leq a^2\|x_{n-1} - x_{n-2}\| \\ &\;\vdots \\ &\leq a^n\|x_1 - x_0\|, \end{aligned} \tag{5}$$

where $a = (1 - \rho\bar{\gamma}(1-\alpha)) \in (0,1)$.
For any number $n, m \in \mathbb{N}$ *and (5), we have*

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=n}^{n+m-1} a^j \|x_1 - x_0\| \\ &\leq \left(\frac{a^n}{1-a}\right)\|x_1 - x_0\|. \end{aligned} \tag{6}$$

Since $a^n \to 0$ *as* $n \to \infty$, *and (6), we have* $\{x_n\}$ *is a Cauchy sequence. Since* H *is a Hilbert space, we get* $\{x_n\}$ *converges to* x^*, *i.e.,*

$$\lim_{n\to\infty} x_n = x^*. \tag{7}$$

Next, we will show that $x^* \in \bigcap_{i=1}^N Fix(T_i) \cap VI(C,A)$. *Since* C *is closed, so we get* $x^* \in C$. *By* $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, *(7), and Lemma 2.3, we have* $x^* \in Fix(T_i)$ *for all* $i = 1, 2, \ldots, N$. *It implies that* $x^* \in \bigcap_{i=1}^N Fix(T_i)$. *From the definition of* y_n^i, $\lim_{n\to\infty} x_n = x^*$, *and* $x^* \in \bigcap_{i=1}^N Fix(T_i)$, *we have*

$$\lim_{n\to\infty} y_n^i = x^*. \tag{8}$$

From the definition of x_n, *(7), and (8), we have*

$$x^* = \alpha x^* + (1-\alpha)P_C(I - \rho A)x^*.$$

It implies that $x^* \in Fix(P_C(I - \rho A))$. *From Lemma 2.2, we have* $x^* \in VI(C,A)$. *Hence, the sequence* $\{x_n\}$ *defined by (3) converges strongly to* $x^* \in \bigcap_{i=1}^N Fix(T_i) \cap VI(C,A)$. □

As direct proof of Theorem 3.1, we obtain the following results.

Corollary 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$. *Let* T *be a nonexpansive mappings of* C *into itself. Let the sequence* $\{x_n\}$ *be generated by* $x_0 \in H$ *and*

$$\begin{cases} y_n = \beta T x_n + (1-\beta)x_n, \\ x_{n+1} = \alpha x_n + (1-\alpha)P_C(I - \rho A)y_n, \forall n \geq 0, \end{cases} \tag{9}$$

where $0 < \alpha, \beta < 1$ *and* $0 < \rho < \|A\|^{-1}$. *Then the following are equivalent.*

(i) *The sequence* $\{x_n\}$ *defined by (9) converges strongly to* $x^* \in Fix(T) \cap VI(C,A)$;

(ii) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Next, in order to prove a strong convergence theorem for κ-strictly pseudo-contractive mappings and variational inequality problem, we need Lemma 3.3. A mapping $T : C \to C$ is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0,1)$ such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \kappa \left\|(I - T)x - (I - T)y\right\|^2,$$

for all $x, y \in C$. Note that the class of strictly pseudo-contractions strictly includes the class of nonexpansive mapping.

Lemma 3.3 (See [2]). *Let $T : C \to H$ be a κ-strict pseudo-contraction. Define $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $Fix(S) = Fix(T)$.*

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mappings of C into itself with $\kappa = \max_{i=1,2,\dots,N} \kappa_i$. Define the mapping $S_i = C \to C$ by $S_i x = \sigma x + (1-\sigma)T_i x$ for every $i = 1, 2, ..., N$, $x \in C$ and $\sigma \in (k, 1)$. Let the sequence $\{x_n\}$ be generated by $x_0 \in H$ and*

$$\begin{cases} y_n^i = \beta S_i x_n + (1-\beta)x_n, \\ x_{n+1} = \alpha x_n + (1-\alpha)P_C(I-\rho A)\displaystyle\sum_{i=1}^N a^i y_n^i, \forall n \geq 0, \end{cases} \tag{10}$$

where $0 < \alpha < 1$, $\kappa \leq \beta < 1$, $0 < \rho < \|A\|^{-1}$, and $\sum_{i=1}^N a^i = 1$. Then the following are equivalent.

(i) *The sequence $\{x_n\}$ defined by (10) converges strongly to $x^* \in \bigcap_{i=1}^N Fix(T_i) \cap VI(C, A)$;*

(ii) $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, *for all $i = 1, 2, \dots, N$.*

Proof. *From Lemma 3.3 and Theorem 3.1, we obtain the desired result.* □

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Iterative algorithms with the regularization for the constrained convex minimization problem and maximal monotone operators

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Iterative algorithms with the regularization for the constrained convex minimization problem and maximal monotone operators

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ABSTRACT
In this paper, we prove a strong convergence theorem for finding a common element of the solution set of a constrained convex minimization problem and the set of solutions of a finite family of variational inclusion problems in Hilbert space. A strong convergence theorem for finding a common element of the solution set of a constrained convex minimization problem and the solution sets of a finite family of zero points of the maximal monotone operator problem in Hilbert space is also obtained. Using our main result, we have some additional results for various types of non-linear problems in Hilbert space.

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1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $g : C \to \mathbb{R}$ be a real-valued convex function. Consider the following constrained convex minimization problem:

$$\min_{x \in C} g(x). \tag{1.1}$$

Assume that the constrained convex minimization problem (1.1) is solvable, that is, it has a solution, and let Ω denote the solution set of (1.1). If g is Frechet differentiable, then $x_0 \in C$ solves (1.1) if and only if $x_0 \in C$ satisfies the following optimality condition:

$$\langle \nabla g(x_0), x - x_0 \rangle \geq 0, \forall x \in C. \tag{1.2}$$

where ∇g denotes the gradient of g. Observe that (1.1) can be rewritten as follows:

$$\langle x_0 - (x_0 - \nabla g(x_0)), x - x_0 \rangle \geq 0, \forall x \in C. \tag{1.3}$$

From (1.3), it is easy to show that the constrained convex minimization problem (1.1) is equivalent to the fixed point problem

$$P_C(x_0 - \beta \nabla g(x_0)) = x_0. \tag{1.4}$$

where $\beta > 0$ is any constant and P_C is the metric projection from H onto C. The gradient projection algorithm (GPA) generates a sequence $\{x_n\}_{n=0}^{\infty}$ using the following the recursion:

$$x_{n+1} = P_C(I - \beta_n \nabla g)x_n, \tag{1.5}$$

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where the parameters β_n are real positive numbers. It is well known that the GPA is a powerful tool for solving constrained convex optimization problems (1.1). Many research papers have increasingly investigated these problems, see, for instance, [1–4] and the references therein.

However, we all know that the minimization problem (1.1) has more than one solution under some conditions, so regularization is needed in finding the unique solution of the minimization problem (1.1). Now, we consider the following regularized minimization problem:

$$\min_{x\in C} g_\alpha(x) = g(x) + \frac{\alpha}{2}\|x\|^2, \tag{1.6}$$

where $\alpha > 0$ is the regularization parameter and g is a convex function with a $1/L$-ism continuous gradient ∇g. The regularized gradient projection algorithm (RGPA) generates a sequence $\{x_n\}_{n=0}^{\infty}$ using the following the recursion:

$$x_{n+1} = P_C(I - \beta\nabla g_{\lambda_n})x_n = P_C(I - \beta(\nabla g + \alpha_n I))x_n, \tag{1.7}$$

where the parameter $\alpha_n > 0$, β is a constant with $0 < \beta < 2/L$. Many authors have extensively studied a strong convergence theorem based on the RGPA under some control conditions, see, for instance, [5,6] and references therein.

A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

A mapping $A : C \to H$ is called α*-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Let $A : C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.8}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$.

Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The *variational inclusion problem* is to find $u \in H$ such that

$$\theta \in Bu + Mu, \tag{1.9}$$

where θ is zero vector in H. The set of the solution of (1.9) is denoted by $VI(H, B, M)$.

A multi-valued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $\langle u - v, x - y\rangle \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \text{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping and the single-valued mapping $J_{M,\lambda} : H \to H$ be defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \forall u \in H,$$

which is called the *resolvent operator* associated with M where λ is a positive number and I is an identity mapping, see [7].

Moreover, let M be a maximal monotone operator on H and define the set of zero points of M as follows:

$$M^{-1}0 = \{x \in H : 0 \in Mu\}. \tag{1.10}$$

It is well known that $M^{-1}0 = Fix(J_{M,\lambda})$. Many research papers have increasingly investigated these problems, see, for instance, [8,9] and the references therein.

In 2015, Tian and Jiao [10] introduced a new iterative algorithm for finding a common element of the solution set of a constrained convex minimization problem and the set of zero points of the maximal monotone operator problem based on the viscosity approximation method and the regularized gradient projection algorithm as follows: Find $x_1 \in C$ and

$$\begin{cases} u_n = J_{M,r_n}(x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_{\lambda_n}(u_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.11}$$

where $P_C(I - \beta\nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\lambda_n \subset (0, \frac{2}{\rho - L})$, $\beta \in (0, \frac{2}{L})$. Under appropriate conditions, they proved that the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.11) converges strongly to a point $q = \Omega \cap M^{-1}0$ which solves the variational inequality

$$\langle (I-f)q, q-z\rangle \leq 0, \forall z \in \Omega \cap M^{-1}0. \tag{1.12}$$

Equivalently, we have $P_{\Omega\cap M^{-1}0} f(q) = q$.

In 2015, Qiu et al. [11] introduced two families of finite maximal monotone mappings by $S_{r_n}^{A_m A_{m-1}\cdots A_1} := J_{A_m,r_n} J_{A_{m-1},r_n}\cdots J_{A_1,r_n}$ and $T_{r_n} := a_0 I + a_1 J_{B_1,r_n} + a_2 J_{B_2,r_n} + \cdots + a_l J_{B_l,r_n}$, where $J_{A_i,r_n} = (I + r_n A_i)^{-1}$, $(i = 1,2,\ldots,m)$, $J_{B_j,r_n} = (I + r_n B_j)^{-1}$, $(j = 1,2,\ldots,l)$, $a_k \in (0,1), k = 1,2,\ldots,l$, $\sum_{i=0}^{N} a_k = 1$ and proved a strong convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping and the set of common zeros of two families of finite maximal monotone mappings in a real Hilbert space.

In 2008, Zhang et al. [7] introduced an iterative scheme for finding a common element of the set of solutions of the variational inclusion problem with multi-valued maximal monotone mapping and inverse-strongly monotone mappings and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha_n x + (1-\alpha_n) S y_n, \forall n \geq 0, \end{aligned}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and λ.

Very recently, Khuangsatung and Kangtunyakarn [12] have modified (1.9) as follows: Find $u \in H$ such that

$$\theta \in \sum_{i=1}^{N} a_i A_i u + M u, \tag{1.13}$$

where $A_i : H \to H$ is a single-valued mapping, $M : H \to 2^H$ is a multi-valued mapping, $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$ and θ is a zero vector, for all $i = 1,2,\ldots,N$. This problem is called *the modified variational inclusion*. The set of solutions (1.13) is denoted by $VI\left(H, \sum_{i=1}^{N} a_i A_i, M\right)$. If $A_i \equiv A$ for all $i = 1,2,\ldots,N$, then (1.13) reduces to (1.9). They also introduced an iterative scheme for finding a common element of the set of fixed points of a κ-strictly pseudononspreading mapping and the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems as follows:

$$\begin{cases} \sum_{i=1}^{N} a_i F_i\left(u_n, y\right) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{M,\lambda}\left(I - \lambda \sum_{i=1}^{N} b_i A_i\right) x_n \\ \qquad + \eta_n(I - \rho_n(I - T))x_n + \delta_n u_n, \forall n \geq 1, \end{cases} \tag{1.14}$$

where $F_i : C \times C \to \mathbb{R}$ is a bifunction satisfying $(A1)$-$(A4)$. Then, they proved that the sequence $\{x_n\}$ generated by (1.14) converges strongly to an element of a set $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$, under some appropriate conditions of $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}$ and $\{\delta_n\}$.

In this paper, motivated by the above-mentioned results, we first prove a strong convergence theorem for finding a common element of the solution set of a constrained convex minimization problem and the set of solutions of a finite family of variational inclusion problems in Hilbert space. Secondly, A strong convergence theorem for finding a common element of the solution set of a constrained convex minimization problem and the solution sets of a finite family of zero points of the maximal monotone operator problem in Hilbert space is also obtained. Using our main result, we prove a strong convergence theorem involving a finite family of equilibrium problems in Hilbert space. Moreover, we utilize our main theorem to prove a strong convergence theorem for a finite family of κ-strictly pseudo-contractive mappings and the constrained convex minimization problem in Hilbert space. In the last section, we give the numerical example to support some of our results.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak and strong convergence by notations "$\rightharpoonup$" and "$\rightarrow$", respectively. In a real Hilbert space H, it is well known that

$$\left\|\alpha x + (1-\alpha)y\right\|^2 = \alpha \left\|x\right\|^2 + (1-\alpha)\left\|y\right\|^2 - \alpha(1-\alpha)\left\|x-y\right\|^2,$$

for all $x, y \in H$ and $\alpha \in [0,1]$. Recall that H satisfies *Opial's condition*[13], i.e. for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \left\|x_n - y\right\|,$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1: *Let H be a real Hilbert space. Then, the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.2 [14]: *Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \geq 0, \forall z \in C.$$

Lemma 2.3 [15]: *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4 [16]: *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 [17]**:** *Let* $\{x_n\}$ *and* $\{z_n\}$ *be the bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ *for all integer* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0$. *Then,* $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Lemma 2.6 [18]**:** *Let* E *be a uniformly convex Banach space,* C *be a nonempty closed convex subset of* E *and* $S: C \to C$ *be a nonexpansive mapping. Then,* $I - S$ *is demi-closed at zero.*

Lemma 2.7 [7]**:** $u \in H$ *is a solution of variational inclusion* (1.9) *if and only if* $u = J_{M,\lambda}(u - \lambda Bu), \forall\lambda > 0$, *i.e.*

$$VI(H,B,M) = Fix(J_{M,\lambda}(I - \lambda B)), \forall\lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, *then* $VI(H,B,M)$ *is closed convex subset in* H.

Lemma 2.8 [7]**:** *The resolvent operator* $J_{M,\lambda}$ *associated with* M *is single-valued, nonexpansive for all* $\lambda > 0$ *and 1-inverse-strongly monotone.*

Lemma 2.9 [12]**:** *Let* H *be a real Hilbert space and let* $M : H \to 2^H$ *be a multi-valued maximal monotone mapping. For every* $i = 1,2,\dots,N$, *let* $A_i : H \to H$ *be* α_i*-inverse strongly monotone mapping with* $\eta = min_{i=1,2,\dots,N}\{\alpha_i\}$ *and* $\bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. *Then,*

$$VI\left(H, \sum_{i=1}^{N} a_i A_i, M\right) = \bigcap_{i=1}^{N} VI(H, A_i, M),$$

where $\sum_{i=1}^{N} a_i = 1$, *and* $0 < a_i < 1$ *for every* $i = 1,2,\dots,N$. *Moreover, we have* $J_{M,\lambda}(I - \lambda\sum_{i=1}^{N} a_i A_i)$ *is a nonexpansive mapping, for all* $0 < \lambda < 2\eta$.

Lemma 2.10 [9]**:** *Let* H *be a real Hilbert space,* C *be a nonempty closed and convex subset of* H, *and let* $A_i, B_j : C \to C (i = 1,2,\dots,m; j = 1,2,\dots,l)$ *be two families of finite maximal monotone mappings such that* $\mathcal{D} := \bigcap_{i=1}^{m} A^{-1}0 \cap \bigcap_{j=1}^{l} B^{-1}0 \neq \emptyset$. *Suppose* $S_{r_n}^{A_m A_{m-1}\cdots A_1} := J_{A_m,r_n} J_{A_{m-1},r_n} \cdots J_{A_1,r_n}$ *and* $W_{r_n} := a_0 I + a_1 J_{B_1,r_n} + a_2 J_{B_2,r_n} + \cdots + a_l J_{B_l,r_n}$, *where* $J_{A_i,r_n} = (I + r_n A_i)^{-1}$, $(i = 1,2,\dots,m)$, $J_{B_j,r_n} = (I + r_n B_j)^{-1}$, $(j = 1,2,\dots,l)$, $a_k \in (0,1), k = 1,2,\dots,l$, $\sum_{i=0}^{N} a_k = 1$, *and* $r_n > 0$. *Then,* $S_{r_n}^{A_m A_{m-1}\cdots A_1} : C \to C$ *and* $W_{r_n} : C \to C$ *are nonexpansive.*

Lemma 2.11 [9]**:** *Let* H, C, $A_i, B_j : C \to C (i = 1,2,\dots,m; j = 1,2,\dots,l)$, $S_{r_n}^{A_m A_{m-1}\cdots A_1}$ *and* W_{r_n} *be the same as those in Lemma* 2.10. *Suppose that* $\mathcal{D} \neq \emptyset$. *Then,* $Fix(S_{r_n}^{A_m A_{m-1}\cdots A_1}) = \bigcap_{i=1}^{m} A^{-1}0$ *and* $Fix(W_{r_n}) = \bigcap_{j=1}^{l} B^{-1}0$, *for* $\forall r > 0$.

3. Main result

Theorem 3.1: *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $M : H \to 2^H$ *be a multi-valued maximal monotone mapping with* $\mathbb{D}(M) = C$ *and* $A_i : C \to H$ *be* α_i*-inverse strongly monotone mapping with* $\eta = min_{i=1,2,\dots,N}\{\alpha_i\}$. *Let* g *be a real-valued convex function of* C *into* $\mathbb{R}$, *and the gradient* ∇g *is* $1/L$*-ism continuous with* $L > 0$, *let* $D : C \to H$ *be a strongly positive bounded linear operator with coefficient* $0 < \bar{\gamma} < 1$ *and let* $f : C \to C$ *be a contractive mapping with* $\alpha \in (0,1)$ *and* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. *Assume that* $\mathcal{F} := \Omega \cap \bigcap_{i=1}^{N} VI(H, A_i, M) \neq \emptyset$. *Suppose that the sequence*

$\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} u_n = J_{M,\mu}\left(I - \mu \sum_{i=1}^{N} a_i A_i\right) x_n \\ x_{n+1} = P_C\left(\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)T_{\lambda_n}(u_n)\right), \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $P_C(I - \rho \nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\lambda_n \subset (0, \frac{2}{\rho} - L)$, $\rho \in (0, \frac{2}{L})$, $\sum_{i=1}^{N} a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(iv) $0 < \mu < 2\eta$, *where* $\eta = min_{i=1,2,\ldots,N}\{\alpha_i\}$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathcal{F}$, which solves uniquely the following variational inequality

$$\langle (D - \gamma f)z, z - x^* \rangle \le 0, \forall x^* \in \mathcal{F}. \tag{3.2}$$

Equivalently, we have $P_{\mathcal{F}}(I - D + \gamma f)z = z$.

***Proof*:** Without loss of generality, by conditions (i) and (ii), we have $\alpha_n \le (1-\beta_n)\|D\|^{-1}$. Since D is a strongly positive linear-bounded self-adjoint operator, it's easy to see that

$$\|(1-\beta_n)I - \alpha_n D\| \le 1 - \beta_n - \alpha_n \bar{\gamma}.$$

It is clear that $\tilde{x} \in C$ solves the minimization problem (1.1) if and only if for each fixed $0 < \rho < 2/L$, $\tilde{x}$ solves the fixed point equation

$$\tilde{x} = P_C(I - \rho \nabla g)\tilde{x} = T\tilde{x},$$

and $\tilde{x} = T\tilde{x}$, that is, $\tilde{x} \in \Omega = Fix(T)$. It is easy to prove that g_{λ_n} is $\frac{1}{L+\lambda_n}$-inverse strongly monotone and T_{λ_n} is nonexpansive mapping (See[10]).

Now, we divide the proof 3.1 into five steps:

Step 1 We show that the sequence $\{x_n\}$ is bounded.

Let $x^* \in \mathcal{F}$. From Lemmas 2.7 and 2.9, we have

$$x^* = J_{M,\mu}\left(I - \mu \sum_{i=1}^{N} a_i A_i\right) x^*.$$

From the nonexpansiveness of $J_{M,\mu}(I - \mu \sum_{i=1}^{N} a_i A_i)$, we have

$$\|u_n - x^*\| = \|J_{M,\mu}\left(I - \mu \sum_{i=1}^{N} a_i A_i\right) x_n - x^*\| \le \|x_n - x^*\|. \tag{3.3}$$

For $x \in C$, note that

$$P_C\left(I - \rho \nabla g_{\lambda_n}\right) x = T_{\lambda_n} x$$

and

$$P_C(I - \rho \nabla g)x = Tx.$$

It follows that

$$\begin{aligned}\|T_{\lambda_n}x - Tx.\| &= \|P_C(I-\rho\nabla g_{\lambda_n})x - P_C(I-\rho\nabla g)x\| \\ &\le \|(I-\rho\nabla g_{\lambda_n})x - (I-\rho\nabla g)x\| \\ &= \rho\|\nabla g_{\lambda_n}(x) - \nabla g(x)\| \\ &= \rho\|\nabla g(x) + \lambda_n x - \nabla g(x)\| \\ &= \lambda_n\rho\|x\|. \end{aligned} \tag{3.4}$$

From the definition of x_n, (3.3) and (3.4), we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|P_C\left(\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n D)T_{\lambda_n}u_n\right) - P_C x^*\| \\ &\le \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n D)T_{\lambda_n}u_n - x^*\| \\ &= \|\alpha_n(\gamma f(x_n)-Dx^*)+\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}u_n-x^*)\| \\ &\le \alpha_n\|\gamma f(x_n)-Dx^*\|+\beta_n\|x_n-x^*\|+\|((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}u_n-x^*)\| \\ &\le \alpha_n\left(\|\gamma f(x_n)-\gamma f(x^*)\|+\|\gamma f(x^*)-Dx^*\|\right)+\beta_n\|x_n-x^*\| \\ &\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|T_{\lambda_n}u_n-x^*\| \\ &\le \alpha_n\left(\gamma\theta\|x_n-x^*\|+\|\gamma f(x^*)-Dx^*\|\right)+\beta_n\|x_n-x^*\| \\ &\quad + (1-\beta_n-\alpha_n\bar{\gamma})\left(\|T_{\lambda_n}u_n-T_{\lambda_n}x^*\|+\|T_{\lambda_n}x^*-x^*\|\right) \\ &\le \alpha_n\gamma\theta\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Dx^*\|+\beta_n\|x_n-x^*\| \\ &\quad + (1-\beta_n-\alpha_n\bar{\gamma})\left(\|u_n-x^*\|+\|T_{\lambda_n}x^*-x^*\|\right) \\ &\le \alpha_n\gamma\theta\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Dx^*\|+\beta_n\|x_n-x^*\| \\ &\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-x^*\|+(1-\beta_n-\alpha_n\bar{\gamma})\lambda_n\rho\|x^*\| \\ &\le \alpha_n\gamma\theta\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Dx^*\|+\beta_n\|x_n-x^*\| \\ &\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-x^*\|+\lambda_n\rho\|x^*\| \\ &= (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Dx^*\|+\lambda_n\rho\|x^*\| \\ &= (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n-x^*\|+\alpha_n(\bar{\gamma}-\gamma\theta)\left(\frac{\|\gamma f(x^*)-Dx^*\|}{\bar{\gamma}-\gamma\theta}+\frac{\lambda_n\rho\|x^*\|}{\alpha_n(\bar{\gamma}-\gamma\theta)}\right).\end{aligned}$$

Since $\lambda_n = o(\alpha_n)$, there exists a real number $M > 0$ such that $\frac{\lambda_n}{\alpha_n} \le M$, then

$$\begin{aligned}\|x_{n+1}-x^*\| &\le (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n-x^*\|+\alpha_n(\bar{\gamma}-\gamma\theta)\left(\frac{\|\gamma f(x^*)-Dx^*\|}{\bar{\gamma}-\gamma\theta}+\frac{M\rho\|x^*\|}{\bar{\gamma}-\gamma\theta}\right) \\ &\le \max\left\{\|x_1-x^*\|, \frac{1}{\bar{\gamma}-\gamma\theta}\left(\|\gamma f(x^*)-Dx^*\|+M\rho\|x^*\|\right)\right\}.\end{aligned}$$

Put $K = \max\left\{\|x_1-x^*\|, \frac{1}{\bar{\gamma}-\gamma\theta}\left(\|\gamma f(x^*)-Dx^*\|+M\rho\|x^*\|\right)\right\}$. By mathematical induction, we have $\|x_n-x^*\| \le K, \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2 We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. From the nonexpansiveness of $J_{M,\mu}(I-\mu\sum_{i=1}^N a_iA_i)$, we have

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|J_{M,\mu}\left(I-\mu\sum_{i=1}^N a_iA_i\right)x_{n+1} - J_{M,\mu}\left(I-\mu\sum_{i=1}^N a_iA_i\right)x_n\| \\ &\le \|x_{n+1}-x_n\|.\end{aligned} \tag{3.5}$$

Putting $y_n = T_{\lambda_n} u_n$. From the nonexpansiveness of T_{λ_n} and (3.5), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|T_{\lambda_n} u_n - T_{\lambda_{n-1}} u_{n-1}\| \\
&\le \|T_{\lambda_n} u_n - T_{\lambda_n} u_{n-1}\| + \|T_{\lambda_n} u_{n-1} - T_{\lambda_{n-1}} u_{n-1}\| \\
&\le \|u_n - u_{n-1}\| + \|T_{\lambda_n} u_{n-1} - T_{\lambda_{n-1}} u_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \|P_C(I - \rho \nabla g_{\lambda_n}) u_{n-1} - P_C(I - \rho \nabla g_{\lambda_{n-1}}) u_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \|(I - \rho \nabla g_{\lambda_n}) u_{n-1} - (I - \rho \nabla g_{\lambda_{n-1}}) u_{n-1}\| \\
&= \|x_n - x_{n-1}\| + \rho \| - (\nabla g(u_{n-1}) + \lambda_n u_{n-1}) + \nabla g(u_{n-1}) + \lambda_{n-1} u_{n-1}\| \\
&= \|x_n - x_{n-1}\| + \rho |\lambda_n - \lambda_{n-1}| \|u_{n-1}\|.
\end{aligned} \tag{3.6}$$

From the definition of x_n, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \|P_C\left(\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D) y_n\right) \\
&\qquad - \ P_C\left(\alpha_{n-1} \gamma f(x_{n-1}) + \beta_{n-1} x_{n-1} + ((1-\beta_{n-1})I - \alpha_{n-1} D) y_{n-1}\right)\| \\
&\quad \le \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D) y_n \\
&\qquad - \ \left(\alpha_{n-1} \gamma f(x_{n-1}) + \beta_{n-1} x_{n-1} + ((1-\beta_{n-1})I - \alpha_{n-1} D) y_{n-1}\right)\| \\
&\quad = \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(x_{n-1}) + \alpha_n \gamma f(x_{n-1}) \\
&\qquad + \ \beta_n x_n - \beta_n x_{n-1} + \beta_n x_{n-1} \\
&\qquad + \ ((1-\beta_n)I - \alpha_n D) y_n - ((1-\beta_n)I - \alpha_n D) y_{n-1} + ((1-\beta_n)I - \alpha_n D) y_{n-1} \\
&\qquad - \ \alpha_{n-1} \gamma f(x_{n-1}) - \beta_{n-1} x_{n-1} - ((1-\beta_{n-1})I - \alpha_{n-1} D) y_{n-1}\| \\
&\quad = \|\alpha_n \gamma (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) \gamma f(x_{n-1}) \\
&\qquad + \ \beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} \\
&\qquad + \ ((1-\beta_n)I - \alpha_n D)(y_n - y_{n-1}) + (((1-\beta_n)I - \alpha_n D) y_{n-1} - ((1-\beta_{n-1})I - \alpha_{n-1} D) y_{n-1}) \\
&\quad = \|\alpha_n \gamma (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) \gamma f(x_{n-1}) \\
&\qquad + \ \beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} + ((1-\beta_n)I - \alpha_n D)(y_n - y_{n-1}) \\
&\qquad + \ (1-\beta_n) y_{n-1} - \alpha_n D y_{n-1} - (1-\beta_{n-1}) y_{n-1} + \alpha_{n-1} D y_{n-1} \\
&\quad = \|\alpha_n \gamma (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) \gamma f(x_{n-1}) \\
&\qquad + \ \beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} + ((1-\beta_n)I - \alpha_n D)(y_n - y_{n-1}) \\
&\qquad + \ (\beta_{n-1} - \beta_n) y_{n-1} + (\alpha_{n-1} - \alpha_n) D y_{n-1} \\
&\quad \le \alpha_n \gamma \|f(x_n) - f(x_{n-1})\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\qquad + \ \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \|((1-\beta_n)I - \alpha_n D)(y_n - y_{n-1})\| \\
&\qquad + \ |\beta_{n-1} - \beta_n| \|y_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|D y_{n-1}\| \\
&\quad \le \alpha_n \gamma \theta \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\qquad + \ \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|y_n - y_{n-1}\| \\
&\qquad + \ |\beta_{n-1} - \beta_n| \|y_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|D y_{n-1}\|.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| \le \ & \alpha_n \gamma \theta \|x_n - x_{n-1}\| + \gamma |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
& + \ \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\
& + \ (1 - \beta_n - \alpha_n \bar{\gamma}) \left(\|x_n - x_{n-1}\| + \rho |\lambda_n - \lambda_{n-1}| \|u_{n-1}\|\right) \\
& + \ |\beta_{n-1} - \beta_n| \|y_{n-1}\| + |\alpha_{n-1} - \alpha_n| \|D y_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \rho(1-\beta_n-\alpha_n\bar{\gamma})|\lambda_n - \lambda_{n-1}|\|u_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n|\|y_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|Dy_{n-1}\|. \qquad (3.8)
\end{aligned}
$$

Applying Lemma 2.4, the conditions (i), (ii), (iii) and (3.8), we have

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.9)
$$

Step 3 We show that $\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|u_n - Tu_n\| = 0$. From the definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1} - T_{\lambda_n}u_n\| &= \|P_C\left(\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)T_{\lambda_n}u_n\right) - P_C\left(T_{\lambda_n}u_n\right)\| \\
&\le \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)T_{\lambda_n}u_n - T_{\lambda_n}u_n\| \\
&= \|\alpha_n\left(\gamma f(x_n) - DT_{\lambda_n}u_n\right) + \beta_n\left(x_n - T_{\lambda_n}u_n\right)\| \\
&\le \alpha_n\|\gamma f(x_n) - DT_{\lambda_n}u_n\| + \beta_n\|x_n - T_{\lambda_n}u_n\| \\
&\le \alpha_n\|\gamma f(x_n) - DT_{\lambda_n}u_n\| + \beta_n\|x_n - x_{n+1}\| + \beta_n\|x_{n+1} - T_{\lambda_n}u_n\|.
\end{aligned}
$$

It follows that

$$
\|x_{n+1} - T_{\lambda_n}u_n\| \le \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - DT_{\lambda_n}u_n\| + \frac{\beta_n}{1-\beta_n}\|x_n - x_{n+1}\|.
$$

By the conditions (i), (ii) and (3.9), we have

$$
\lim_{n\to\infty} \left\|x_{n+1} - T_{\lambda_n}u_n\right\| = 0. \qquad (3.10)
$$

Since $\|T_{\lambda_n}u_n - x_n\| \le \|T_{\lambda_n}u_n - x_{n+1}\| + \|x_{n+1} - x_n\|$, (3.9) and (3.10), we have

$$
\lim_{n\to\infty} \left\|T_{\lambda_n}u_n - x_n\right\| = 0. \qquad (3.11)
$$

From the nonexpansiveness of $J_{M,\lambda}$, we have

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \|J_{M,\mu}\left(I - \mu\sum_{i=1}^N a_iA_i\right)x_n - J_{M,\mu}\left(I - \mu\sum_{i=1}^N a_iA_i\right)x^*\|^2 \\
&\le \|\left(I - \mu\sum_{i=1}^N a_iA_i\right)x_n - \left(I - \mu\sum_{i=1}^N a_iA_i\right)x^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|(x_n - x^*) - \mu\left(\sum_{i=1}^{N} a_i A_i x_n - \sum_{i=1}^{N} a_i A_i x^*\right)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\mu \sum_{i=1}^{N} a_i \langle x_n - x^*, A_i x_n - A_i x^*\rangle + \mu^2 \sum_{i=1}^{N} a_i \|A_i x_n - A_i x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\mu \sum_{i=1}^{N} a_i \alpha_i \|A_i x_n - A_i x^*\|^2 + \mu^2 \sum_{i=1}^{N} a_i \|A_i x_n - A_i x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\mu\eta \sum_{i=1}^{N} a_i \|A_i x_n - A_i x^*\|^2 + \mu^2 \sum_{i=1}^{N} a_i \|A_i x_n - A_i x^*\|^2 \\
&= \|x_n - x^*\|^2 + \mu \sum_{i=1}^{N} a_i(\mu - 2\eta)\|A_i x_n - A_i x^*\|^2. \qquad (3.12)
\end{aligned}
$$

From the definition of x_n, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|P_C\left(\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)T_{\lambda_n} u_n\right) - P_C x^*\|^2 \\
&\quad \leq \|\alpha_n \gamma f(x_n) + \beta_n x_n + T_{\lambda_n} u_n - \beta_n T_{\lambda_n} u_n - \alpha_n D T_{\lambda_n} u_n - x^*\|^2 \\
&\quad = \|\alpha_n \gamma f(x_n) - \alpha_n D x^* + \alpha_n D x^* + \beta_n x_n - \beta_n T_{\lambda_n} u_n \\
&\qquad + \ (1-\alpha_n D)T_{\lambda_n} u_n - (1-\alpha_n D)x^* + (1-\alpha_n D)x^* - x^*\|^2 \\
&\quad = \|\alpha_n(\gamma f(x_n) - D x^*) + \beta_n(x_n - T_{\lambda_n} u_n) + (1-\alpha_n D)(T_{\lambda_n} u_n - x^*)\|^2 \\
&\quad \leq \|(1-\alpha_n D)(T_{\lambda_n} u_n - x^*) + \beta_n(x_n - T_{\lambda_n} u_n)\|^2 \\
&\qquad + \ 2\alpha_n \langle \gamma f(x_n) - D x^*, \alpha_n(\gamma f(x_n) - D x^*) + \beta_n(x_n - T_{\lambda_n} u_n) \\
&\qquad + (1-\alpha_n D)(T_{\lambda_n} u_n - x^*)\rangle \\
&\quad \leq \left(\|(1-\alpha_n D)(T_{\lambda_n} u_n - x^*)\| + \beta_n \|x_n - T_{\lambda_n} u_n\|\right)^2 \\
&\qquad + \ 2\alpha_n^2 \langle \gamma f(x_n) - D x^*, \gamma f(x_n) - D x^*\rangle \\
&\qquad + \ 2\alpha_n \beta_n \langle \gamma f(x_n) - D x^*, x_n - T_{\lambda_n} u_n\rangle \\
&\qquad + \ 2\alpha_n \langle \gamma f(x_n) - D x^*, (1-\alpha_n D)(T_{\lambda_n} u_n - x^*)\rangle \\
&\quad \leq \left(\|1-\alpha_n D\| \|T_{\lambda_n} u_n - x^*\| + \beta_n \|x_n - T_{\lambda_n} u_n\|\right)^2 \\
&\qquad + \ 2\alpha_n^2 \|\gamma f(x_n) - D x^*\|^2 + 2\alpha_n \beta_n \|\gamma f(x_n) - D x^*\| \|x_n - T_{\lambda_n} u_n\| \\
&\qquad + \ 2\alpha_n \|\gamma f(x_n) - D x^*\| \|(1-\alpha_n D)(T_{\lambda_n} u_n - x^*)\| \\
&\quad \leq \left((1-\alpha_n \bar{\gamma})\|u_n - x^*\| + \beta_n \|x_n - T_{\lambda_n} u_n\|\right)^2 \\
&\qquad + \ 2\alpha_n^2 \|\gamma f(x_n) - D x^*\|^2 + 2\alpha_n \beta_n \|\gamma f(x_n) - D x^*\| \|x_n - T_{\lambda_n} u_n\| \\
&\qquad + \ 2\alpha_n(1-\alpha_n \bar{\gamma})\|\gamma f(x_n) - D x^*\| \|T_{\lambda_n} u_n - x^*\| \\
&\quad \leq (1-\alpha_n \bar{\gamma})^2 \|u_n - x^*\|^2 + \beta_n^2 \|x_n - T_{\lambda_n} u_n\|^2 + 2(1-\alpha_n \bar{\gamma})\beta_n \|u_n - x^*\| \|x_n - T_{\lambda_n} u_n\| \\
&\qquad + \ 2\alpha_n^2 \|\gamma f(x_n) - D x^*\|^2 + 2\alpha_n \beta_n \|\gamma f(x_n) - D x^*\| \|x_n - T_{\lambda_n} u_n\| \\
&\qquad + \ 2\alpha_n(1-\alpha_n \bar{\gamma})\|\gamma f(x_n) - D x^*\| \|T_{\lambda_n} u_n - x^*\| \\
&\quad \leq \|u_n - x^*\|^2 + \beta_n^2 \|x_n - T_{\lambda_n} u_n\|^2 + 2(1-\alpha_n \bar{\gamma})\beta_n \|u_n - x^*\| \|x_n - T_{\lambda_n} u_n\| \\
&\qquad + \ 2\alpha_n^2 \|\gamma f(x_n) - D x^*\|^2 + 2\alpha_n \beta_n \|\gamma f(x_n) - D x^*\| \|x_n - T_{\lambda_n} u_n\| \\
&\qquad + \ 2\alpha_n(1-\alpha_n \bar{\gamma})\|\gamma f(x_n) - D x^*\| \|T_{\lambda_n} u_n - x^*\|. \qquad (3.13)
\end{aligned}
$$

From (3.12) and (3.13), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 &+ \mu \sum_{i=1}^{N} a_i(\mu-2\eta)\|A_ix_n - A_ix^*\|^2 + \beta_n^2\|x_n - T_{\lambda_n}u_n\|^2 \\
&+ 2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-x^*\|\|x_n-T_{\lambda_n}u_n\| \\
&+ 2\alpha_n^2\|\gamma f(x_n)-Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n)-Dx^*\|\|x_n-T_{\lambda_n}u_n\| \\
&+ 2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dx^*\|\|T_{\lambda_n}u_n - x^*\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mu \sum_{i=1}^{N} a_i(2\eta-\mu)\|A_ix_n - A_ix^*\|^2 \\
&\quad\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \beta_n^2\|x_n-T_{\lambda_n}u_n\|^2 \\
&\qquad + 2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-x^*\|\|x_n-T_{\lambda_n}u_n\| \\
&\qquad + 2\alpha_n^2\|\gamma f(x_n)-Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n)-Dx^*\|\|x_n-T_{\lambda_n}u_n\| \\
&\qquad + 2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dx^*\|\|T_{\lambda_n}u_n - x^*\| \\
&\quad\le (\|x_n-x^*\| + \|x_{n+1}-x^*\|)\|x_{n+1}-x_n\| + \beta_n^2\|x_n-T_{\lambda_n}u_n\|^2 \\
&\qquad + 2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-x^*\|\|x_n-T_{\lambda_n}u_n\| \\
&\qquad + 2\alpha_n^2\|\gamma f(x_n)-Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n)-Dx^*\|\|x_n-T_{\lambda_n}u_n\| \\
&\qquad + 2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dx^*\|\|T_{\lambda_n}u_n - x^*\|.
\end{aligned}
\tag{3.14}
$$

From the condition (i), (ii), (iv), (3.9), (3.11), and (3.14), we have

$$
\lim_{n\to\infty} \left\|A_ix_n - A_ix^*\right\| = 0, \forall i = 1,2,3,\ldots,N. \tag{3.15}
$$

Since $J_{m,\mu}$ is 1-inverse strongly monotone, we have

$$
\begin{aligned}
\|u_n-x^*\|^2 &= \|J_{M,\mu}\left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x_n - J_{M,\mu}\left(I-\mu\sum_{i=1}^{N} a_iA_iA_i\right)x^*\|^2 \\
&\le \left\langle \left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x_n - \left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x^*, u_n - x^*\right\rangle \\
&= \frac{1}{2}\left(\|\left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x_n - \left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x^*\|^2 + \|u_n-x^*\|^2\right. \\
&\quad \left. -\|\left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x_n - \left(I-\mu\sum_{i=1}^{N} a_iA_i\right)x^* - u_n + x^*\|^2\right) \\
&\le \frac{1}{2}\left(\|x_n-x^*\|^2 + \|u_n-x^*\|^2\right. \\
&\quad \left. -\|(x_n-u_n) - \mu\left(\sum_{i=1}^{N} a_iA_ix_n - \sum_{i=1}^{N} a_iA_ix^*\right)\|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - x_n\|^2 \\
&\quad - \mu^2\|\sum_{i=1}^{N} a_iA_ix_n - \sum_{i=1}^{N} a_iA_ix^*\|^2 \\
&\quad + 2\mu\langle x_n - u_n, \sum_{i=1}^{N} a_iA_ix_n - \sum_{i=1}^{N} a_iA_ix^*\rangle\Big) \\
&\leq \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - x_n\|^2 \\
&\quad - \mu^2\|\sum_{i=1}^{N} a_iA_ix_n - \sum_{i=1}^{N} a_iA_ix^*\|^2 \\
&\quad + 2\mu\|u_n - x_n\|\|\sum_{i=1}^{N} a_iA_ix_n - \sum_{i=1}^{N} a_iA_ix^*\|\Big).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\mu\|u_n - x_n\|\|\sum_{i=1}^{N} a_iA_ix_n - \sum_{i=1}^{N} a_iA_ix^*\| \\
&\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\mu\|u_n - x_n\|\sum_{i=1}^{N} a_i\|A_ix_n - A_ix^*\|. \qquad (3.16)
\end{aligned}
$$

From (3.13) and (3.16), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|u_n - x^*\|^2 + \beta_n^2\|x_n - T_{\lambda_n}u_n\|^2 + 2(1-\alpha_n\bar{\gamma})\beta_n\|u_n - x^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n^2\|\gamma f(x_n) - Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n) - Dx^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n) - Dx^*\|\|T_{\lambda_n}u_n - x^*\| \\
&\leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\mu\|u_n - x_n\|\sum_{i=1}^{N} a_i\|A_ix_n - A_ix^*\| \\
&\quad + \beta_n^2\|x_n - T_{\lambda_n}u_n\|^2 + 2(1-\alpha_n\bar{\gamma})\beta_n\|u_n - x^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n^2\|\gamma f(x_n) - Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n) - Dx^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n) - Dx^*\|\|T_{\lambda_n}u_n - x^*\|. \qquad (3.17)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|u_n - x_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu\|u_n - x_n\|\sum_{i=1}^{N} a_i\|A_ix_n - A_ix^*\| \\
&\quad + \beta_n^2\|x_n - T_{\lambda_n}u_n\|^2 + 2(1-\alpha_n\bar{\gamma})\beta_n\|u_n - x^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n^2\|\gamma f(x_n) - Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n) - Dx^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n) - Dx^*\|\|T_{\lambda_n}u_n - x^*\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + 2\mu\|u_n - x_n\| \sum_{i=1}^{N} a_i\|A_i x_n - A_i x^*\| \\
&\quad + \beta_n^2\|x_n - T_{\lambda_n}u_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - x^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n^2\|\gamma f(x_n) - Dx^*\|^2 + 2\alpha_n\beta_n\|\gamma f(x_n) - Dx^*\|\|x_n - T_{\lambda_n}u_n\| \\
&\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - Dx^*\|\|T_{\lambda_n}u_n - x^*\|.
\end{aligned}
$$

From the condition (i), (ii), (iv), (3.9), (3.11) and (3.15), we have

$$
\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\left\|J_{M,\mu}(I - \mu\sum_{i=1}^{N} a_i A_i)x_n - x_n\right\| = 0. \tag{3.18}
$$

Observe that

$$
\|T_{\lambda_n}u_n - u_n\| \leq \|T_{\lambda_n}u_n - x_n\| + \|x_n - u_n\|. \tag{3.19}
$$

It follows from (3.11), (3.19) and (3.18) that

$$
\lim_{n\to\infty}\left\|T_{\lambda_n}u_n - u_n\right\| = 0. \tag{3.20}
$$

Step 4 We will show that $\limsup_{n\to\infty}\left\langle(\gamma f - D)z, x_n - z\right\rangle \leq 0$, where $z = P_{\mathcal{F}}(I - (D - \gamma f))z$.

To show this, choose a subsequence $\left\{x_{n_k}\right\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty}\left\langle(\gamma f - D)z, x_n - z\right\rangle = \lim_{k\to\infty}\left\langle(\gamma f - D)z, x_{n_k} - z\right\rangle. \tag{3.21}
$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup q$ as $k \to \infty$. From (3.18), we obtain $u_{n_k} \rightharpoonup q$ as $k \to \infty$.

First, we will show that $q \in \Omega$. Assume that $q \notin \Omega$. Since $\Omega = Fix(T)$, then we have $q \neq Tq$. Since $u_{n_k} \rightharpoonup q$ as $k \to \infty$, by Opial's condition, the condition (i), $\lambda_n = o(\alpha_n)$, and (3.20), we obtain

$$
\begin{aligned}
\liminf_{k\to\infty}\|u_{n_k} - q\| &< \liminf_{k\to\infty}\|u_{n_k} - Tq\| \\
&\leq \liminf_{k\to\infty}(\|u_{n_k} - T_{\lambda_{n_k}}u_{n_k}\| + \|T_{\lambda_{n_k}}u_{n_k} - T_{\lambda_{n_k}}q\| + \|T_{\lambda_{n_k}}q - Tq\|) \\
&\leq \liminf_{k\to\infty}(\|u_{n_k} - T_{\lambda_{n_k}}u_{n_k}\| + \|u_{n_k} - q\| + \lambda_{n_k}\rho\|q\|) \\
&\leq \liminf_{k\to\infty}(\|u_{n_k} - T_{\lambda_{n_k}}u_{n_k}\| + \|u_{n_k} - q\| + \alpha_{n_k}M\rho\|q\|) \\
&\leq \liminf_{k\to\infty}\|u_{n_k} - q\|.
\end{aligned}
$$

This is a contradiction. Then, we have

$$
q \in Fix(T) = \Omega.
$$

Next, we will show that $q \in \bigcap_{i=1}^{N} VI(H, A_i, M)$. Assume that $q \notin \bigcap_{i=1}^{N} VI(H, A_i, M)$. By Lemmas 2.7 and 2.9. Then, $q \neq J_{M,\mu}(I - \mu\sum_{i=1}^{N} a_i A_i)q$. By the nonexpansiveness of $J_{M,\lambda}((I - \lambda\sum_{i=1}^{N} a_i A_i))$, (3.18) and Opial's condition, we obtain

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-q\| &< \liminf_{k\to\infty}\|x_{n_k}-J_{M,\mu}\left(\left(I-\mu\sum_{i=1}^{N}a_iA_i\right)\right)q\|\\ &\le \liminf_{k\to\infty}(\|x_{n_k}-J_{M,\mu}\left(\left(I-\mu\sum_{i=1}^{N}a_iA_i\right)\right)x_{n_k}\|\\ &\quad+\|J_{M,\mu}\left(\left(I-\mu\sum_{i=1}^{N}a_iA_i\right)\right)x_{n_k}-J_{M,\mu}\left(\left(I-\mu\sum_{i=1}^{N}a_iA_i\right)\right)q\|\\ &\le \liminf_{k\to\infty}\|x_{n_k}-q\|.\end{aligned}$$

This is a contradiction. Then, we have

$$q\in\bigcap_{i=1}^{N}VI(H,A_i,M).$$

Therefore, $q\in\mathcal{F}=\Omega\cap\bigcap_{i=1}^{N}VI(H,A_i,M)$.

Since $x_{n_k}\rightharpoonup q$ as $k\to\infty$ and $q\in\mathcal{F}$. By (3.21) and Lemma 2.2, we have

$$\begin{aligned}\limsup_{n\to\infty}\left\langle(\gamma f-D)z,x_n-z\right\rangle &= \lim_{k\to\infty}\left\langle(\gamma f-D)z,x_{n_k}-z\right\rangle\\ &=\left\langle(\gamma f-D)z,q-z\right\rangle\\ &\le 0.\end{aligned}\tag{3.22}$$

Step 5 Finally, we will show that $\lim_{n\to\infty}x_n=z$, where $z=P_{\mathcal{F}}(I-(D-\gamma f))z$. Putting $m_n=\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n D)T_{\lambda_n}u_n$, then $x_{n+1}=P_C m_n$. Since

$$\begin{aligned}&\|x_{n+1}-z\|^2\\ &=\langle P_C m_n-z,P_C m_n-z\rangle\\ &=\langle P_C m_n-m_n+m_n-z,P_C m_n-z\rangle\\ &=\langle P_C m_n-m_n,P_C m_n-z\rangle+\langle m_n-z,x_{n+1}-z\rangle\\ &\le\langle m_n-z,x_{n+1}-z\rangle\\ &=\langle\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n D)T_{\lambda_n}u_n-z,x_{n+1}-z\rangle\\ &=\langle\alpha_n(\gamma f(x_n)-Dz)+\beta_n(x_n-z)+((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}u_n-z),x_{n+1}-z\rangle\\ &=\alpha_n\langle\gamma f(x_n)-Dz,x_{n+1}-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle\\ &\quad+\langle((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}u_n-z),x_{n+1}-z\rangle\\ &=\alpha_n\langle\gamma f(x_n)-\gamma f(z),x_{n+1}-z\rangle+\alpha_n\langle\gamma f(z)-Dz,x_{n+1}-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle\\ &\quad+\langle((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}u_n-T_{\lambda_n}z),x_{n+1}-z\rangle\\ &\quad+\langle((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}z-z),x_{n+1}-z\rangle\\ &\le\alpha_n\|\gamma f(x_n)-\gamma f(z)\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Dz,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\ &\quad+\|((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}u_n-T_{\lambda_n}z)\|\|x_{n+1}-z\|\\ &\quad+\|((1-\beta_n)I-\alpha_n D)(T_{\lambda_n}z-z)\|\|x_{n+1}-z\|\\ &\le\alpha_n\gamma\theta\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Dz,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\ &\quad+(1-\beta_n-\alpha_n\bar{\gamma})\|T_{\lambda_n}u_n-T_{\lambda_n}z\|\|x_{n+1}-z\|\\ &\quad+(1-\beta_n-\alpha_n\bar{\gamma})\|T_{\lambda_n}z-z\|\|x_{n+1}-z\|\\ &\le\alpha_n\gamma\theta\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Dz,x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-z\|\|x_{n+1}-z\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\lambda_n\rho\|z\|\|x_{n+1}-z\| \\
&= (\alpha_n\gamma\theta+\beta_n+1-\beta_n-\alpha_n\bar{\gamma})\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle\gamma f(z)-Dz, x_{n+1}-z\rangle \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\lambda_n\rho\|z\|\|x_{n+1}-z\| \\
&\le (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle\gamma f(z)-Dz, x_{n+1}-z\rangle \\
&\quad + \lambda_n\rho\|z\|\|x_{n+1}-z\| \\
&\le (1-\alpha_n(\bar{\gamma}-\gamma\theta))\frac{1}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) + \alpha_n\big(\langle\gamma f(z)-Dz, x_{n+1}-z\rangle \\
&\quad + \frac{\lambda_n}{\alpha_n}\rho\|z\|\|x_{n+1}-z\|\big).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\le \frac{(1-\alpha_n(\bar{\gamma}-\gamma\theta))}{(1+\alpha_n(\bar{\gamma}-\gamma\theta))}\|x_n-z\|^2 + \frac{2\alpha_n}{(1+\alpha_n(\bar{\gamma}-\gamma\theta))}\big(\langle\gamma f(z)-Dz, x_{n+1}-z\rangle \\
&\quad + \frac{\lambda_n}{\alpha_n}\rho\|z\|\|x_{n+1}-z\|\big) \\
&\le (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n-z\|^2 + \frac{2\alpha_n(\bar{\gamma}-\gamma\theta)}{(1+\alpha_n(\bar{\gamma}-\gamma\theta))(\bar{\gamma}-\gamma\theta)}\big(\langle\gamma f(z)-Dz, x_{n+1}-z\rangle \\
&\quad + \frac{\lambda_n}{\alpha_n}\rho\|z\|\|x_{n+1}-z\|\big) \\
&= (1-p_n)\|x_n-z\|^2 + p_nq_n,
\end{aligned}
$$

where $p_n=\alpha_n(\bar{\gamma}-\gamma\theta)$ and $q_n=\frac{2}{(1+\alpha_n(\bar{\gamma}-\gamma\theta))(\bar{\gamma}-\gamma\theta)}\big(\langle\gamma f(z)-Dz, x_{n+1}-z\rangle+\frac{\lambda_n}{\alpha_n}\rho\|z\|\|x_{n+1}-z\|$. From the condition (i), (3.34), $\lambda_n=o(\alpha_n)$ and Lemma 2.4, we can conclude that the sequence $\{x_n\}$ converges strongly to $z=P_{\mathcal{F}}(I-(D-\gamma f))z$. □

Next, we introduce and prove a strong convergence theorem for finding a common element of the solution set of a constrained convex minimization problem and the solution sets of a finite family of zero points of the maximal monotone operator problem in Hilbert space as follows:

Theorem 3.2: *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i=1,2,\dots,N$, let $M_i: H\to 2^H$ be a finite family of multi-valued maximal monotone mappings with $\mathbb{D}(M)=C$ and Let g be a real-valued convex function of C into $\mathbb{R}$, and the gradient ∇g is $1/L$-ism continuous with $L>0$, let $D: C\to H$ be a strongly positive bounded linear operator with coefficient $0<\bar{\gamma}<1$, and let $f: C\to C$ be a contractive mapping with $\alpha\in(0,1)$ and $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. Assume that $\mathcal{F}:=\Omega\cap\bigcap_{i=1}^N M_i^{-1}0\neq\emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1\in C$ and*

$$
\begin{cases} u_n = W_{r_n}x_n \\ x_{n+1} = P_C\left(\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nD)T_{\lambda_n}(u_n)\right), \forall n\in\mathbb{N}, \end{cases} \tag{3.23}
$$

where $W_{r_n}:=a_0I+a_1J_{M_1,r_n}+a_2J_{M_2,r_n}+\cdots+a_NJ_{M_N,r_n}$, with $J_{M_i,r_n}=(I+r_nM_i)^{-1}$, $a_i\in(0,1)$, $\sum_{i=0}^N a_i=1$, and $0<a_i<1$ for every $i=1,2,\dots,N$, $P_C(I-\rho\nabla g_{\lambda_n})=T_{\lambda_n}$, $\nabla g_{\lambda_n}=\nabla g+\lambda_nI$, $\lambda_n\subset(0,\frac{2}{\rho}-L)$, $\rho\in(0,\frac{2}{L})$. Let $\{\alpha_n\}$, $\{\beta_n\}\subset[0,1]$, satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^\infty\alpha_n=\infty$, $\sum_{n=1}^\infty|\alpha_{n+1}-\alpha_n|<\infty$;
(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;
(iii) $\lambda_n=o(\alpha_n)$, $\sum_{n=1}^\infty|\lambda_{n+1}-\lambda_n|<\infty$;
(iv) $r_n\in[c,d]\subset(0,1)$, $\sum_{n=1}^\infty|r_{n+1}-r_n|<\infty$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathcal{F}$, which solves uniquely the following variational inequality

$$\langle (D - \gamma f)z, z - x^* \rangle \leq 0, \forall x^* \in \mathcal{F}. \tag{3.24}$$

Equivalently, we have $P_{\mathcal{F}}(I - D + \gamma f)z = z$.

***Proof*:** Some parts of the proof are also the same as Theorem 3.1. Now, we divide the proof 3.2 into five steps:

Step 1 We show that the sequence $\{x_n\}$ is bounded.

Let $x^* \in \mathcal{F}$. From Lemmas 2.10 and 2.11, we have

$$x^* = W_{r_n} x^*.$$

and

$$\|u_n - x^*\| = \|W_{r_n} x_n - x^*\| \leq \|x_n - x^*\|. \tag{3.25}$$

By continuing in the same direction as in step 1 of Theorem 3.1. It implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2 We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Using in the same direction as of Theorem 2.2 in [9], we have

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \|W_{r_n} x_n - W_{r_{n-1}} x_{n-1}\| \\
&\leq a_0 \|x_n - x_{n-1}\| + \sum_{i=1}^{N} a_i \left(\|x_n - x_{n-1}\| + \frac{|r_n - r_{n-1}|}{c} \|J_{M_i, r_n} x_n - x_{n-1}\| \right) \\
&\leq \|x_n - x_{n-1}\| + \sum_{i=1}^{N} a_i \frac{|r_n - r_{n-1}|}{c} (\|J_{M_i, r_n} x_n\| + \|x_{n-1}\|).
\end{aligned} \tag{3.26}$$

Putting $y_n = T_{\lambda_n} u_n$. From the nonexpansiveness of T_{λ_n} and (3.26), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|T_{\lambda_n} u_n - T_{\lambda_{n-1}} u_{n-1}\| \\
&\leq \|T_{\lambda_n} u_n - T_{\lambda_n} u_{n-1}\| + \|T_{\lambda_n} u_{n-1} - T_{\lambda_{n-1}} u_{n-1}\| \\
&\leq \|u_n - u_{n-1}\| + \|T_{\lambda_n} u_{n-1} - T_{\lambda_{n-1}} u_{n-1}\| \\
&\leq \left(\|x_n - x_{n-1}\| + \sum_{i=1}^{N} a_i \frac{|r_n - r_{n-1}|}{c} (\|J_{M_i, r_n} x_n\| + \|x_{n-1}\|) \right) \\
&\quad + \|P_C(I - \rho \nabla g_{\lambda_n}) u_{n-1} - P_C(I - \rho \nabla g_{\lambda_{n-1}}) u_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \sum_{i=1}^{N} a_i \frac{|r_n - r_{n-1}|}{c} (\|J_{M_i, r_n} x_n\| + \|x_{n-1}\|) \\
&\quad + \|(I - \rho \nabla g_{\lambda_n}) u_{n-1} - (I - \rho \nabla g_{\lambda_{n-1}}) u_{n-1}\| \\
&= \|x_n - x_{n-1}\| + \sum_{i=1}^{N} a_i \frac{|r_n - r_{n-1}|}{c} (\|J_{M_i, r_n} x_n\| + \|x_{n-1}\|) \\
&\quad + \rho \| - (\nabla g(u_{n-1}) + \lambda_n u_{n-1}) + \nabla g(u_{n-1}) + \lambda_{n-1} u_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \sum_{i=1}^{N} a_i \frac{|r_n - r_{n-1}|}{c} (\|J_{M_i, r_n} x_n\| + \|x_{n-1}\|) \\
&\quad + \rho |\lambda_n - \lambda_{n-1}| \|u_{n-1}\|.
\end{aligned} \tag{3.27}$$

From the definition of x_n and by continuing in the same direction as in step 2 of Theorem 3.1, we have

$$\begin{aligned}\|x_{n+1}-x_n\| \le{}& \alpha_n\gamma\theta\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\ &+\ \beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-y_{n-1}\| \\ &+\ |\beta_{n-1}-\beta_n|\|y_{n-1}\|+|\alpha_{n-1}-\alpha_n|\|Dy_{n-1}\|. \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \le{}& \alpha_n\gamma\theta\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\ &+\ \beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+(1-\beta_n-\alpha_n\bar{\gamma})\|y_n-y_{n-1}\| \\ &+\ |\beta_{n-1}-\beta_n|\|y_{n-1}\|+|\alpha_{n-1}-\alpha_n|\|Dy_{n-1}\| \\ \le{}& \alpha_n\gamma\theta\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\ &+\ \beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\ &+\ (1-\beta_n-\alpha_n\bar{\gamma})\Big(\|x_n-x_{n-1}\|+\sum_{i=1}^{N}a_i\frac{|r_n-r_{n-1}|}{c}(\|J_{M_i,r_n}x_n\|+\|x_{n-1}\|) \\ &+\ \rho|\lambda_n-\lambda_{n-1}|\|u_{n-1}\|\Big)+|\beta_{n-1}-\beta_n|\|y_{n-1}\|+|\alpha_{n-1}-\alpha_n|\|Dy_{n-1}\| \\ ={}& (1-\alpha_n(\bar{\gamma}-\gamma\theta))\|x_n-x_{n-1}\|+\gamma|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\ &+\ |\beta_n-\beta_{n-1}|\|x_{n-1}\|+(1-\beta_n-\alpha_n\bar{\gamma})\sum_{i=1}^{N}a_i\frac{|r_n-r_{n-1}|}{c}(\|J_{M_i,r_n}x_n\|+\|x_{n-1}\|) \\ &+\ \rho(1-\beta_n-\alpha_n\bar{\gamma})|\lambda_n-\lambda_{n-1}|\|u_{n-1}\| \\ &+\ |\beta_{n-1}-\beta_n|\|y_{n-1}\|+|\alpha_{n-1}-\alpha_n|\|Dy_{n-1}\|. \end{aligned} \tag{3.29}$$

From the condition (i)–(iv), (3.29), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.30}$$

Step 3 We show that $\lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}(1-\beta_n)\|u_n-Tu_n\|=0$.

By continuing in the same direction as in step 3 of Theorem 3.1, we have

$$\lim_{n\to\infty}\|x_n-u_n\| = \lim_{n\to\infty}\|x_n-W_{r_n}x_n\|=0. \tag{3.31}$$

and

$$\lim_{n\to\infty}\|Tu_n-u_n\|=0, \tag{3.32}$$

where $T\equiv P_C(I-\rho\nabla g)$.

Step 4 We will show that $\limsup_{n\to\infty}\langle(\gamma f-D)z, x_n-z\rangle\le 0$, where $z=P_{\mathcal{F}}(I-(D-\gamma f))z$.

To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f-D)z, x_n-z\rangle = \lim_{k\to\infty}\langle(\gamma f-D)z, x_{n_k}-z\rangle. \tag{3.33}$$

Without loss of generality, we can assume that $x_{n_k}\rightharpoonup q$ as $k\to\infty$. From (3.31), we obtain $u_{n_k}\rightharpoonup q$ as $k\to\infty$.

First, by continuing in the same direction as in step 4 of Theorem 3.1, we have $q\in Fix(T)=\Omega$.

Next, we will show that $q \in \bigcap_{i=1}^{N} M_i^{-1}0$. Assume that $q \notin \bigcap_{i=1}^{N} M_i^{-1}0$. By Lemmas 2.10 and 2.11. Then, $q \neq W_{r_{n_k}} q$. By the nonexpansiveness of $W_{r_{n_k}}$, (3.31) and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - q\| &< \liminf_{k\to\infty} \|x_{n_k} - W_{r_{n_k}} q\| \\
&\leq \liminf_{k\to\infty} (\|x_{n_k} - W_{r_{n_k}} x_{n_k}\| + \|W_{r_{n_k}} x_{n_k} - W_{r_{n_k}} q\|) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - q\|.
\end{aligned}$$

This is a contradiction. Then, we have

$$q \in \bigcap_{i=1}^{N} M_i^{-1}0.$$

Therefore, $q \in \mathcal{F} = \Omega \cap \bigcap_{i=1}^{N} M^{-1}0$.

Since $x_{n_k} \rightharpoonup q$ as $k \to \infty$ and $q \in \mathcal{F}$. By (3.21) and Lemma 2.2, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\gamma f - D)z, x_n - z\rangle &= \lim_{k\to\infty} \langle (\gamma f - D)z, x_{n_k} - z\rangle \\
&= \langle (\gamma f - D)z, q - z\rangle \\
&\leq 0. \qquad (3.34)
\end{aligned}$$

Step 5. Finally, we will show that $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathcal{F}}(I - (D - \gamma f))z$. By continuing in the same direction as in step 5 of Theorem 3.1, we can conclude that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(I - (D - \gamma f))z$. $\square$

4. Application

In this section, we prove a strong convergence theorem involving a finite family of equilibrium problems in Hilbert space. Moreover, we utilize our main theorem to prove a strong convergence theorem for a finite family of κ-strictly pseudo-contractive mappings and the constrained convex minimization problem in Hilbert space.

To obtain this result, we recall some definitions, lemmas and remarks as follows:

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium point. The set of solution of equilibrium problem is denoted by

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \qquad (4.1)$$

In 2013, Suwannaut and Kangtunyakarn [19] have modified (4.1) as follows:

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(x, y) \geq 0, \forall y \in C\right\}, \qquad (4.2)$$

where $F_i : C \times C \to \mathbb{R}$ is for bifunctions and $a_i > 0$ with $\sum_{i=1}^{N} a_i = 1$ for every $i = 1, 2, \ldots, N$. It is obvious that (4.2) reduces to (4.1), if $F_i = F$, for all $i = 1, 2, \ldots, N$.

For finding solutions of the equilibrium problem, assume a bifunction $F : C \times C \to \mathbb{R}$ to satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\left(tz + (1 - t)x, y\right) \leq F(x, y);$$

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 4.1 [20]: *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 4.2 [21]: *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$, define a mapping $\Phi_r : H \to C$ as follows:*

$$\Phi_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) *Φ_r is single valued;*
(ii) *Φ_r is firmly nonexpansive, i.e. for any $x, y \in H$,*

$$\left\|\Phi_r(x) - \Phi_r(y)\right\|^2 \leq \left\langle \Phi_r(x) - \Phi_r(y), x - y\right\rangle;$$

(iii) *$Fix\left(\Phi_r\right) = EP(F)$;*
(iv) *$EP(F)$ is closed and convex.*

Lemma 4.3 [19]: *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4) with $\bigcap_{i=1}^{N} EP\left(F_i\right) \neq \emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\left(F_i\right),$$

where $a_i \in (0,1)$ for every $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Remark 1 [19]: From Lemma 4.3

$$Fix(\Phi_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP\left(F_i\right),$$

where $a_i \in (0,1)$, for each $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} a_i = 1$.

Lemma 4.4 [22]: *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$. Let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{F(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(F) = A_F^{-1}0$ and A_F is a maximal monotone operator with $\mathbb{D}(A_F) \subset C$. Further, for any $x \in H$ and $r > 0$, the resolvent Φ_r of F coincides with the resolvent of A_F; i.e.

$$\Phi_r(x) = (I + rA_F)^{-1}x.$$

Remark 2: Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, 3, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying $(A1) - (A4)$ with $\bigcap_{i=1}^{N} EP\left(F_i\right) \neq \emptyset$.

Define $A_{\sum_{i=1}^N a_iF_i} : H \to 2^H$ by

$$A_{\sum_{i=1}^N a_iF_i}x = \begin{cases} \left\{\sum_{i=1}^N a_iF_i(x,y) \geq \langle y-x, z\rangle, \forall y \in C\right\}, & x \in C, \\ \emptyset, & x \notin C, \end{cases}$$

where $0 \leq a_i \leq 1$, for every $i = 1, 2, \ldots, N$, and $\sum_{i=1}^N a_i = 1$. From Lemma 4.3 and 4.4, we have $A^{-1}_{\sum_{i=1}^N a_iF_i}0 = \bigcap_{i=1}^N EP(F_i)$ and $A_{\sum_{i=1}^N a_iF_i}$ is a maximal monotone operator with $\mathbb{D}(A_{\sum_{i=1}^N a_iF_i}) \subset C$ for all $i = 1, 2, \ldots, N$. Further, for any $x \in H$ and $r > 0$, the resolvent Φ_r of $\sum_{i=1}^N a_iF_i$ coincides with the resolvent of $A_{\sum_{i=1}^N a_iF_i}$; i.e.

$$\Phi_r(x) = \left(I + rA_{\sum_{i=1}^N a_iF_i}\right)^{-1} x.$$

Let $B : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. It is well-known that $VI(H, B, M) = M^{-1}0$, where $B = 0$. From above fact, we have the following theorem.

Theorem 4.5: *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and Φ_μ be the resolvent of F_i. Let g be a real-valued convex function of C into $\mathbb{R}$, and the gradient ∇g is $1/L$-ism continuous with $L > 0$, let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $0 < \bar{\gamma} < 1$, and let $f : C \to C$ be a contractive mapping with $\alpha \in (0, 1)$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Assume that $\mathcal{F} := \Omega \cap \bigcap_{i=1}^N EP(F_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and*

$$\begin{cases} \sum_{i=1}^N a_iF_i\left(u_n, y\right) + \frac{1}{\mu}\left\langle y - u_n, u_n - x_n\right\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C\left(\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)T_{\lambda_n}(u_n)\right), \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $P_C(I - \rho\nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\lambda_n \subset (0, \frac{2}{\rho} - L)$, $\rho \in (0, \frac{2}{L})$, $\sum_{i=1}^N a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$;
(iv) $0 < \mu < 2\eta$, *where* $\eta = min_{i=1,2,\ldots,N}\{\alpha_i\}$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathcal{F}$, which solves uniquely the following variational inequality

$$\langle (D - \gamma f)z, z - x^*\rangle \leq 0, \forall x^* \in \mathcal{F}. \tag{4.4}$$

Equivalently, we have $P_{\mathcal{F}}(I - D + \gamma f)z = z$.

***Proof*:** For every $i = 1, 2, \ldots, N$, put $A_i = 0$ in Theorem 3.1. For a bifunction $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$, we can define a maximal monotone operator $A_{\sum_{i=1}^N a_iF_i}$ with $\mathbb{D}(A_{\sum_{i=1}^N a_iF_i}) = C$. Put $M = A_{\sum_{i=1}^N a_iF_i}$, in Theorem 3.1. Then, by Remark 2, we have $\Phi_\mu x_n = (I + \mu A_{\sum_{i=1}^N a_iF_i})^{-1} x_n = J_{A_{\sum_{i=1}^N a_iF_i},\mu} x_n$. So, from Theorem 3.1, we obtain the desired result. □

Recall that let $S : C \to C$ be a mapping. Then, S is said to be ξ*-strictly pseudo-contractive* if there exists a constant $\xi \in [0, 1)$ such that

$$\left\|Sx - Sy\right\|^2 \leq \left\|x - y\right\|^2 + \xi \left\|(I - S)x - (I - S)y\right\|^2, \forall x, y \in C.$$

Now, we consider a property of finite family of strictly pseudo-contractive mappings in Hilbert space as follows:

Proposition 4.6 [23]: *Let C be a nonempty closed convex subset of a real Hilbert space H.*

(i) *Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $S_i : C \to H$ is a ξ_i-strict pseudo-contraction for some $0 \leq \xi_i < 1$. Assume $\{a_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N a_i = 1$. Then, $\sum_{i=1}^N a_i S_i$ is a ξ-strict pseudo-contraction, with $\xi = \max_{i=1,2,\dots,N}\{\xi_i\}$.*

(ii) *Let $\{S_i\}_{i=1}^N$ and $\{a_i\}_{i=1}^N$ be given as in (i) above. Suppose that $\{S_i\}_{i=1}^N$ has a common fixed point. Then,*

$$Fix\left(\sum_{i=1}^N a_i S_i\right) = \bigcap_{i=1}^N Fix(S_i).$$

Theorem 4.7: *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping with $\mathbb{D}(M) = C$. For every $i = 1, 2, \dots, N$, $S_i : C \to H$ be ξ_i-strictly pseudo-contractive mappings with $\xi = \max_{i=1,2,\dots,N}\{\xi_i\}$. Let g be a real-valued convex function of C into $\mathbb{R}$, and the gradient ∇g is $1/L$-ism continuous with $L > 0$, let $D : C \to H$ be a strongly positive bounded linear operator with coefficient $0 < \bar{\gamma} < 1$, and let $f : C \to C$ be a contractive mapping with $\alpha \in (0, 1)$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Assume that $\mathcal{F} := \Omega \cap \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by $x_1 \in C$ and*

$$\begin{cases} u_n = (1-\mu)x_n + \mu \sum\limits_{i=1}^N a_i S_i x_n \\ x_{n+1} = P_C\left(\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)T_{\lambda_n}(u_n)\right), \forall n \in \mathbb{N}, \end{cases} \tag{4.5}$$

where $P_C(I - \rho\nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\lambda_n \subset (0, \frac{2}{\rho} - L)$, $\rho \in (0, \frac{2}{L})$, $\sum_{i=1}^N a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \dots, N$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $\mu \in (0, 1-\xi) \subset (0, 1)$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathcal{F}$, which solves uniquely the following variational inequality

$$\langle (D - \gamma f)z, z - x^* \rangle \leq 0, \forall x^* \in \mathcal{F}. \tag{4.6}$$

Equivalently, we have $P_{\mathcal{F}}(I - D + \gamma f)z = z$.

***Proof*:** Put $A_i = I - S_i$ and $M = 0$ for all $i = 1, 2, \dots, N$. Since $A_i = I - S_i$ for all $i = 1, 2, \dots, N$, then we have that A_i is $\frac{1-\xi}{2}$-inverse strongly monotone. Now, we show that $\bigcap_{i=1}^N VI(H, A_i, M) = \bigcap_{i=1}^N Fix(S_i)$. Since $A_i = I - S_i$, $M = 0$, Lemma 2.9, and Proposition 4.6, then

$$\begin{aligned} x \in \bigcap_{i=1}^N VI(H, A_i, M) \Leftrightarrow x \in VI\left(H, \sum_{i=1}^N a_i A_i, M\right) &\Leftrightarrow 0 \in \sum_{i=1}^N a_i A_i x + Mx \\ &\Leftrightarrow 0 = \sum_{i=1}^N a_i A_i x \\ &\Leftrightarrow 0 = \sum_{i=1}^N a_i (I - S_i)x \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow x = \sum_{i=1}^{N} a_i S_i x\\
&\Leftrightarrow x \in Fix\left(\sum_{i=1}^{N} a_i S_i\right)\\
&\Leftrightarrow x \in \bigcap_{i=1}^{N} Fix(S_i).
\end{aligned}$$

It implies that

$$\Omega \cap \bigcap_{i=1}^{N} VI(H, A_i, M) = \Omega \cap \bigcap_{i=1}^{N} Fix(S_i).$$

From the definition of $J_{M,\mu}$, we have

$$\begin{aligned}
J_{M,\mu}\left(I - \mu \sum_{i=1}^{N} a_i A_i\right) u_n &= (I + \mu M)^{-1}\left(I - \mu \sum_{i=1}^{N} a_i A_i\right) x_n\\
&= x_n - \mu \sum_{i=1}^{N} a_i A_i x_n\\
&= x_n - \mu \sum_{i=1}^{N} a_i (I - S_i) x_n\\
&= (1-\mu) x_n + \mu \sum_{i=1}^{N} a_i S_i x_n.
\end{aligned}$$

So, from Theorem 3.1, we obtain the desired result. □

5. Numerical results

The purpose of this section is to give a numerical example to support our some result. The following example is supported by Theorem 3.1.

Example 5.1: Let $H = \mathbb{R}$ be the set of real numbers and $C = [0, 100]$. Define $g : [0, 100] \to \mathbb{R}$ by $\frac{x^2}{4}$. Clearly, g is convex and differentiable with $g'(x) = \frac{x}{2}$. It implies that ∇g is 1-ism, that is $L = 1$. From the problem (1.1), we have

$$\min_{x\in[0,100]} \frac{x^2}{4}. \tag{5.1}$$

Observe that the optimal solution x^* to the minimization problem (5.1) is $x^* = 0$. Putting $\rho = \frac{1}{5}$, then we can give the parameters $\lambda_n = \frac{1}{25n^2}$, for every $n \in \mathbb{N}$. For every $i = 1, 2, \ldots, N$, let mappings $A_i : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, be defined by

$$A_i x = \frac{2i}{5}x, Dx = \frac{x}{5}, f(x) = \frac{x}{6}, \text{ for all } x, y \in \mathbb{R},$$

and suppose that $J_{\mu}^{M} = I$, $a_i = \frac{6}{7^i} + \frac{1}{N7^N}$, and $\gamma = \frac{1}{2}$, $\alpha_n = \frac{1}{5n}$, $\beta_n = \frac{n}{8n+9}$, for all $n \in \mathbb{N}$. Observe that A_i is $\frac{1}{2i}$-ism with $\eta = \frac{1}{2N}$, for all $i = 1, 2, \ldots, N$. It's easy to see that all parameters and sequences satisfy conditions of Theorem 3.1. For every $n \in \mathbb{N}$, we rewrite (3.1) as follows:

Table 1. The values of the sequences $\{u_n\}$ and $\{x_n\}$ with initial values $x_1 = 60$, $n = 200$ and $N = 100$.

n	u_n	x_n
1	59.7600000	60.0000000
2	52.3571828	52.5674526
3	46.7706993	46.9585335
4	42.0294924	42.1982856
⋮	⋮	⋮
197	0.0000004	0.0000004
198	0.0000004	0.0000004
199	0.0000003	0.0000003
200	0.0000003	0.0000003

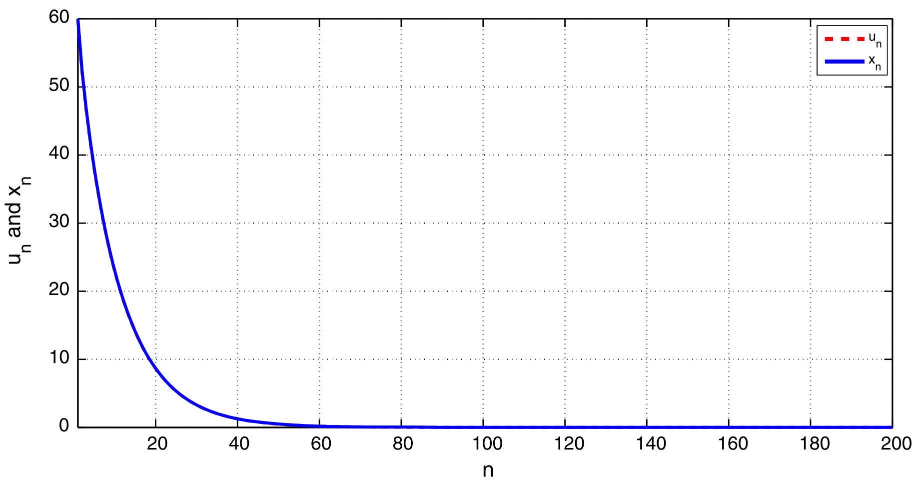

Figure 1. The behaviour of sequences $\{u_n\}$ and $\{x_n\}$ with initial values $x_1 = 60$, $n = 200$ and $N = 100$.

$$\begin{cases} u_n = x_n - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{6}{7^i} + \frac{1}{N7^N}\right)A_i x_n \\ x_{n+1} = P_{[0,100]}\left(\frac{1}{60n}x_n + \frac{n}{8n+9}x_n + \left(\left(1 - \frac{n}{8n+9}\right)I - \frac{1}{5n}D\right)P_{[0,100]}\left(u_n - \frac{1}{5}\left(\frac{u_n}{2} + \frac{1}{25n^2}u_n\right)\right)\right). \end{cases} \tag{5.2}$$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (5.2) converge strongly to 0.

Using the algorithm (5.2) and choosing $x_1 = 60$, $n = 200$ and $N = 100$. The numerical results for the sequences x_n and u_n are shown in the following Table 1 and Figure 1.

6. Conclusion

(1) Table 1 shows that the sequences $\{u_n\}$ and $\{x_n\}$ converge to 0.
(2) Theorem 3.1 guarantees the convergence of $\{u_n\}$ and $\{x_n\}$ to 0 in Example 5.1.

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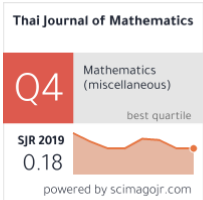

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Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

Iterative Methods for Solving the Monotone Inclusion Problem and the Fixed Point Problem in Banach Spaces

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Abstract In this work, we propose two iterative algorithms for solving the monotone inclusion problem and the fixed point problem of a relatively nonexpansive mapping in the framework of Banach spaces. We prove the strong convergence theorems of the proposed algorithms under some suitable assumptions. Furthermore, some numerical experiments of proposed algorithms to compressed sensing in signal recovery are presented. Our results improve and generalize many recent and important results in the literature.

MSC: 47H09; 47H10; 47J25; 47J05

Keywords: maximal monotone operator; Banach space; strong convergence; extragradient algorithm

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1. Introduction

Let E be a real Banach space. Consider the following so-called *monotone inclusion problem*: find $x^* \in E$ such that

$$0 \in (A+B)x^*, \tag{1.1}$$

where $A : E \to E$ and $B : E \to 2^E$ are single and set-valued mappings, respectively and 0 is a zero vector in E. In particular case, when $A = 0$, then the problem (1.1) becomes the inclusion problem introduced by Rockafellar [1] and when $E = \mathbb{R}^n$, then the problem

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(1.1) becomes the generalized equation introduced by Robinson [2]. The set of solutions of the problem (1.1) is denoted by $(A+B)^{-1}0$. Many practical nonlinear problems arising in applied sciences such as in machine learning, image processing, statistical regression and linear inverse problem can be formulated as this problem (see [3–5]).

A well-known method for solving the problem (1.1) in Hilbert spaces H, is the *forward-backward algorithm* [6] which is defined by the following manner:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = J_\lambda^B(x_n - \lambda A x_n), \ \forall n \geq 1, \end{cases} \tag{1.2}$$

where $J_\lambda^B := (I + \lambda B)^{-1}$ is a resolvent of B for $\lambda > 0$. Here, I denotes the identity operator of H. It was proved that the sequence generated by (1.2) converge weakly to a point in $(A+B)^{-1}0$ under the assumption that A is α-cocoercivity, that is,

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in H$$

and λ is chosen in $(0, 2\alpha)$.

In order to get strong convergence, Takashashi et al. [7] introduced the following modified forward–backward algorithm in Hilbert spaces H:

$$\begin{cases} x_1, u \in H, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n) J_{\lambda_n}^B(x_n - \lambda_n A x_n)), \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where A is an α-cocoercive mapping on H and $\{\lambda_n\} \subset (0, \infty)$. They also proved the strong convergence of the generated by (1.3) converges strongly to a point in $(A+B)^{-1}0$ under appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

López et al. [8] established a strong convergence theorem of the forward-backward algorithm (1.2) in a q-uniformly smooth and uniformly convex Banach spaces E. They introduced a modified forward-backward algorithm with errors a_n and b_n in the following way:

$$\begin{cases} x_1, u \in E, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(J_{\lambda_n}^B(x_n - \lambda_n(Ax_n + a_n)) + b_n), \ \forall n \geq 1, \end{cases} \tag{1.4}$$

where $J_{\lambda_n}^B := (I + \lambda_n B)^{-1}$ is the resolvent of an m-accretive operator B, A is an α-cocoercive mapping, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$. They also proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point in $(A+B)^{-1}0$.

In recent years, various modifications of forward-backward algorithm have been constructed and modified by many authors in several settings (see, *e.g.*, [9–16]). It can be seen that, the cocoercivity of A of most of methods is strong assumption. To avoid this strong assumption, Tseng [17] introduced the following algorithm in Hilbert spaces H, later it is known as *Tseng's splitting algorithm*:

$$\begin{cases} x_1 \in H, \\ y_n = J_{\lambda_n}^B(I - \lambda_n A)x_n, \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \ \forall n \geq 1, \end{cases} \tag{1.5}$$

where A is Lipschitz continuous with a constant $L > 0$. It was shown that the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of (1.1) provided the step-size λ_n is chosen in $\left(0, \frac{1}{L}\right)$.

On the other hand, the fixed point problem is problem of finding a point $x^* \in E$ such that

$$x^* = Tx^*, \tag{1.6}$$

where $T : E \to E$ is a nonlinear mapping. The set of solutions of problem (1.6) is denoted by $F(T) = \{x \in E : x = Tx\}$. In real life, many mathematical models have been formulated as this problem.

In this paper, we study the following problem: find $x^* \in E$ such that

$$x^* \in F(T) \cap (A+B)^{-1}0. \tag{1.7}$$

Currently, there have been many authors who interested in finding a common solution of the fixed point problem (1.6) and the monotone inclusion problem (1.1) (see, *e.g.*, [16, 18–23]).

Motivated by the works in the literature, we introduce two Halpern-Tseng type for solving the monotone inclusion problem and the fixed point problem of a relatively nonexpansive mapping in the framework of Banach spaces. We prove the strong convergence results of the proposed methods under some appropriate conditions. Finally, we provide numerical experiments to compressed sensing in signal recovery. The results presented in this paper are improve and generalize many known results in this direction.

2. Preliminaries

Let E be a real Banach space with its dual space E^*. We denote $\langle x, f\rangle$ by the value of a functional f in E^* at x in E, that is, $\langle x, f\rangle = f(x)$. For a sequence $\{x_n\}$ in E, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The set of all real numbers is denoted by $\mathbb{R}$, while $\mathbb{N}$ stands for the set of nonnegative integers. Let S_E denote the unit sphere of E. The space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. The space E is said to be *uniformly smooth* if the limit (2.1) converges uniformly in $x, y \in S_E$. It is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. The space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S_E, \|x-y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. Let $p \geq 2$. The space E is said to be *p-uniformly convex* if there is a $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Let $1 < q \leq 2$. The space E is said to be *q-uniformly smooth* if there exists a $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$, where ρ_E is the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in S_E\right\}$$

for all $t \geq 0$. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is observe that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [24]). If E is uniformly convex then E is reflexive and strictly convex and if E is uniformly smooth then E is reflexive and smooth (see [25]). Moreover, we know that for every $p > 1$, L_p and ℓ_p spaces are $\min\{p, 2\}$-uniformly smooth and $\max\{p, 2\}$-uniformly convex, while Hilbert space is 2-uniformly smooth and 2-uniformly convex (see [26] for more details).

Definition 2.1. Let C be a nonempty subset of E. Recall that a mapping $A : C \to E^*$ is said to be:

(i) *cocoercive* if there exists a constant $\gamma > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2$ for all $x, y \in C$;
(ii) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$;
(iii) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in C$;
(iv) *hemicontinuous* if for each $x, y \in C$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Remark 2.2. It is easy to see that if A is cocoercive, then A is monotone and Lipschitz continuous but converse is not true in general.

Definition 2.3. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*.

If E is a Hilbert space, then $J = I$ is the identity mapping on E. It is known that E is smooth if and only if J is single-valued from E into E^* and if E is a reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E. Moreover, if E is uniformly smooth then J is norm-to-norm uniformly continuous on bounded subsets of E (see [25] for more details).

Lemma 2.4. [27, 28] *(i) Let E be a 2-uniformly smooth Banach space. Then there exists a constant $\kappa > 0$ such that*

$$\|x - y\|^2 \leq \|x\|^2 - 2\langle y, Jx\rangle + \kappa\|y\|^2,\ \forall x, y \in E.$$

(ii) Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that

$$\|x - y\|^2 \geq \|x\|^2 - 2\langle y, Jx\rangle + c\|y\|^2,\ \forall x, y \in E.$$

Remark 2.5. It is well-known that $\kappa = c = 1$ whenever E is a Hilbert space. Moreover, we refer to [28] for the exact values of the constants κ and c.

Next, we recall the following Lyapunov function which introduced in [29]:

Definition 2.6. Let E be a smooth Banach space. The Lyapunov functional $\phi : E\times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall x, y \in E.$$

In the particular case in which E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2,\ \forall x, y \in E$$

and

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z),\ \forall x, y, z \in E,\ \alpha \in [0, 1]. \quad (2.2)$$

In addition, the function ϕ satisfies the following *three point identity*:

$$\phi(x,y) = \phi(x,z) - \phi(y,z) + 2\langle y-x, Jy - Jz\rangle, \ \forall x,y,z \in E.$$

Lemma 2.7. [30] *Let E be a 2-uniformly convex Banach space. Then there exists a constant $c > 0$ such that*

$$c\|x-y\|^2 \le \phi(x,y), \ \forall x,y \in E,$$

where c is the constant in Lemma 2.4 (ii).

Lemma 2.8. [31] *Let E be a uniformly convex Banach space. Then there exists a continuous strictly increasing convex function $g : [0,2r) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz) \le \alpha\phi(x,y) + (1-\alpha)\phi(x,z) - \alpha(1-\alpha)g(\|Jy - Jz\|)$$

for all $\alpha \in [0,1]$, $x \in E$ and $y,z \in B_r := \{\omega : \|\omega\| \le r\}$ for some $r > 0$.

The following important fact can be found in [32]. For two sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E. Then

$$\|x_n - y_n\| \to 0 \ \Leftrightarrow \ \|Jx_n - Jy_n\| \to 0 \ \Leftrightarrow \ \phi(x_n,y_n) \to 0. \tag{2.3}$$

Let C be a nonempty subset of a smooth Banach space E. A point $p \in C$ is a fixed point of T if $p = Tp$ and we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *relatively nonexpansive* if it satisfies the following conditions:

(i) $F(T) \neq \emptyset$;

(ii) $\phi(p,Tx) \le \phi(p,x)$ for all $p \in F(T)$ and $x \in C$;

(iii) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, it follows that $p \in F(T)$.

Remark 2.9. If T satisfies (i) and (ii), then T is called *relatively quasi-nonexpansive*. In a Hilbert space H, we know that $\phi(x,y) = \|x-y\|^2$ for all $x,y \in H$. Hence, if $T : C \to C$ is relatively quasi-nonexpansive, then it is quasi-nonexpansive, that is, $\|Tx-p\| \le \|x-p\|$ for all $p \in F(T)$ and $x \in C$.

Lemma 2.10. [33] *Let E be a strictly convex and smooth Banach space. Let C be a closed and convex subset of E. If $T : C \to C$ be a relatively nonexpansive mapping, then $F(T)$ is closed and convex.*

We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in [29]:

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \ \forall x \in E, \ x^* \in E^*.$$

Obviously, $V(x,x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.11. [29] *Let E be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*), \ \forall x \in E, \ x^*,y^* \in E^*.$$

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed convex subset of E. Then we know that for any $x \in E$, there exists a unique point $z \in C$ such that

$$\phi(z,x) = \min_{y\in C} \phi(y,x).$$

Such a mapping $\Pi_C : E \to C$ defined by $z = \Pi_C(x)$ is called the *generalized projection*. If E is a Hilbert space, then Π_C is coincident with the metric projection denoted by P_C.

Lemma 2.12. [29] *Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty, closed, and convex subset of E. For each $x \in E$ and $z \in C$. Then the following statements hold:*

(i) $z = \Pi_C(x)$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$, $\forall y \in C$.
(ii) $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x)$, $\forall y \in C$.

Let $B : E \to 2^{E^*}$ be a multi-valued mapping. The effective domain of B is denoted by $D(B) = \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $R(B) = \bigcup\{Bx : x \in D(B)\}$. The set of zeros of B is denoted by $B^{-1}0 = \{x \in D(B) : 0 \in Bx\}$. A multi-valued mapping B from E into E^* is said to be *monotone* if

$$\langle x - y, u - v \rangle \geq 0, \ \forall x, y \in D(B), \ u \in Bx \text{ and } v \in By.$$

A monotone operator B on E is said to be *maximal* if its graph $G(B) = \{(x, y) \in E \times E^* : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone operator on E. In other words, the maximality of B is equivalent to $R(J + \lambda B) = E^*$ for $\lambda > 0$ (see [34, Theorem 1.2]). It is known that if B is maximal monotone, then $B^{-1}0$ is closed and convex (see [35]). For a maximal monotone operator B, we define the resolvent of B by $J_\lambda^B(x) = (J + \lambda B)^{-1}Jx$ for $x \in E$ and $\lambda > 0$. It is also known that $B^{-1}0 = F(J_\lambda^B)$.

Lemma 2.13. [34] *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded mapping. Let $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A + B$ is a maximal monotone mapping.*

Lemma 2.14. *Let E be a reflexive, strictly convex and smooth Banach space. Let $A : E \to E^*$ be a mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Then the following statements hold:*

(i) Define a mapping $T_\lambda x := J_\lambda^B \circ J^{-1}(J - \lambda A)x$ for $x \in E$ and $\lambda > 0$, then $F(T_\lambda) = (A + B)^{-1}0$.
(ii) $(A + B)^{-1}0$ is closed and convex.

Proof. (i) Let $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned} x = T_\lambda x \quad &\Leftrightarrow \quad x = J_\lambda^B \circ J^{-1}(J - \lambda A)x \\ &\Leftrightarrow \quad x = (J + \lambda B)^{-1} J \circ J^{-1}(J - \lambda A)x \\ &\Leftrightarrow \quad Jx - \lambda Ax \in Jx + \lambda Bx \\ &\Leftrightarrow \quad 0 \in (A + B)x \\ &\Leftrightarrow \quad x \in (A + B)^{-1}0. \end{aligned}$$

Hence $F(T_\lambda) = (A + B)^{-1}0$.

(ii) By Lemma 2.13, we know that $A + B$ is maximal monotone, then we can show that the set $(A + B)^{-1}0 = \{x \in E : 0 \in (A + B)x\}$ is closed and convex. ∎

Lemma 2.15. [36] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence of real numbers such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.16. [37] *Let* $\{a_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists an increasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k := \max\{j \le k : a_j \le a_{j+1}\}$.

3. Main Results

In this section, we introduce two Halpern-Tseng type for finding a common solution of the monotone inclusion problem and the fixed point problem in Banach spaces. From now on, let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone operator. Let $T : E \to E$ be a relatively nonexpansive mapping. Assume that $\Omega := F(T) \cap (A+B)^{-1}0 \neq \emptyset$. To prove the strong convergence results, we also need to assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, such that $\{\beta_n\} \subset [a,b] \subset (0,1)$ for some $a, b > 0$ and $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 1 Halpern-Tseng type algorithm

Step 0. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n). \tag{3.1}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.2}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n Jz_n + (1-\beta_n)JTz_n)). \tag{3.3}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.1. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3. Then*

$$\phi(p, z_n) \le \phi(p, x_n) - \Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n), \ \forall p \in (A+B)^{-1}0,$$

where c *and* κ *are the constants in Lemma 2.4.*

Proof. Let $p \in (A+B)^{-1}0$. By Lemma 2.4 (ii), we have

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n))) \\
&= V(p, Jy_n - \lambda_n(Ay_n - Ax_n)) \\
&= \|p\|^2 - 2\langle p, Jy_n - \lambda_n(Ay_n - Ax_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Ax_n)\|^2 \\
&\leq \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda_n\langle p, Ay_n - Ax_n\rangle + \|Jy_n\|^2 - 2\lambda_n\langle y_n, Ay_n - Ax_n\rangle \\
&\quad + \kappa\|\lambda_n(Ay_n - Ax_n)\|^2 \\
&= \|p\|^2 - 2\langle p, Jy_n\rangle + \|y_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, y_n) - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) + \phi(x_n, y_n) + 2\langle x_n - p, Jy_n - Jx_n\rangle - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle \\
&\quad + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) + \phi(x_n, y_n) - 2\langle y_n - x_n, Jy_n - Jx_n\rangle + 2\langle y_n - p, Jy_n - Jx_n\rangle \\
&\quad - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) - \phi(y_n, x_n) + 2\langle y_n - p, Jy_n - Jx_n\rangle - 2\lambda_n\langle y_n - p, Ay_n - Ax_n\rangle \\
&\quad + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&= \phi(p, x_n) - \phi(y_n, x_n) + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
&\quad - 2\langle y_n - p, Jx_n - Jy_n + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.4)
\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}
\phi(p, z_n) \leq\ & \phi(p, x_n) - \Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n) + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 \\
& - 2\langle y_n - p, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.5)
\end{aligned}$$

We now show that

$$\langle y_n - p, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0.$$

From the definition of $\{y_n\}$, we note that $Jx_n - \lambda_n Ax_n \in Jy_n + \lambda_n By_n$. Since B is maximal monotone, there exists $v_n \in By_n$ such that $Jx_n - \lambda_n Ax_n = Jy_n + \lambda_n v_n$, it follows that

$$v_n = \frac{1}{\lambda_n}\big(Jx_n - Jy_n - \lambda_n Ax_n\big). \qquad (3.6)$$

Since $0 \in (A+B)p$ and $Ay_n + v_n \in (A+B)y_n$, it follows from Lemma 2.13 that $A+B$ is maximal monotone. Hence

$$\langle y_n - p, Ay_n + v_n\rangle \geq 0. \qquad (3.7)$$

Substituting (3.6) into (3.7), we have

$$\frac{1}{\lambda_n}\langle y_n - p, Jx_n - Jy_n - \lambda_n Ax_n + \lambda_n Ay_n\rangle \geq 0,$$

which implies that

$$\langle y_n - p, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0. \qquad (3.8)$$

Combining (3.5) and (3.8), we have

$$\phi(p, z_n) \leq \phi(p, x_n) - \Big(1 - \frac{\kappa\lambda_n^2 L^2}{c}\Big)\phi(y_n, x_n). \qquad (3.9)$$

∎

Theorem 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Suppose that $\{\lambda_n\}$ be a sequence in $\left(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\right)$ such that $\{\lambda_n\} \subset [a', b'] \subset \left(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\right)$ for some $a', b' > 0$. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = \Pi_\Omega(u)$.*

Proof. We first show that $\{x_n\}$ is bounded. Let $z \in \Omega$. Since $\lambda_n \in \left(0, \frac{\sqrt{c}}{\sqrt{\kappa}L}\right)$, we have $1 - \frac{\kappa\lambda_n^2 L^2}{c} > 0$. This implies by Lemma 3.1 that

$$\phi(z, z_n) \leq \phi(z, x_n). \tag{3.10}$$

Put $w_n = J^{-1}(\beta_n J z_n + (1-\beta_n) JTz_n)$ for all $n \in \mathbb{N}$. Thus by (2.2) and (3.10), we have

$$\begin{aligned} \phi(z, w_n) &\leq \beta_n \phi(z, z_n) + (1-\beta_n)\phi(z, Tz_n) \\ &\leq \beta_n \phi(z, z_n) + (1-\beta_n)\phi(z, z_n) \\ &\leq \phi(z, x_n). \end{aligned} \tag{3.11}$$

Using (3.11), we obtain

$$\begin{aligned} \phi(z, x_{n+1}) &\leq \alpha_n \phi(z, u) + (1-\alpha_n)\phi(z, w_n) \\ &\leq \alpha_n \phi(z, u) + (1-\alpha_n)\phi(z, x_n) \\ &\leq \max\{\phi(z, u), \phi(z, x_n)\} \\ &\;\;\vdots \\ &\leq \max\{\phi(z, u), \phi(z, x_1)\}. \end{aligned}$$

This implies that $\{\phi(z, x_n)\}$ is bounded. Applying Lemma 2.7, we have $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Let $x^* = \Pi_\Omega(u)$. From Lemma 2.8 and (3.9), we have

$$\begin{aligned} \phi(x^*, w_n) &\leq \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, Tz_n) - \beta_n(1-\beta_n) g(\|Jz_n - JTz_n\|) \\ &\leq \beta_n \phi(x^*, z_n) + (1-\beta_n)\phi(x^*, z_n) - \beta_n(1-\beta_n) g(\|Jz_n - JTz_n\|) \\ &\leq \beta_n \phi(x^*, z_n) + (1-\beta_n)\left\{\phi(x^*, x_n) - \left(1 - \frac{\kappa\lambda_n^2 L^2}{c}\right)\phi(y_n, x_n)\right\} \\ &\quad -\beta_n(1-\beta_n) g(\|Jz_n - JTz_n\|) \\ &\leq \phi(x^*, x_n) - (1-\beta_n)\left(1 - \frac{\kappa\lambda_n^2 L^2}{c}\right)\phi(y_n, x_n) \\ &\quad -\beta_n(1-\beta_n) g(\|Jz_n - JTz_n\|). \end{aligned} \tag{3.12}$$

Then we have

$$\begin{aligned} &\phi(x^*, x_{n+1}) \\ \leq\; &\alpha_n \phi(x^*, u) + (1-\alpha_n)\left\{\phi(x^*, x_n) - (1-\beta_n)\left(1 - \frac{\kappa\lambda_n^2 L^2}{c}\right)\phi(y_n, x_n)\right. \\ &\left. -\beta_n(1-\beta_n) g(\|Jz_n - JTz_n\|)\right\} \\ =\; &\alpha_n \phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n) - (1-\alpha_n)(1-\beta_n)\left(1 - \frac{\kappa\lambda_n^2 L^2}{c}\right)\phi(y_n, x_n) \\ &-(1-\alpha_n)\beta_n(1-\beta_n) g(\|Jz_n - JTz_n\|). \end{aligned}$$

This implies that

$$
\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\Big(1-\frac{\kappa\lambda_n^2L^2}{c}\Big)\phi(y_n,x_n)+(1-\alpha_n)\beta_n(1-\beta_n)g(\|Jz_n-JTz_n\|)\\
\leq\ & \phi(x^*,x_n)-\phi(x^*,x_{n+1})+\alpha_nK, \qquad (3.13)
\end{aligned}
$$

where $K=\sup_{n\in\mathbb{N}}\{|\phi(x^*,u)-\phi(x^*,x_n)|\}$.

The rest of the proof will be divided into two cases:

Case 1. Suppose that there exists $N\in\mathbb{N}$ such that $\phi(x^*,x_{n+1})\leq\phi(x^*,x_n)$ for all $n\geq N$. This implies that $\lim_{n\to\infty}\phi(x^*,x_n)$ exists. By our assumptions, we have from (3.13) that

$$\lim_{n\to\infty}\phi(y_n,x_n)=0 \text{ and } \lim_{n\to\infty}g(\|Jz_n-JTz_n\|)=0. \qquad (3.14)$$

Consequently,

$$\lim_{n\to\infty}\|x_n-y_n\|=0 \text{ and } \lim_{n\to\infty}\|Jz_n-JTz_n\|=0. \qquad (3.15)$$

Moreover, we also have

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0. \qquad (3.16)$$

Since A is Lipschitz continuous, we have

$$\lim_{n\to\infty}\|Ax_n-Ay_n\|=0 \qquad (3.17)$$

and hence

$$
\begin{aligned}
\|Jz_n-Jy_n\| &= \lambda_n\|Ax_n-Ay_n\|\\
&\to 0. \qquad (3.18)
\end{aligned}
$$

Combining (3.16) and (3.18), we obtain

$$
\begin{aligned}
\|Jx_n-Jz_n\| &\leq \|Jx_n-Jy_n\|+\|Jy_n-Jz_n\|\\
&\to 0. \qquad (3.19)
\end{aligned}
$$

Moreover from (3.15) and (3.19), we obtain

$$
\begin{aligned}
\|Jx_{n+1}-Jx_n\| &\leq \|Jx_{n+1}-Jw_n\|+\|Jw_n-Jz_n\|+\|Jz_n-Jx_n\|\\
&= \alpha_n\|Ju-Jw_n\|+(1-\beta_n)\|JTz_n-Jz_n\|+\|Jz_n-Jx_n\|\\
&\to 0. \qquad (3.20)
\end{aligned}
$$

Then we have from (3.19) and (3.20) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0 \qquad (3.21)$$

and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.22)$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}\in E$ and

$$\limsup_{n\to\infty}\langle x_n-x^*,Ju-Jx^*\rangle=\lim_{k\to\infty}\langle x_{n_k}-x^*,Ju-Jx^*\rangle.$$

From (3.21), we also have $z_{n_k} \rightharpoonup \hat{x}$. Since $\|z_n - Tz_n\| \to 0$ and $I - T$ is demi-closed at zero, we have $\hat{x} \in F(T)$. We next show that $\hat{x} \in (A+B)^{-1}0$. Let $(v, w) \in G(A+B)$, we have $w - Av \in Bv$. Since

$$(J - \lambda_{n_k} A)x_{n_k} \in (J + \lambda_{n_k} B)y_{n_k}.$$

It follows that

$$\frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \in By_{n_k}.$$

Since B is maximal monotone, we have

$$\left\langle v - y_{n_k}, w - Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \right\rangle \geq 0$$

Using the monotonicity of A, we have

$$\begin{aligned}
\langle v - y_{n_k}, w \rangle &\geq \left\langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \right\rangle \\
&= \langle v - y_{n_k}, Av - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k} \rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle \\
&\quad + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k} \rangle.
\end{aligned}$$

Since $y_{n_k} \rightharpoonup \hat{x}$, it follows from (3.16) and (3.17) that

$$\langle v - \hat{x}, w \rangle \geq 0.$$

By the monotonicity of $A + B$, we get $0 \in (A + B)\hat{x}$, that is, $\hat{x} \in (A + B)^{-1}0$. So $\hat{x} \in \Omega := F(T) \cap (A + B)^{-1}0$. Thus we have

$$\limsup_{n\to\infty} \langle x_n - x^*, Ju - Jx^* \rangle = \langle \hat{x} - x^*, Ju - Jx^* \rangle \leq 0.$$

From (3.22), we also have

$$\limsup_{n\to\infty} \langle x_{n+1} - x^*, Ju - Jx^* \rangle \leq 0. \tag{3.23}$$

Finally, we show that $x_n \to x^*$. By Lemma 2.11, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n)) \\
&= V(x^*, \alpha_n Ju + (1 - \alpha_n)Jw_n) \\
&\leq V(x^*, \alpha_n Ju + (1 - \alpha_n)Jw_n - \alpha_n(Ju - Jx^*)) \\
&\quad + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\
&= V(x^*, \alpha_n Jx^* + (1 - \alpha_n)Jw_n) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\
&= \phi(x^*, J^{-1}(\alpha_n Jx^* + (1 - \alpha_n)Jw_n)) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\
&\leq \alpha_n \phi(x^*, x^*) + (1 - \alpha_n)\phi(x^*, w_n) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle \\
&\leq (1 - \alpha_n)\phi(x^*, x_n) + 2\alpha_n \langle x_{n+1} - x^*, Ju - Jx^* \rangle.
\end{aligned} \tag{3.24}$$

This together with (3.23) and (3.24), so we can conclude by Lemma 2.15 that $\phi(x^*, x_n) \to 0$. Therefore, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\phi(x^*, x_{n_i})\}$ of $\{\phi(x^*, x_n)\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. By Lemma 2.16, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \tag{3.25}$$

and

$$\phi(x^*, x_k) \leq \phi(x^*, x_{m_k}). \tag{3.26}$$

As proved in the **Case 1**, we obtain

$$\begin{aligned}
&(1-\alpha_{m_k})(1-\beta_{m_k})\Big(1-\frac{\kappa\lambda_{m_k}^2 L^2}{c}\Big)\phi(y_{m_k}, x_{m_k}) \\
&+(1-\alpha_{m_k})\beta_{m_k}(1-\beta_{m_k})g(\|Jz_{m_k} - JTz_{m_k}\|) \\
\leq\ & \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) + \alpha_{m_k} K \\
\leq\ & \alpha_{m_k} K,
\end{aligned}$$

where $K = \sup_{k\in\mathbb{N}}\{|\phi(x^*, u) - \phi(x^*, x_{m_k})|\}$. By our assumptions, we have

$$\lim_{k\to\infty} \phi(y_{m_k}, x_{m_k}) = 0 \text{ and } \lim_{k\to\infty} g(\|Jz_{m_k} - JTz_{m_k}\|) = 0.$$

Consequently,

$$\lim_{k\to\infty} \|x_{m_k} - y_{m_k}\| = 0 \text{ and } \lim_{k\to\infty} \|Jz_{m_k} - JTz_{m_k}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we can show that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty} \langle x_{m_k+1} - x^*, Ju - Jx^*\rangle \leq 0.$$

From (3.24) and (3.25), we have

$$\begin{aligned}
\phi(x^*, x_{m_k+1}) &\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k}) + \alpha_{m_k}\langle x_{m_k+1} - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_{m_k})\phi(x^*, x_{m_k+1}) + \alpha_{m_k}\langle x_{m_k+1} - x^*, Ju - Jx^*\rangle.
\end{aligned}$$

This implies that

$$\phi(x^*, x_{m_k+1}) \leq \langle x_{m_k+1} - x^*, Ju - Jx^*\rangle.$$

Then we have

$$\limsup_{k\to\infty} \phi(x^*, x_{m_k+1}) \leq 0. \tag{3.27}$$

Combining (3.26) and (3.27) we obtain

$$\limsup_{k\to\infty} \phi(x^*, x_k) \leq 0.$$

Hence $\limsup_{k\to\infty} \phi(x^*, x_k) = 0$ and so $x_k \to x^*$. This completes the proof. ∎

If we take $T = I$ in Theorem 3.2, then we obtain the following result regarding the monotone quasi-inclusion problem (1.1).

Corollary 3.3. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Assume that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, u \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(Jy_n - \lambda_n(Ay_n - Ax_n))), \ \forall n \geq 1, \end{cases} \tag{3.28}$$

where $\{\lambda_n\}$ is a sequence in $\left(0, \frac{1}{L}\right)$ such that $\{\lambda_n\} \subset [a', b'] \subset \left(0, \frac{1}{L}\right)$ for some $a', b' > 0$. Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.28) converges strongly to $x^* \in (A+B)^{-1}0$, where $x^* = \Pi_{(A+B)^{-1}0}(u)$.

We next propose a strong convergence theorem of another modification of Tseng's splitting algorithm with line search for solving the monotone inclusion problem and the fixed point problem in Banach spaces. It is noted that this proposed algorithm does not required to know the Lipschitz constant of the Lipschitz continuous mapping.

Algorithm 2 Halpern-Tseng type algorithm with Armijo-type line search

Step 0. Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in \left(0, \sqrt{\frac{c}{\kappa}}\right)$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n Ax_n), \tag{3.29}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{3.30}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.31}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n Jz_n + (1-\beta_n)JTz_n)). \tag{3.32}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.4. *The Armijo line search rule defined by (3.30) is well defined and*

$$\min\{\gamma, \frac{\mu l}{L}\} \leq \lambda_n \leq \gamma.$$

Proof. Since A is L-Lipschitz continuous on E, we have

$$\|Ax_n - A(J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n))\| \leq L\|x_n - J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n)\|.$$

Using the fact that $L > 0$ and $\mu > 0$, we get

$$\frac{\mu}{L}\|Ax_n - A(J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n))\| \leq \mu\|x_n - J^B_{\gamma l^{m_n}} J^{-1}(Jx_n - \gamma l^{m_n} Ax_n)\|.$$

This implies that (3.30) holds for all $\gamma l^{m_n} \leq \frac{\mu}{L}$ and so λ_n is well defined. Obviously, $\lambda_n \leq \gamma$. If $\lambda_n = \gamma$, then the lemma is proved. Otherwise, if $\lambda_n < \gamma$, then we have from

(3.30) that

$$\|Ax_n - A(J^B_{\frac{\lambda_n}{l}}J^{-1}(Jx_n - \frac{\lambda_n}{l}Ax_n))\| \quad > \quad \frac{\mu}{\frac{\lambda_n}{l}}\|x_n - J^B_{\frac{\lambda_n}{l}}J^{-1}(Jx_n - \frac{\lambda_n}{l}Ax_n)\|.$$

Again by the L-Lipschitz continuity of A, we obtain $\lambda_n > \frac{\mu l}{L}$. This completes the proof. ∎

Lemma 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then*

$$\phi(p, z_n) \leq \phi(p, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\Big)\phi(y_n, x_n), \ \forall p \in (A+B)^{-1}0,$$

where c and κ are the constants in Lemma 2.4.

Proof. From (3.30), we see that $\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_n}\|x_n - y_n\|$. By using the same arguments as in the proof of Lemma 3.1, we can show that this lemma holds. ∎

Theorem 3.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$.*

Proof. By using the same arguments as in the proof of Theorem 3.2, we immediately obtain the proof. ∎

If we take $T = I$ in Theorem 3.6, then we obtain the following result regarding the monotone quasi-inclusion problem (1.1).

Corollary 3.7. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let the mapping $A : E \to E^*$ be monotone and L-Lipschitz continuous and $B : E \to 2^{E^*}$ be a maximal monotone operator. Assume that $(A+B)^{-1}0 \neq \emptyset$. Given $\gamma > 0$, $l \in (0,1)$ and $\mu \in \big(0, \sqrt{\frac{c}{\kappa}}\big)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, u \in E, \\ y_n = J^B_{\lambda_n}J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)(Jy_n - \lambda_n(Ay_n - Ax_n))), \ \forall n \geq 1, \end{cases} \tag{3.33}$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n\|Ax_n - Ay_n\| \leq \mu\|x_n - y_n\|.$$

Suppose that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.33) converges strongly to $x^* \in (A+B)^{-1}0$, where $x^* = \Pi_{(A+B)^{-1}0}(u)$.

4. Numerical Experiments

In this section, we provide numerical experiments to the signal recovery in compressed sensing by using our proposed algorithms. Moreover, we also compare the mentioned algorithms with Tseng's splitting algorithm (1.5). In signal recovery, compressed sensing can be modeled as the following under determinated linear equation system:

$$y = Cx + \varepsilon \tag{4.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $C : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear observation operator. It is known that to solve (4.1) can be seen as solving the LASSO problem [5]:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Cx-y\|_2^2+\lambda\|x\|_1, \tag{4.2}$$

where $\lambda > 0$. In this case, we set $A = \nabla f$ the gradient of f, where $f(x) = \frac{1}{2}\|Cx - y\|_2^2$ and $B = \partial g$ the subdifferential of g, where $g(x) = \lambda\|x\|_1$. Then the LASSO problem (4.2) can be considered as the monotone quasi-inclusion problem (1.1). It is known that $\nabla f(x) = C^t(Cx - y)$ and it is $\|C\|^2$-Lipschitz continuous and monotone (see [3]). Moreover, ∂g is maximal monotone (see [1]).

In this experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $C \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio (SNR)=40. The restoration accuracy is measured by the mean squared error (MSE) as follows:

$$E_n = \frac{1}{N}\|x_n - x\|_2^2 < 10^{-5}, \tag{4.3}$$

where x_n is an estimated signal of x. In our numerical test, we compare our Algorithm 3 and Algorithm 3 ($T = I$) with Tseng's splitting algorithm (1.5).

We take $\alpha_n = \frac{1}{15(n+5)}$ and $\lambda_n = \frac{0.3}{\|C\|^2}$ in Algorithm 3 and take $\lambda_n = \frac{0.3}{\|C\|^2}$ in Tseng's splitting algorithm (1.5). For Alogorithm 3, we take $\alpha_n = \frac{1}{15(n+5)}$, $\gamma = 5$, $\mu = 0.5$, $l = 0.3$. The point u is chosen to be $(1, 1, 1, \ldots, 1) \in \mathbb{R}^N$ and the starting point x_1 is randomly generated in $\mathbb{R}^N$. We perform the numerical test with the following four cases:

Case 1: $N = 512$, $M = 256$ and $m = 10$;
Case 2: $N = 1024$, $M = 512$ and $m = 30$;
Case 3: $N = 2048$, $M = 1024$ and $m = 60$;
Case 4: $N = 4096$, $M = 2048$ and $m = 100$.

The numerical results are reported as follows:

TABLE 1. The comparison of the proposed algorithms with Tseng's splitting algorithm

		Algorithm 3	Algorithm 3	Tseng's splitting algorithm
Case 1	No. of Iter.	1,850	4,864	5,689
Case 2	No. of Iter.	3,320	10,186	12,753
Case 3	No. of Iter.	7,126	19,076	24,666
Case 4	No. of Iter.	14,889	40,743	48,652

We next demonstrate the graphs of original signal and recovered signal by Algorithm 3, Algorithm 3 and Tseng's splitting algorithm. The number of iterations are reported in the Figures 1-8, respectively.

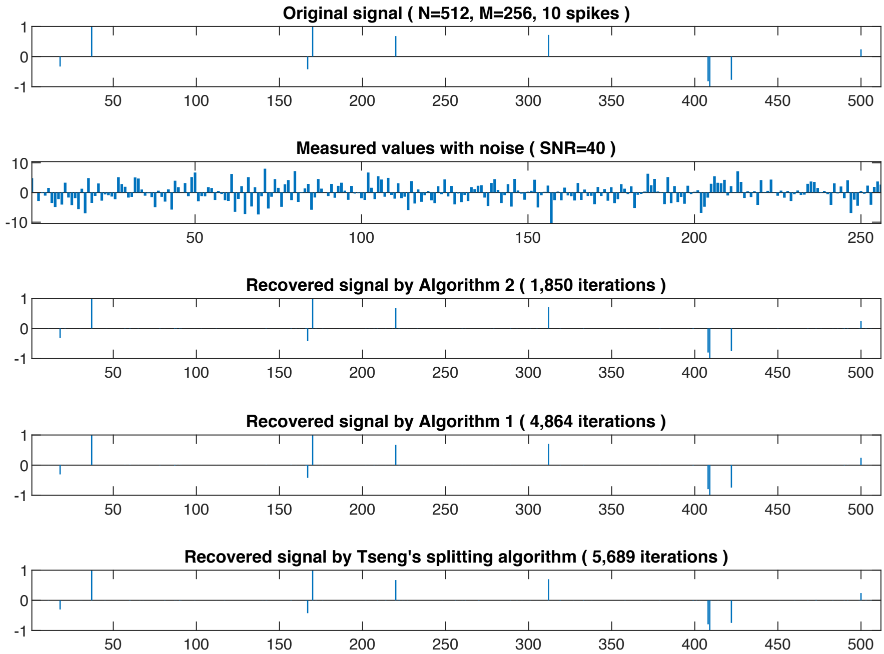

Figure 1: The comparison of recovered signal by using different algorithms in **Case 1**.

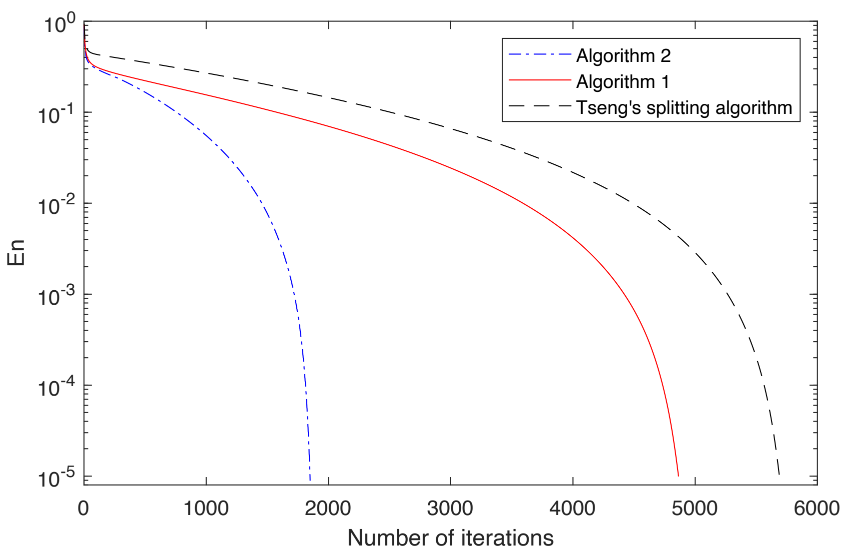

Figure 2: The plotting of MSE versus number of iterations in **Case 1**.

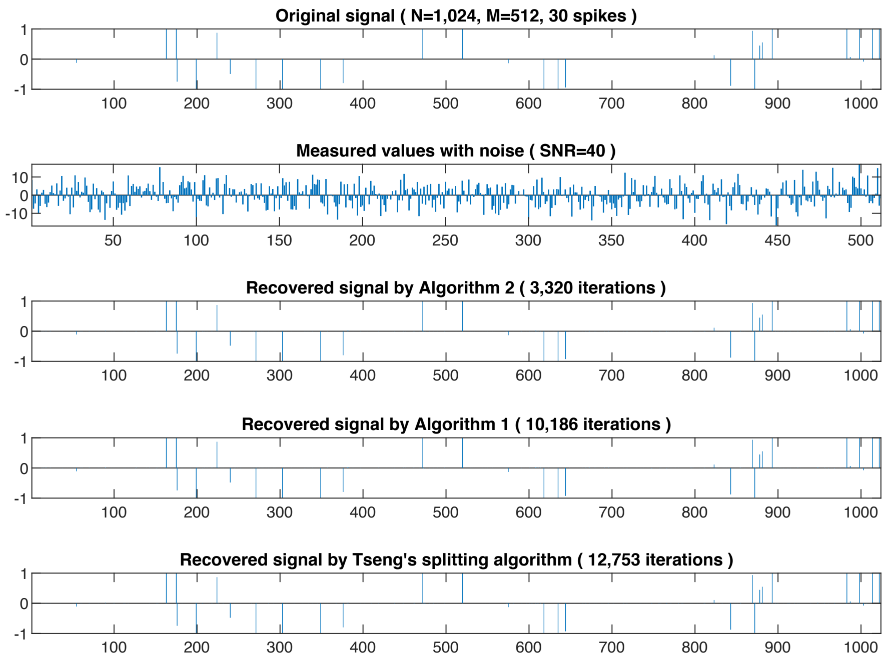

Figure 3: The comparison of recovered signal by using different algorithms in **Case 2**.

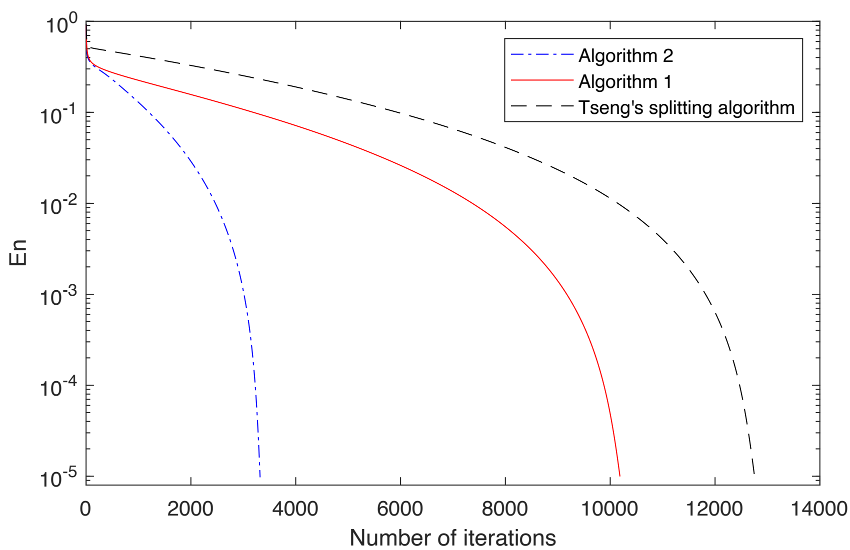

Figure 4: The plotting of MSE versus number of iterations in **Case 2**.

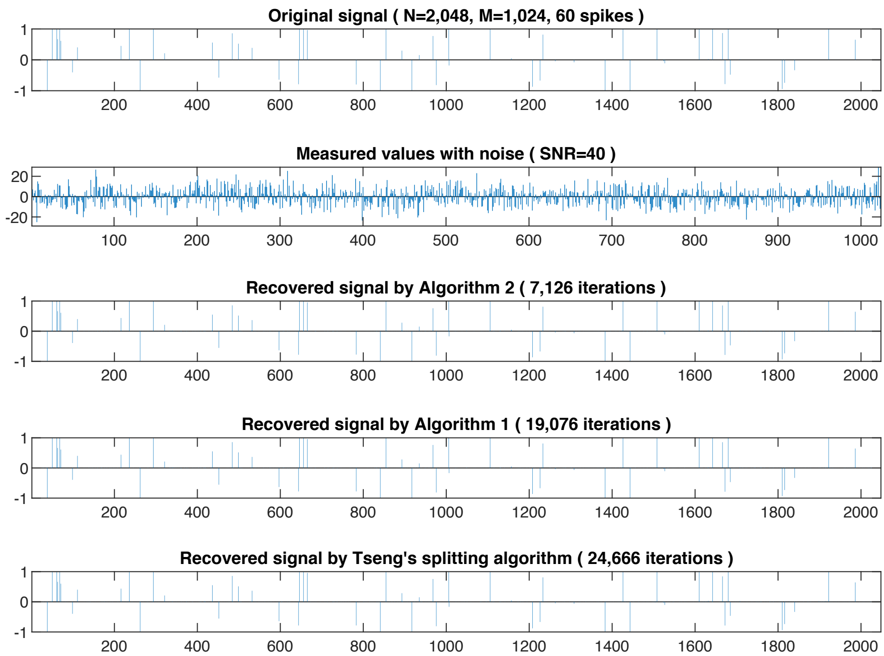

Figure 5: The comparison of recovered signal by using different algorithms in **Case 3**.

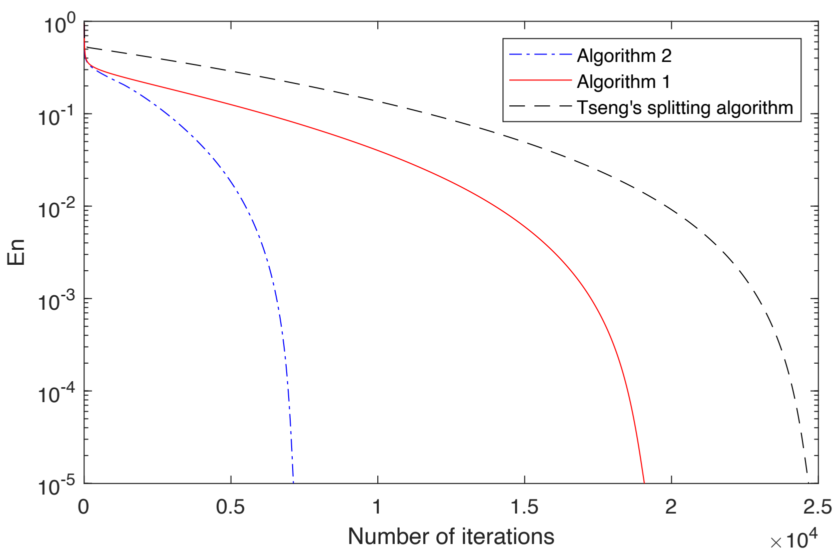

Figure 6: The plotting of MSE versus number of iterations in **Case 3**.

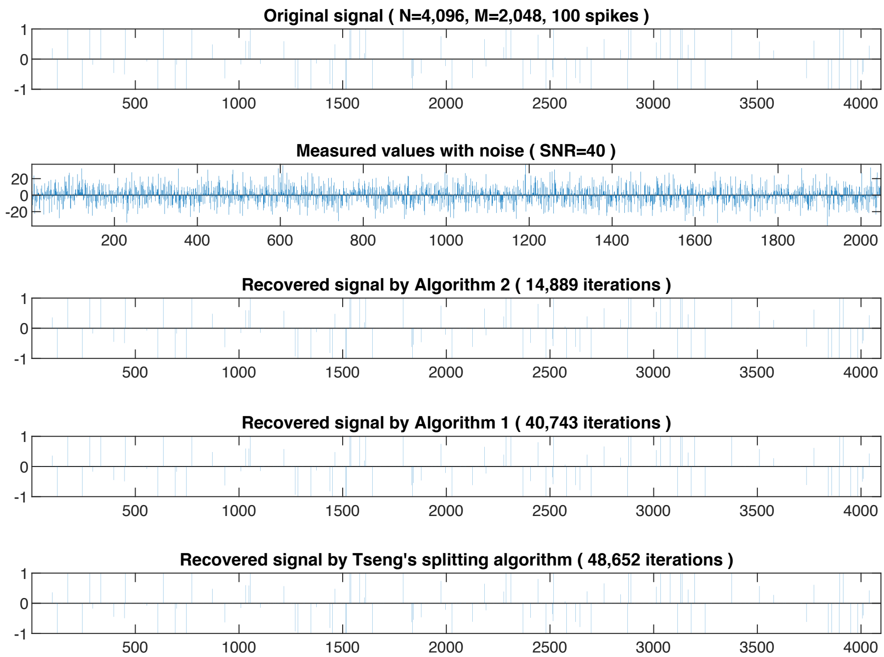

Figure7: The comparison of recovered signal by using different algorithms in **Case 4**.

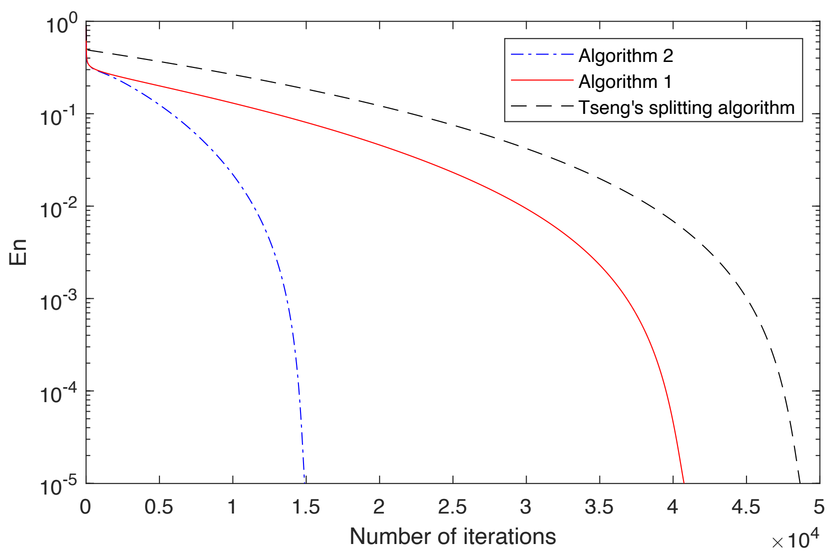

Figure 8: The plotting of MSE versus number of iterations in **Case 4**.

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Iterative algorithms for the split combination of variational inequalities and various nonlinear mappings

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Abstract— **In this paper, we prove a strong convergence theorem for finding a common element of the sets of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and the sets of fixed points of a finite family of nonexpansive mappings and the sets of solutions of the split combination of variational inequalities.**

Keywords— **Split combination of variational inequalities, Fixed point problem, Strictly pseudo- contractive mapping, Nonexpansive mapping.**

I. INTRODUCTION

The fixed point theory plays an important role in nonlinear functional analysis and becomes a very useful tool in various fields. In the past few decades, fixed point theorem has been applied in real world problems such as intensity modulated radiation therapy (IMRT), signal processing and image reconstruction.

Throughout this article, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. A mapping $f:C\to C$ is said to be a contraction if there exists a constant $\alpha\in(0,1)$ such that

$$\|f(x)-f(y)\|\le\alpha\|x-y\|,\forall x,y\in C.$$

A mapping $A:C\to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax-Ay,x-y\rangle\ge\alpha\|Ax-Ay\|^2,$$

for all $x,y\in C$. Let $T:C\to C$ be a nonlinear mapping. A point $x\in C$ is called a fixed point of T if $Tx=x$. The set of fixed points of T is the set $F(T):=\{x\in C:Tx=x\}$.

Definition 1.1. Let $T:C\to C$ be a nonlinear mapping, then

(i) T is said to be nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|,\ \forall x,y\in C,$$

(ii) T is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa\in[0,1)$ such that $\|Tx-Ty\|^2\le\|x-y\|^2+\kappa\|(I-T)x-(I-T)y\|^2,\ \forall x,y\in C.$ It is well known that if $T:H\to H$ is a nonexpansive mapping, we have $\langle Ty-Tx,(I-T)x-(I-T)y\rangle\le\frac{1}{2}\|(I-T)x-(I-T)y\|^2,\ \forall x,y\in H.$ Moreover, we also know that $\langle y-Tx,(I-T)x\rangle\le\frac{1}{2}\|(I-T)x\|^2,$ for all $x\in H$ and $y\in F(T)$.

In 2006, Marino and Xu [1] introduced the general iterative method based on the viscosity approximation method proposed by Moudafi [2] in 2000 as follows:

$$\begin{cases}x_0\in H_1\\ x_{n+1}=(I-\alpha_nD)Tx_n+\alpha_n\xi f(x_n),n\ge 0,\end{cases}\tag{1.1}$$

where T is a nonexpansive mapping, f is a contractive mapping on H, D is a strongly positive bounded linear self-adjoint operator and $\{\alpha_n\}$ is a sequence in $(0,1)$. They also proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.1).

The Split Variational Inequality Problem (SVIP), which was introduced and studied by Censer et. al. [3] in 2012, has a great impact and influence in the classes of mathematical problems and widely studied in many fields of pure and applied sciences. They introduced SVIP as follows: Let H_1 and H_2 be two real Hilbert spaces and C,Q be a nonempty closed convex subset of H_1 and H_2, respectively. The split variational inequality problems is to find $x^*\in C$ such that $\langle f(x^*x^*),x-x^*\rangle\ge 0$ for all $x\in C$ and such that

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$y^* = Ax^* \in Q$ solves $\langle g(y^*), y - y^* \rangle \geq 0$ for all $y \in Q$, where $f : C \to H_1$ and $g : Q \to H_2, A : H_1 \to H_2$ be a bounded linear operator. Many authors have increasingly investigated the split variational inequality problem (SVIP), see for instance [4, 5] and references therein.

In 2016, Kangtunyakarn [6] have modified SVIP which is called the split combination of variational inequality problem as follows: For every $i = 1, 2$, let C_i be a nonempty closed convex subset of H_i. The split combination of variational inequality problem is to find a point $x^* \in C_I$ such that

$$\langle (a_1 A_1 + (1 - a_1) B_1) x^*, x - x^* \rangle \geq 0, \ \forall x \in C_1, \tag{1.2}$$

and find a point $y^* = Ax^*$ such that

$$\langle (a_2 A_2 + (1 - a_2) B_2) y^*, y - y^* \rangle \geq 0, \ \forall y \in C_2, \tag{1.3}$$

where $A : H_1 \to H_2$ is a bounded linear operator with adjoint A of A, $A_i, B_i : C_i \to H_i$ and $a_i \in [0,1]$ for all $i = 1, 2$. The set of all solutions of (1.2) and (1.3) is denoted by $\Omega = \{x^* \in VI(C_1, (a_1 A_1 + (1 - a_1) B_1))$ and $Ax^* \in VI(C_2, (a_2 A_2 + (1 - a_2) B_2))\}$, where $a_i \in [0,1]$ for all $i = 1, 2$.

In this paper, motivated by the above-mentioned results, we prove a strong convergence theorem for finding a common element of the sets of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and the sets of fixed points of a finite family of nonexpansive mappings and the sets of solutions of the split combination of variational inequalities.

II. Preliminary

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subset of H_1 and H_2, respectively. Recall that H_1 satisfies Opial's condition [9], i.e., for any sequence $\{x_n\}$ with $x_n \xrightarrow{w} x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

holds for every $y \in H_1$ with $y \neq x$.

For a proof of the our main results, we will use the following lemmas.

Lemma 2.1 ([7]). Given $x \in H_1$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \ \forall z \in C.$$

Lemma 2.2 ([8]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1 - \alpha_n) s_n + \delta_n, \ \ \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty;$

(2) $\limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then, $\lim_{n \to \infty} s_n = 0.$

Definition 2.1 ([10]). Let C be a nonempty convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S^A : C \to C$ as follows:

$$U_0 = I$$
$$U_1 = S_1(\alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I),$$
$$U_2 = S_2(\alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I),$$
$$U_3 = S_3(\alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I),$$
$$\vdots$$
$$U_{N-1} = S_{N-1}(\alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I)$$
$$S^A = U_N = S_N(\alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I).$$

This mapping is called the S^A-mapping generated by $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots \alpha_N$.

Lemma 2.3 ([10]). Let C be a nonempty convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \neq \phi$ and $\kappa = \max_{i=1,2,\ldots,N} \{\kappa_i\}$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1), \alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, \ldots, N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots \alpha_N$. Then $F(S^A) = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$ and S^A is a nonexpansive mapping.

Lemma 2.4 ([1]). Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.5 ([6]). For every i = 1, 2, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i and let A : $H_1 \to H_2$ be a bounded linear operator. For every i = 1, 2, ..., N, let $A_i, B_i : C_1 \to H_1$ be α_i, β_i-inverse strongly monotone mapping with $\eta = \min_{i=1,2,\dots,N}\{\eta_i\}$. Assume that Ω is a nonempty. Then the following are equivalent:

(1) $x^* \in \Omega$;

(2) $x^* = P_{C_1}(I - \lambda_1(a_1A_1 + (1-a_1)B_1))(x^* - \gamma A^*(I - P_{C_2}(I - \lambda_2(a_2A_2 + (1-a_2)B_2)))Ax^*)$,

where $\lambda_i \in (0, 2\eta_i)$, $a_i \in (0,1)$, for all i = 1,2 and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A.

III. Main Results

Theorem 3.1. For every i = 1, 2, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i and let A : $H_1 \to H_2$ be a bounded linear operator. For every $i = 1,2,\dots,N$, let $A_i, B_i : C_1 \to H_1$ be α_i, β_i -inverse strongly monotone mapping with $\eta = \min_{i=1,2,\dots,N}\{\alpha_i, \beta_i\}$. Let Ω be a solution of (1.2) and (1.3) and $\Omega \neq \varnothing$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\psi = \cap_{i=1}^N F(T_i) \cap \cap_{i=1}^N F(S_i) \cap \Omega \neq \phi$ and $\kappa = \max_{i=1,2,\dots,N}\{\kappa_i\}$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where I = [0, 1] , $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\dots,N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1), \alpha_2^j \in [\kappa, 1)$ for all $j = 1,2,\dots,N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Let $f: C_1 \to H_1$ be a contractive mapping with $\alpha \in (0, 1)$ and let D be a strongly positive bounded linear operator with coefficient $\overline{\xi} \in (0,1)$ with 0 $0 < \xi < \frac{\overline{\xi}}{\alpha}$. Let the sequence { x_n } be generated by $x_1 \in C_1$ and

$$\begin{cases} u_n = P_{C_1}(I - \lambda_1 M_1)(x_n - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))Ax_n) \\ x_{n+1} = \alpha_n \xi f(x_n) + (I - \alpha_n D)(\beta_n S^A x_n + (1-\beta_n)u_n) \forall n \in \mathbb{N} \end{cases} \quad (3.1)$$

where $M_i = (a_iA_i + (1-a_i)B_i)$, $\lambda_i \in (0, 2\eta_i)$, $a_i \in (0,1)$, for all $i = 1,2$, $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I - D + \xi f)(z)$.

Proof. Since $\alpha_n \to 0$ as n $n \to 0$, without loss of generality, we may assume that $\alpha_n \leq \|D\|^{-1}$, for all $n \in \mathbb{N}$.

We divide the proof into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded. Let $z \in \Psi$. From Lemma 2.5, we have

$$z = P_{C_1}(I - \lambda_1 M_1)(z - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))Az).$$

Putting $y_n = \beta_n S^A x_n + (1-\beta_n)u_n$. From Lemma 2.3 and applying in Lemma 2.5, we have

$$\begin{aligned} \|y_n - z\| &= \|\beta_n S^A x_n + (1-\beta_n)u_n - z\| \\ &\leq \beta_n \|S^A x_n - z\| + (1-\beta_n)\|u_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \quad (3.2)$$

From the definition of x_n and (3.2), we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n \xi f(x_n) + 1(I - \alpha_n D)y_n - z\| \\ &= \|\alpha_n(\xi f(x_n) - Dz) + (I - \alpha_n D)(y_n - z)\| \\ &\leq \alpha_n \|\xi f(x_n) - Dz\| + \|I - \alpha_n D\|\|y_n - z\| \\ &\leq \alpha_n(\xi\|f(x_n) - h(z)\| + \|\xi f(z) - Dz\|) + (1 - \alpha_n \xi)\|x_n - z\| \\ &\leq (1 - \alpha_n)(\overline{\xi} - \xi\alpha))\|x_n - z\| + \alpha_n \|\xi f(z) - Dz\| \\ &\leq \max\left\{\|x_1 - z\|, \frac{\|\xi f(z) - Dz\|}{\xi - \xi\alpha}\right\}. \end{aligned}$$

By mathematical induction, we have $\|x_n - z\| \leq K, \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. By continuting in the same direction as in Step2 of Theorem 3.1 in , we have

$$\|u_n-u_{n-1}\|^2 \le \|x_n-x_{n-1}\|^2. \tag{3.3}$$

From the definition of y_n and (3.3), we have

$$\|y_n-y_{n-1}\|=\|\beta_n S^A x_n+(1-\beta_n)u_n-\beta_{n-1}S^A x_{n-1}-(1-\beta_{n-1})u_{n-1}\|$$
$$=\|\beta_n(S^A x_n-S^A x_{n-1})+(\beta_n-\beta_{n-1})S^A x_{n-1}+(1-\beta_n)(u_n-u_{n-1})+(\beta_{n-1}-\beta_n)u_{n-1}\|$$

$$\le\beta_n\|S^A x_n-S^A x_{n-1}\|+|\beta_n-\beta_{n-1}|\|S^A x_{n-1}\|+(1-\beta_n)\|u_n-u_{n-1}\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|$$
$$\le\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|S^A x_{n-1}\|+(1-\beta_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|$$
$$=\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|S^A x_{n-1}\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|. \tag{3.4}$$

From the definition of x_n and (3.4), we have

$$\|x_{n+1}-x_n\|=\|\alpha_n\xi f(x_n)+(I-\alpha_n D)y_n-\alpha_{n-1}\xi f(x_{n-1})-(I-\alpha_{n-1}D)y_{n-1}\|$$
$$\le\alpha_n\xi\|f(x_n)-f(x_{n-1})\|+\xi|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|$$
$$+\|(I-\alpha_n D)\|\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Dy_{n-1}\|$$
$$\le\alpha_n\xi\alpha\|x_n-x_{n-1}\|+\xi|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|$$
$$+(I-\alpha_n\xi)\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Dy_{n-1}\|$$
$$\le\alpha_n\xi\alpha\|x_n-x_{n-1}\|+\xi|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|$$
$$+|\alpha_n-\alpha_{n-1}|\|Dy_{n-1}\|$$
$$+(I-\alpha_n\bar{\xi})(\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|S^A x_{n-1}\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|)$$
$$\le(1-\alpha_n(\bar{\xi}-\xi\alpha))\|x_n-x_{n-1}\|+\xi|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|$$
$$+|\alpha_n-\alpha_{n-1}|\|Dy_{n-1}\|$$
$$+|\beta_n-\beta_{n-1}|\|S^A x_{n-1}\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|). \tag{3.5}$$

Applying Lemma 2.2, (3.5) and the conditions (i), (ii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

Step 3. We show that $\lim_{n\to\infty}\|S^A x_n-x_n\|=\lim_{n\to\infty}\|u_n-x_n\|=0$.

From the definition of x_n, we have

$$\|x_{n+1}-y_n\|=\|\alpha_n\xi f(x_n)+(I-\alpha_n D)y_n-y_n\|$$
$$\le\alpha_n\|\xi f(x_n)-\alpha_n Dy_n\|$$

From above and the condition (i), we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.7}$$

Observe that

$$\|x_n-y_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\| \tag{3.8}$$

From (3.6), (3.7), and (3.8), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.9}$$

From the definition of y_n, we have

$$\|y_n-z\|^2=\|\beta_n S^A x_n+(1-\beta_n)u_n-z\|^2$$
$$=\beta_n\|S^A x_n-z\|^2+(1-\beta_n)\|u_n-z\|^2-\beta_n(1-\beta_n)\|S^A x_n-u_n\|^2$$
$$\le\|x_n-z\|^2-\beta_n(1-\beta_n)\|S^A x_n-u_n\|^2$$

It implies that

$$\beta_n(1-\beta_n)\|S^A x_n-u_n\|^2\le\|x_n-z\|^2-\|y_n-z\|^2$$
$$=(\|x_n-z\|+\|y_n-z\|)\|x_n-y_n\|.$$

From the condition (ii) and (3.9), we have

$$\lim_{n\to\infty}\|S^A x_n-y_n\|=0. \tag{3.10}$$

Observe that

$$\|S^A x_n-x_n\|\le\|S^A x_n-y_n\|+\|y_n-x_n\|. \tag{3.11}$$

From (3.9), (3.10), and (3.11), we have

$$\lim_{n\to\infty}\|S^A x_n-x_n\|=0. \tag{3.12}$$

Since

$$y_n-S^A x_n=\beta_n S^A x_n+(1-\beta_n)u_n-S^A x_n,$$

then

$$y_n-S^A x_n=(1-\beta_n)(u_n-S^A x_n),$$

From above , (3.10), and the condition (ii), we have

$$\lim_{n\to\infty}\|S^A x_n-u_n\|=0. \tag{3.13}$$

Observe that

$$\|u_n-x_n\|\le\|u_n-S^A x_n\|+\|S^A x_n-x_n\| \tag{3.14}$$

From (3.12), (3.13), and (3.14), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.15}$$

Step 4. We will show that $\limsup_{n\to\infty}(((\xi f-D)z, x_n-z\rangle\le 0$, where $z=P_\Psi(I-D+\xi f)z$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle (\xi f - D)z, x_n - z\right\rangle = \lim_{k\to\infty}\left\langle (\xi f - D)z, x_{n_k} - z\right\rangle. \qquad (3.16)$$

Without loss of generality, we can assume that $x_{n_k} \xrightarrow{w} \omega$ as $k \to \infty$. From (3.15), we obtain $u_{n_k} \xrightarrow{w} \omega$ as $k \to \infty$.

Next, we will show that $\omega \in \Omega$. Assume that $\omega \notin \Omega$. By Lemma 2. 5, we have

$$\omega \neq P_{C_1}(I - \lambda_1 M_1)(\omega - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))A\omega).$$

By the Opial's condition and (3.15), we obtain

$$\liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\| < \liminf_{k\to\infty}\left\|x_{n_k} - P_{C_1}(I - \lambda_1 M_1)(\omega - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))A\omega)\right\|$$

$$\leq \liminf_{k\to\infty}(\left\|x_{n_k} - P_{C_1}(I - \lambda_1 M_1)(x_{nk} - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))Ax_{n_k})\right\|$$

$$+\left\|P_{C_1}(I - \lambda_1 M_1)(x_{n_k} - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))Ax_{n_k})\right.$$

$$\left. -P_{C_1}(I - \lambda_1 M_1)(\omega - \gamma A^*(I - P_{C_2}(I - \lambda_2 M_2))A\omega))\right\|)$$

$$\leq \liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\|.$$

This is a contradiction. Then we have

$\omega \in \Omega.$

Next, we will show that $\omega \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$. From Lemma 2.3, then $F(S^A) = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$. Assume that $\omega \notin F(S^A)$. That is $\omega \notin S^A\omega$. By the nonexpansiveness of S^A, the Opial's condition, and (3.12), we obtain

$$\liminf_{k\to\infty}\left\|x_{n_K} - \omega\right\| < \liminf_{k\to\infty}\left\|x_{n_K} - S^A\omega\right\|$$

$$\leq \liminf_{k\to\infty}(\left\|x_{n_k} - S^A x_{n_k}\right\| + \left\|S^A x_{n_k} - S^A\omega\right\|)$$

$$\leq \liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\|.$$

This is a contradiction. Then we have

$$\omega \in F(S^A) = \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i).$$

Therefore $\omega \in \psi = \Omega \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$.

Since $x_{n_k} \xrightarrow{w} \omega$ as $k \to \infty$ and $\omega \in \psi$. By (3.16) and Lemma 2.1, we have

$$\limsup_{n\to\infty}\left\langle (\xi f - D)z, x_n - z\right\rangle = \lim_{k\to\infty}\left\langle (\xi f - D)z, x_{n_k} - z\right\rangle$$

$$= \left\langle (\xi f - D)z, \omega - z)\right\rangle$$

$$\leq 0. \qquad (3.17)$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z$, where $z = P_\Psi(I - D + \xi f)z$. From the definition of x_n, we have

$$\left\|x_{n+1} - z\right\|^2 = \left\|\alpha_n(\xi f(x_n) - Dz) + (I - \alpha_n D)(y_n - z)\right\|^2$$

$$\leq \left\|(I - \alpha_n D)(y_n - z)\right\|^2 + 2\alpha_n\left\langle \xi f(x_n) - Dz, x_{n+1} - z\right\rangle$$

$$\leq (I - \alpha_n \bar{\xi})^2\left\|y_n - z\right\|^2 + 2\alpha_n\left\langle \xi f(x_n) - \xi f(z), x_{n+1} - x_0\right\rangle + 2\alpha_n\left\langle \xi h(z) - Dz, x_{n+1} - x_0\right\rangle$$

$$\leq (I - \alpha_n \bar{\xi})^2\left\|y_n - z\right\|^2 + 2\alpha_n\xi\left\|f(x_n) - f(z)\right\|\left\|x_{n+1} - z\right\| + 2\alpha_n\left\langle \xi f(z) - Dz, x_{n+1} - x_0\right\rangle$$

$$\leq (I - \alpha_n \bar{\xi})^2\left\|x_n - z\right\|^2 + 2\alpha_n\xi\alpha\left\|x_n - z\right\|\left\|x_{n+1} - z\right\| + 2\alpha_n\left\langle \xi f(z) - Dz, x_{n+1} - x_0\right\rangle$$

$$= (1 - \alpha_n \bar{\xi})^2\left\|x_n - z\right\|^2 + \alpha_n\xi\alpha\left\|x_n - z\right\|^2 + \alpha_n\xi\alpha\left\|x_{n+1} - z\right\|^2 + 2\alpha_n\left\langle \xi f(z) - Dz, x_{n+1} - x_0\right\rangle$$

It implies that

$$\left\|x_{n+1} - z\right\|^2 \leq \frac{1 - 2\alpha_n\bar{\xi} + (\alpha_n\bar{\xi})^2 + \alpha_n\xi\alpha}{1 - \alpha_n\xi\alpha}\left\|x_n - z\right\|^2 + \frac{2\alpha_n}{1 - \alpha_n\xi\alpha}(\xi f(z) - Dz, x_{n+1} - z)$$

$$= \left(1 - \frac{2\alpha_n(\bar{\xi} - \xi\alpha)}{1 - \alpha_n\xi\alpha}\right)\left\|x_n - z\right\|^2 + \frac{2\alpha_n(\bar{\xi} - \xi\alpha)}{1 - \alpha_n\xi\alpha}\left(\frac{\alpha_n\bar{\xi}^2}{2(\bar{\xi} - \xi\alpha)}\left\|x_n - z\right\|^2 + \frac{1}{\bar{\xi} - \xi\alpha}\left\langle \xi f(z) - Dz, x_{n+1} - z\right\rangle\right).$$

From the condition (i), (3.17) and Lemma 2.2, we can conclude that the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I - D + \xi f)z$. This completes the proof.

As direct proof of Theorem 3.1, we obtain the following result.

Corollary 3.2. For every i = 1, 2, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i and let A : $H_1 \to H_2$ be a bounded linear operator. Let $A_1 : C_1 \to H_1$ be α_i , β_i - inverse strongly monotone mapping with $\eta = \min_{i=1,2,\ldots,N}\{\alpha_i\}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i - strict pseudo- contractions of C_1 into itself with $\kappa = \max_{i=1,2,\ldots,N}\{\kappa_i\}$., and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C_1 into itself with $\psi = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap \bar{\Omega} \neq \phi$ where $\bar{\Omega} = \{x \in VI(C_1, A_1) : Ax \in VI(C_2, A_2)\}$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where I = [0, 1] , $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1, \alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (\kappa, 1], \alpha_3^N \in [\kappa, 1), \alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, \ldots, N$. Let S^A be the S^A- mapping generated by $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $f: C_1 \to H_1$ be a contractive mapping with $\alpha \in (0, 1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0,1)$ with 0 $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence { x_n } be generated by $x_1 \in C_1$ and

$$\begin{cases} u_n = P_{C_1}(I - \lambda_1 A_1)(x_n - \gamma A^*(I - P_{C_2}(I - \lambda_2 A_2))Ax_n) \\ x_{n+1} = \alpha_n \xi f(x_n) + (I - \alpha_n D)(\beta_n S^A x_n + (1 - \beta_n)u_n) \quad \forall n \in \mathbb{N}, \end{cases} \qquad (3.18)$$

where $\lambda_i \in (0, 2\alpha_i)$ for all i = 1, 2, $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty, \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty;$

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1,\ \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty.$

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I - D + \xi f)(z)$.

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ลำดับเมทริกซ์ในพจน์ของพหุนามเกาส์เซียนเพลล์ พหุนามเกาส์เซียนโมดิฟายด์เพลล์ จำนวนเกาส์เซียนเพลล์ จำนวนเกาส์เซียนเพลล์-ลูคัส จำนวนเกาส์เซียนโมดิฟายด์เพลล์ พหุนามเพลล์ พหุนามเพลล์-ลูคัส และพหุนามโมดิฟายด์เพลล์

Matrix Sequences in Terms of Gaussian Pell Polynomial, Gaussian Modified Pell Polynomial, Gaussian Pell Number, Gaussian Pell-Lucas Number, Gaussian Modified Pell Number, Pell Polynomial, Pell-Lucas Polynomial and Modified Pell Polynomial

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บทคัดย่อ

ในบทความนี้เราได้ศึกษาลำดับเมทริกซ์พหุนามเกาส์เซียนเพลล์ ลำดับเมทริกซ์พหุนามเกาส์เซียนโมดิฟายด์เพลล์ ลำดับเมทริกซ์เกาส์เซียนเพลล์ ลำดับเมทริกซ์เกาส์เซียนเพลล์-ลูคัส ลำดับเมทริกซ์เกาส์เซียนโมดิฟายด์เพลล์ ลำดับเมทริกซ์พหุนามเพลล์ ลำดับเมทริกซ์พหุนามเพลล์-ลูคัส และลำดับเมทริกซ์พหุนามโมดิฟายด์เพลล์ พร้อมทั้งพิสูจน์เอกลักษณ์บางอย่างของความสัมพันธ์ระหว่างลำดับเมทริกซ์และเอกลักษณ์บางอย่างของผลบวก

คำสำคัญ : ความสัมพันธ์เวียนเกิด ; ลำดับเมทริกซ์ ; สูตรไบเนต ; พจน์ที่ n

Abstract

In this paper, we study Gaussian Pell polynomial, Gaussian modified Pell polynomial, Gaussian Pell, Gaussian Pell-Lucas, Gaussian modified Pell, Pell polynomial, Pell-Lucas polynomial, and modified Pell polynomial matrix sequences. Furthermore, we prove some identities of the relation between matrix sequences and summations.

Keywords : recurrence relations ; matrix sequences ; Binet's formulas ; n^{th} terms

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Introduction

The Fibonacci numbers $\boldsymbol{F_n}$, Lucas numbers $\boldsymbol{L_n}$, Pell numbers $\boldsymbol{P_n}$, and Pell-Lucus numbers $\boldsymbol{Q_n}$ are examples of the famous number generated by recurrence relation. Their Binet's formulas are $F_n = \dfrac{r_1^{\,n} - r_2^{\,n}}{r_1 - r_2}$, $L_n = r_1^{\,n} + r_2^{\,n}$, $P_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, and $Q_n = \alpha^n + \beta^n$, where n is an integer, $r_1 = \frac{1}{2}\left(1+\sqrt{5}\right)$, $r_2 = \frac{1}{2}\left(1-\sqrt{5}\right)$ are roots of $t^2 - t - 1 = 0$ and $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$ are roots of $t^2 - 2t - 1 = 0$. So $r_1 - r_2 = \sqrt{5}$, $r_1 r_2 = -1$, $\alpha \neq \beta$, $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$, and $\alpha\beta = -1$. (Horadam, A.F., 1961), (Daykin, D.E. & Dresel, L.A.G., 1967), (Horadam, A.F., 1984).

In 1985, Alwyn F. Horadam and Brother J.M. Mahon studied properties of the sequences of Pell polynomials $\boldsymbol{P_n(x)}$ and Pell-Lucas polynomials $\boldsymbol{Q_n(x)}$. For a natural number n, these sequences are defined by the recurrence relations

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \tag{1}$$

and

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), \tag{2}$$

with initial conditions $P_0(x) = 0$, $P_1(x) = 1$, $Q_0(x) = 2$, and $Q_1(x) = 2x$.

The definitions of negative subscript are extended by

$$P_{-n}(x) = (-1)^{n+1} P_n(x), \text{ for } n \geq 1, \tag{3}$$

and

$$Q_{-n}(x) = (-1)^{n} Q_n(x), \text{ for } n \geq 1. \tag{4}$$

So, Binet's formulas can be derived as follows

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \tag{5}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x), \tag{6}$$

where $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$ are the roots of $t^2 - 2xt - 1 = 0$. Then $\alpha(x) \neq \beta(x)$, $\alpha(x) + \beta(x) = 2x$, $\alpha(x) - \beta(x) = 2\sqrt{x^2+1}$ and $\alpha(x)\beta(x) = -1$. By (5) and (6), we have the following elementary identity:

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x). \tag{7}$$

The particular cases of the polynomials are $P_n(1) = P_n$, $Q_n(1) = Q_n$, $P_n\left(\frac{1}{2}\right) = F_n$, and $Q_n\left(\frac{1}{2}\right) = L_n$.

In 2012, Hasan Huseyin Gulec and Necati Taskara studied the (s,t)-Pell matrix sequence $\{\boldsymbol{P_n(s,t)}\}_{n\in\mathbb{N}}$ and (s,t)-Pell-Lucas matrix sequence $\{\boldsymbol{Q_n(s,t)}\}_{n\in\mathbb{N}}$ consisting of elements of the (s,t)-Pell numbers and (s,t)-Pell-Lucas numbers defined by

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t), \text{ for } n \geq 2, \tag{8}$$

and

$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t), \text{ for } n \geq 2, \tag{9}$$

with initial conditions $P_0(s,t)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $P_1(s,t)=\begin{pmatrix}2s & 1\\ t & 0\end{pmatrix}$, $Q_0(s,t)=\begin{pmatrix}2s & 2\\ 2t & -2s\end{pmatrix}$, and $Q_1(s,t)=\begin{pmatrix}4s^2+2t & 2s\\ 2st & 2t\end{pmatrix}$,

where $s^2+t>0$, $s>0$, $t\neq 0$, and s, t are real numbers.

In 2016, Serpil Halici and Sinan Öz introduced the complex Pell and complex Pell – Lucas sequences, namely Gaussian Pell sequence $\{\boldsymbol{GP_n}\}_{n\in\mathbb{N}}$ and Gaussian Pell-Lucas sequence $\{\boldsymbol{GQ_n}\}_{n\in\mathbb{N}}$, which are defined by recurrence relations

$$GP_n = 2GP_{n-1}+GP_{n-2}, \text{ for } n\geq 2, \tag{10}$$

and

$$GQ_n = 2GQ_{n-1}+GQ_{n-2}, \text{ for } n\geq 2, \tag{11}$$

with initial conditions $GP_0=i$, $GP_1=1$, $GQ_0=2-2i$ and $GQ_1=2+2i$. Their well-known Binet's formulas are

$$GP_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}+i\frac{\alpha\beta^n-\beta\alpha^n}{\alpha-\beta}, \tag{12}$$

and

$$GQ_n = \alpha^n+\beta^n-i\alpha\beta^n-i\beta\alpha^n. \tag{13}$$

Also, Gaussian Pell and Gaussian Pell-Lucas are related to Pell and Pell-Lucas. Some identities of the sequences are

$$GP_n = P_n+iP_{n-1}, \text{ for } n\geq 1, \tag{14}$$

and

$$GQ_n = Q_n+iQ_{n-1}, \text{ for } n\geq 1, \tag{15}$$

with initial conditions $P_0=0$, $P_1=1$, $Q_0=2$ and $Q_1=2$.

In 2018, Tulay Yagmar and Nusret karaaslan defined Gaussian modified Pell numbers $\boldsymbol{Gq_n}$ and Gaussian modified Pell polynomials $\boldsymbol{Gq_n(x)}$ by

$$Gq_n = 2Gq_{n-1}+Gq_{n-2}, \text{ for } n\geq 2, \tag{16}$$

and

$$Gq_n(x)=2xGq_{n-1}(x)+Gq_{n-2}(x), \text{ for } n\geq 2, \tag{17}$$

with initial conditions $Gq_0=1-i$, $Gq_1=1+i$, $Gq_0(x)=1-xi$ and $Gq_1(x)=x+i$. Then, their Binet's formulas are

$$Gq_n = \frac{\alpha^n+\beta^n}{2}-i\frac{\alpha\beta^n+\beta\alpha^n}{2}, \tag{18}$$

and

$$Gq_n(x)=\frac{\alpha^n(x)+\beta^n(x)}{2}-i\frac{\alpha(x)\beta^n(x)+\beta(x)\alpha^n(x)}{2}. \tag{19}$$

In 2018, Serpil Halici and Sinan Oz introduced Gaussian Pell polynomials $\boldsymbol{GP_n(x)}$, which is defined recurrently by

$$GP_{n+1}(x)=2xGP_n(x)+GP_{n-1}(x), \text{ for } n\geq 1, \tag{20}$$

with initial conditions $GP_0(x)=i$ and $GP_1(x)=1$. Their well-known Binet's formula is

$$GP_n(x)=\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^n(x)-\beta(x)\alpha^n(x)}{\alpha(x)-\beta(x)}. \tag{21}$$

That authors observed that relation between Gaussian Pell polynomial and Pell polynomial is

$$GP_n(x) = P_n(x) + iP_{n-1}(x), \text{ for } n \geq 1. \tag{22}$$

In 2019, Nusret Karaaslan studied modified Pell polynomials $q_n(x)$ defined by

$$q_n(x) = 2xq_{n-1}(x) + q_{n-2}(x), \text{ for } n \geq 2, \tag{23}$$

with initial conditions $q_0(x) = 1$ and $q_1(x) = x$. Then, the Binet's formula is

$$q_n(x) = \frac{\alpha^n(x) + \beta^n(x)}{2}. \tag{24}$$

In particular, if $x = 1$, then $q_n(1)$ is the modified Pell number q_n.

Methods

In this section, firstly, we find the first few terms of the recurrence relations GP_n, GQ_n, Gq_n, $Gq_n(x)$, $GP_n(x)$, and $q_n(x)$, which the extension of negative subscripts is created by rewriting as $GP_{n-2} = GP_n - 2GP_{n-1}$, $GQ_{n-2} = GQ_n - 2GQ_{n-1}$, $Gq_{n-2} = Gq_n - 2Gq_{n-1}$, $Gq_{n-2}(x) = Gq_n(x) - 2xGq_{n-1}(x)$, $GP_{n-2}(x) = GP_n(x) - 2xGP_{n-1}(x)$, and $q_{n-2}(x) = q_n(x) - 2xq_{n-1}(x)$ as below.

Table 1 The first few terms of GP_n, GQ_n, Gq_n, $Gq_n(x)$, $GP_n(x)$, and $q_n(x)$ for $-2 \leq n \leq 1$.

n:	-2	-1	0	1
GP_n:	$-2+5i$	$1-2i$	i	1
GQ_n:	$6-14i$	$-2+6i$	$2-2i$	$2+2i$
Gq_n:	$3-7i$	$-1+3i$	$1-i$	$1+i$
$Gq_n(x)$:	$(2x^2+1)-(4x^3+3x)i$	$-x+(2x^2+1)i$	$1-xi$	$x+i$
$GP_n(x)$:	$-2x+(4x^2+1)i$	$1-2xi$	i	1
$q_n(x)$:	$2x^2+1$	$-x$	1	x

After that, we define the recurrence relation of a 2x2 matrix for all integer $n \geq -1$ in which the component of each matrix consists of numbers and polynomials of these sequences, and the index starts at -1.

Definition 1 Let $n \in \mathbb{N}$, x is a scalar-value polynomial, $x > 0$, and $x^2 + 1 > 0$. Then the Gaussian Pell polynomial matrix sequence $\{MGP_n(x)\}_{n\in\mathbb{N}}$ and Gaussian modified Pell polynomial matrix sequence $\{MGq_n(x)\}_{n\in\mathbb{N}}$ are defined by

$$MGP_n(x) = 2xMGP_{n-1}(x) + MGP_{n-2}(x), \tag{25}$$

and

$$MGq_n(x) = 2xMGq_{n-1}(x) + MGq_{n-2}(x), \tag{26}$$

respectively, with initial conditions $MGP_0(x)=\begin{pmatrix}1 & i\\ i & 1-2xi\end{pmatrix}$, $MGP_1(x)=\begin{pmatrix}2x+i & 1\\ 1 & i\end{pmatrix}$, $MGq_0(x)=\begin{pmatrix}x+i & 1-xi\\ 1-xi & -x+(2x^2+1)i\end{pmatrix}$, and $MGq_1(x)=\begin{pmatrix}2x^2+1+xi & x+i\\ x+i & 1-xi\end{pmatrix}$.

Definition 2 Let $n\in\mathbb{N}$. Then the Gaussian Pell matrix sequence $\{\boldsymbol{MGP_n}\}_{n\in\mathbb{N}}$, Gaussian Pell-Lucas matrix sequence $\{\boldsymbol{MGQ_n}\}_{n\in\mathbb{N}}$, and Gaussian modified Pell matrix sequence $\{\boldsymbol{MGq_n}\}_{n\in\mathbb{N}}$ are defined by

$$MGP_n = 2MGP_{n-1} + MGP_{n-2}, \tag{27}$$

and

$$MGQ_n = 2MGQ_{n-1} + MGQ_{n-2}, \tag{28}$$

and

$$MGq_n = 2MGq_{n-1} + MGq_{n-2}, \tag{29}$$

respectively, with initial conditions $MGP_0=\begin{pmatrix}1 & i\\ i & 1-2i\end{pmatrix}$, $MGP_1=\begin{pmatrix}2+i & 1\\ 1 & i\end{pmatrix}$, $MGQ_0=\begin{pmatrix}2+2i & 2-2i\\ 2-2i & -2+6i\end{pmatrix}$, $MGQ_1=\begin{pmatrix}6+2i & 2+2i\\ 2+2i & 2-2i\end{pmatrix}$, $MGq_0=\begin{pmatrix}1+i & 1-i\\ 1-i & -1+3i\end{pmatrix}$, and $MGq_1=\begin{pmatrix}3+i & 1+i\\ 1+i & 1-i\end{pmatrix}$.

Definition 3 Let $n\in\mathbb{N}$, x is a scalar-value polynomial, $x>0$, and $x^2+1>0$. Then the Pell polynomial matrix sequence $\{\boldsymbol{MP_n(x)}\}_{n\in\mathbb{N}}$, Pell-Lucas polynomial matrix sequence $\{\boldsymbol{MQ_n(x)}\}_{n\in\mathbb{N}}$, and modified Pell polynomial matrix sequence $\{\boldsymbol{Mq_n(x)}\}_{n\in\mathbb{N}}$ are defined by

$$MP_n(x) = 2xMP_{n-1}(x) + MP_{n-2}(x), \tag{30}$$

and

$$MQ_n(x) = 2xMQ_{n-1}(x) + MQ_{n-2}(x), \tag{31}$$

and

$$Mq_n(x) = 2xMq_{n-1}(x) + Mq_{n-2}(x), \tag{32}$$

respectively, with initial conditions $MP_0(x)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $MP_1(x)=\begin{pmatrix}2x & 1\\ 1 & 0\end{pmatrix}$, $MQ_0(x)=\begin{pmatrix}2x & 2\\ 2 & -2x\end{pmatrix}$, $MQ_1(x)=\begin{pmatrix}4x^2+2 & 2x\\ 2x & 2\end{pmatrix}$, $Mq_0(x)=\begin{pmatrix}x & 1\\ 1 & -x\end{pmatrix}$, and $Mq_1(x)=\begin{pmatrix}2x^2+1 & x\\ x & 1\end{pmatrix}$.

Note that, for all integer $n<0$, we find negative subscripts of matrix sequences in which the extension of definition is obtained by rewriting

$$MGP_{n-2}(x) = MGP_n(x) - 2xMGP_{n-1}(x), \tag{33}$$

$$MGq_{n-2}(x) = MGq_n(x) - 2xMGq_{n-1}(x), \tag{34}$$

$$MGP_{n-2} = MGP_n - 2MGP_{n-1}, \tag{35}$$

$$MGQ_{n-2} = MGQ_n - 2MGQ_{n-1}, \tag{36}$$

$$MGq_{n-2} = MGq_n - 2MGq_{n-1}\,, \tag{37}$$

$$MP_{n-2}(x) = MP_n(x) - 2xMP_{n-1}(x)\,, \tag{38}$$

$$MQ_{n-2}(x) = MQ_n(x) - 2xMQ_{n-1}(x)\,, \tag{39}$$

and

$$Mq_{n-2}(x) = Mq_n(x) - 2xMq_{n-1}(x)\,. \tag{40}$$

Results

In this section, the first step, we find the n^th general terms of the matrix sequences, which correspond to the following theorem and corollary.

Theorem 4 For $n \in \mathbb{N}$. Then the n^th terms of $\{MGP_n(x)\}_{n\in\mathbb{N}}$ and $\{MGq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MGP_n(x) = \begin{pmatrix} GP_{n+1}(x) & GP_n(x) \\ GP_n(x) & GP_{n-1}(x) \end{pmatrix}, \tag{41}$$

and

$$MGq_n(x) = \begin{pmatrix} Gq_{n+1}(x) & Gq_n(x) \\ Gq_n(x) & Gq_{n-1}(x) \end{pmatrix}. \tag{42}$$

Proof. We will show that $MGP_n(x) = \begin{pmatrix} GP_{n+1}(x) & GP_n(x) \\ GP_n(x) & GP_{n-1}(x) \end{pmatrix}$ for $n \in \mathbb{N}$.

Since, $MGP_0(x) = \begin{pmatrix} 1 & i \\ i & 1-2xi \end{pmatrix}$, it follows that (41) is true.

Since, $MGP_1(x) = \begin{pmatrix} 2x+i & 1 \\ 1 & i \end{pmatrix}$, it follows that (41) is true.

By iterating this procedure and considering induction steps, let us assume that the equality in (41) holds for all $n \le k \in \mathbb{N}$. To finish the proof.

Next, we have to show that (41) also holds for $n = k+1$ by considering (20) and (25).

Then

$$\begin{aligned} MGP_{k+1}(x) &= 2xMGP_k(x) + MGP_{k-1}(x) \\ &= 2x\begin{pmatrix} GP_{k+1}(x) & GP_k(x) \\ GP_k(x) & GP_{k-1}(x) \end{pmatrix} + \begin{pmatrix} GP_k(x) & GP_{k-1}(x) \\ GP_{k-1}(x) & GP_{k-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} 2xGP_{k+1}(x) + GP_k(x) & 2xGP_k(x) + GP_{k-1}(x) \\ 2xGP_k(x) + GP_{k-1}(x) & 2xGP_{k-1}(x) + GP_{k-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{k+2}(x) & GP_{k+1}(x) \\ GP_{k+1}(x) & GP_k(x) \end{pmatrix}. \end{aligned}$$

Thus, $n = k+1$ is true.

The similar proof of (41) is used to prove (42).

Therefore, the proof is complete.

Corollary 5 For $n \in \mathbb{N}$. Then the n[th] terms of $\{MGP_n\}_{n\in\mathbb{N}}$, $\{MGQ_n\}_{n\in\mathbb{N}}$, and $\{MGq_n\}_{n\in\mathbb{N}}$ are given by

$$MGP_n = \begin{pmatrix} GP_{n+1} & GP_n \\ GP_n & GP_{n-1} \end{pmatrix}, \tag{43}$$

and

$$MGQ_n = \begin{pmatrix} GQ_{n+1} & GQ_n \\ GQ_n & GQ_{n-1} \end{pmatrix}, \tag{44}$$

and

$$MGq_n = \begin{pmatrix} Gq_{n+1} & Gq_n \\ Gq_n & Gq_{n-1} \end{pmatrix}. \tag{45}$$

Proof. Take $x=1$ in (41) and (42), we have (43) and (45).

The similar proof of Theorem 4 is used for (44).

Corollary 6 For $n \in \mathbb{N}$. Then the n[th] terms of $\{MP_n(x)\}_{n\in\mathbb{N}}$, $\{MQ_n(x)\}_{n\in\mathbb{N}}$, and $\{Mq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MP_n(x) = \begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{pmatrix}, \tag{46}$$

and

$$MQ_n(x) = \begin{pmatrix} Q_{n+1}(x) & Q_n(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix}, \tag{47}$$

and

$$Mq_n(x) = \begin{pmatrix} q_{n+1}(x) & q_n(x) \\ q_n(x) & q_{n-1}(x) \end{pmatrix}. \tag{48}$$

Proof. The similar proof of Theorem 4 is used for (46), (47), and (48).

Next, we find Binet's formulas of the matrix sequences that lead to some identities. These formulas correspond to the following theorem and corollary.

Theorem 7 Let $n \in \mathbb{N}$. Then, the Binet's formula for $\{MGP_n(x)\}_{n\in\mathbb{N}}$ and $\{MGq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MGP_n(x) = \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right) - \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right), \tag{49}$$

and

$$MGq_n(x) = \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(MGq_1(x)-\beta(x)MGq_0(x)\right) - \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(MGq_1(x)-\alpha(x)MGq_0(x)\right). \tag{50}$$

Proof. Let c_1, c_2 be the 2x2 matrices and $\alpha(x) = x+\sqrt{x^2+1}$, $\beta(x) = x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$. Then the general term of (25) is

$$MGP_n(x) = c_1\alpha^n(x) + c_2\beta^n(x). \tag{51}$$

Take $n=0$ and $n=1$ in (51), we get

$$MGP_0(x)=c_1+c_2, \tag{52}$$

and

$$MGP_1(x)=c_1\alpha(x)+c_2\beta(x). \tag{53}$$

By using (52), (53) and scalar multiplication to find c_1 and c_2, we obtain

$$c_1=\frac{1}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big), \tag{54}$$

and

$$c_2=-\frac{1}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big). \tag{55}$$

By using (54), (55) in (51), we get

$$MGP_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\beta(x)MGP_0(x)\big)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\big(MGP_1(x)-\alpha(x)MGP_0(x)\big).$$

The similar proof of (49) is used to prove (50).

Therefore, the proof is complete.

Corollary 8 Let $n\in\mathbb{N}$. Then, the Binet's formula for $\{MGP_n\}_{n\in\mathbb{N}}$, $\{MGQ_n\}_{n\in\mathbb{N}}$, and $\{MGq_n\}_{n\in\mathbb{N}}$ are given by

$$MGP_n=\frac{\alpha^n}{\alpha-\beta}(MGP_1-\beta MGP_0)-\frac{\beta^n}{\alpha-\beta}(MGP_1-\alpha MGP_0), \tag{56}$$

and

$$MGQ_n=\frac{\alpha^n}{\alpha-\beta}(MGQ_1-\beta MGQ_0)-\frac{\beta^n}{\alpha-\beta}(MGQ_1-\alpha MGQ_0), \tag{57}$$

and

$$MGq_n=\frac{\alpha^n}{\alpha-\beta}(MGq_1-\beta MGq_0)-\frac{\beta^n}{\alpha-\beta}(MGq_1-\alpha MGq_0). \tag{58}$$

Proof. Take $x=1$ in (49) and (50), we have (56) and (58).

The similar proof of Theorem 7 is used to (57).

Corollary 9 Let $n\in\mathbb{N}$. Then, the Binet's formula for $\{MP_n(x)\}_{n\in\mathbb{N}}$, $\{MQ_n(x)\}_{n\in\mathbb{N}}$, and $\{Mq_n(x)\}_{n\in\mathbb{N}}$ are given by

$$MP_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\big(MP_1(x)-\beta(x)MP_0(x)\big)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\big(MP_1(x)-\alpha(x)MP_0(x)\big), \tag{59}$$

and

$$MQ_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\big(MQ_1(x)-\beta(x)MQ_0(x)\big)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\big(MQ_1(x)-\alpha(x)MQ_0(x)\big), \tag{60}$$

and

$$Mq_n(x)=\frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\big(Mq_1(x)-\beta(x)Mq_0(x)\big)-\frac{\beta^n(x)}{\alpha(x)-\beta(x)}\big(Mq_1(x)-\alpha(x)Mq_0(x)\big). \tag{61}$$

Proof. The similar proof of Theorem 7 is used to prove (59), (60), and (61).

Next, we find the n^th^ power of $MP_1(x)$ and MP_1, for any integer $n \geq 0$, which are symmetry. They correspond to the following lemma.

Lemma 10 For $n \in \mathbb{N}$, the n^th^ power of $MP_1(x)$ and MP_1 are given by

(i) $\left(MP_1(x)\right)^n = MP_n(x)$,

(ii) $\left(MP_1\right)^n = MP_n$.

Proof. Since, $MP_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it follows that (i) is true.

Since, $MP_1(x) = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}$, it follows that (i) is true.

We assume the result is true for a positive integer $n = k$ then

$$MP_k(x) = \begin{pmatrix} P_{k+1}(x) & P_k(x) \\ P_k(x) & P_{k-1}(x) \end{pmatrix} = \left(MP_1(x)\right)^k.$$

We consider a positive integer $n = k+1$.

Then
$$\begin{aligned}
\left(MP_1(x)\right)^{k+1} &= \left(MP_1(x)\right)^k MP_1(x) \\
&= \begin{pmatrix} P_{k+1}(x) & P_k(x) \\ P_k(x) & P_{k-1}(x) \end{pmatrix} \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2xP_{k+1}(x) + P_k(x) & P_{k+1}(x) \\ 2xP_k(x) + P_{k-1}(x) & P_k(x) \end{pmatrix} \\
&= \begin{pmatrix} P_{k+2}(x) & P_{k+1}(x) \\ P_{k+1}(x) & P_k(x) \end{pmatrix} \\
&= MP_{k+1}(x).
\end{aligned}$$

Thus, the statement is true when $n = k+1$.

Take $x = 1$ in (i), we have (ii).

Therefore, the proof is complete.

After that, we find some identities of the relations between the studied sequences of numbers and polynomials, which corresponds to the following lemma.

Lemma 11 For $m, n \in \mathbb{N}$, the following results hold.

(i) $GP_{m+1}(x)P_n(x) + GP_m(x)P_{n-1}(x) = GP_{m+n}(x)$,

(ii) $Gq_{m+1}(x)P_n(x) + Gq_m(x)P_{n-1}(x) = Gq_{m+n}(x)$,

(iii) $GP_{m+1}P_n + GP_mP_{n-1} = GP_{m+n}$,

(iv) $GQ_{m+1}P_n + GQ_mP_{n-1} = GQ_{m+n}$,

(v) $Gq_{m+1}P_n + Gq_mP_{n-1} = Gq_{m+n}$,

(vi) $P_{m+1}(x)P_n(x)+P_m(x)P_{n-1}(x)=P_{m+n}(x)$,

(vii) $Q_{m+1}(x)P_n(x)+Q_m(x)P_{n-1}(x)=Q_{m+n}(x)$,

(viii) $q_{m+1}(x)P_n(x)+q_m(x)P_{n-1}(x)=q_{m+n}(x)$,

(ix) $GP_{n+1}(x)+GP_{n-1}(x)=2Gq_n(x)$,

(x) $GP_{n+1}+GP_{n-1}=2Gq_n$,

(xi) $P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x)$,

(xii) $P_n(x)Q_{n+1}(x)+P_{n-1}(x)Q_n(x)=P_{2n+1}(x)+P_{2n-1}(x)=Q_{2n}(x)=P_{n+1}(x)Q_n(x)+P_n(x)Q_{n-1}(x)$,

(xiii) $P_n(x)Q_n(x)+P_{n-1}(x)Q_{n-1}(x)=P_{2n}(x)+P_{2n-2}(x)=Q_{2n-1}(x)$.

Proof. Since (5), (21) and $\alpha(x)=x+\sqrt{x^2+1}$, $\beta(x)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$, we have

$$
\begin{aligned}
&GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x)\\
&=\left(\frac{\alpha^{m+1}(x)-\beta^{m+1}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{m+1}(x)-\beta(x)\alpha^{m+1}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}\right)\\
&\qquad+\left(\frac{\alpha^m(x)-\beta^m(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^m(x)-\beta(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}\right)\\
&=\frac{1}{(\alpha(x)-\beta(x))^2}\big(\alpha^{m+n+1}(x)-\alpha^{m+1}(x)\beta^n(x)-\alpha^n(x)\beta^{m+1}(x)+\beta^{m+n+1}(x)\\
&\qquad+\alpha^{m+n-1}(x)-\alpha^m(x)\beta^{n-1}(x)-\alpha^{n-1}(x)\beta^m(x)+\beta^{m+n-1}(x)\\
&\qquad+i\alpha^{n+1}(x)\beta^{m+1}(x)-i\alpha(x)\beta^{m+n+1}(x)-i\beta(x)\alpha^{m+n+1}(x)+i\alpha^{m+1}(x)\beta^{n+1}(x)\\
&\qquad+i\alpha^n(x)\beta^m(x)-i\alpha(x)\beta^{m+n-1}(x)-i\beta(x)\alpha^{m+n-1}(x)+i\alpha^m(x)\beta^n(x)\big)\\
&=\frac{1}{\alpha(x)-\beta(x)}\Bigg(\alpha^{m+n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}-\alpha^m(x)\beta^n(x)\frac{\alpha(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}-\alpha^n(x)\beta^m(x)\frac{\beta(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\qquad+\beta^{m+n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}+i\alpha^n(x)\beta^m(x)\frac{\alpha(x)\beta(x)+1}{\alpha(x)-\beta(x)}-i\alpha(x)\beta^{m+n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\qquad-i\beta(x)\alpha^{m+n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}+i\alpha^m(x)\beta^n(x)\frac{\alpha(x)\beta(x)+1}{\alpha(x)-\beta(x)}\Bigg)\\
&=\frac{1}{\alpha(x)-\beta(x)}\big(\alpha^{m+n}(x)-\beta^{m+n}(x)+i\alpha(x)\beta^{m+n}(x)-i\beta(x)\alpha^{m+n}(x)\big)\\
&=\frac{\alpha^{m+n}(x)-\beta^{m+n}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{m+n}(x)-\beta(x)\alpha^{m+n}(x)}{\alpha(x)-\beta(x)}\\
&=GP_{m+n}(x).
\end{aligned}
$$

Thus, $GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x)=GP_{m+n}(x)$.

The similar proof of (i) is used for (ii), (iii), (iv), (v), (vi), (vii), and (viii).

Next, we will show that $GP_{n+1}(x)+GP_{n-1}(x)=2Gq_n(x)$.

By (21), we have

$$
\begin{aligned}
&GP_{n+1}(x)+GP_{n-1}(x)\\
&\quad=\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{n+1}(x)-\beta(x)\alpha^{n+1}(x)}{\alpha(x)-\beta(x)}+\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}+i\frac{\alpha(x)\beta^{n-1}(x)-\beta(x)\alpha^{n-1}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\alpha^{n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}-\beta^{n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\qquad\qquad+i\alpha(x)\beta^{n}(x)\frac{\beta(x)+\beta^{-1}(x)}{\alpha(x)-\beta(x)}-i\beta(x)\alpha^{n}(x)\frac{\alpha(x)+\alpha^{-1}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\alpha^{n}(x)+\beta^{n}(x)-i\alpha(x)\beta^{n}(x)-i\beta(x)\alpha^{n}(x)\\
&\quad=2Gq_{n}(x).
\end{aligned}
$$

Thus, $GP_{n+1}(x)+GP_{n-1}(x)=2Gq_n(x)$.

Take $x=1$ in (ix), we have (x).

Next, we will show that $P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x)$.

By (5) and (6), we have

$$
\begin{aligned}
&P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)\\
&\quad=\left(\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\alpha^{n+1}(x)+\beta^{n+1}(x)\right)+\left(\frac{\alpha^{n}(x)-\beta^{n}(x)}{\alpha(x)-\beta(x)}\right)\cdot\left(\alpha^{n}(x)+\beta^{n}(x)\right)\\
&\quad=\frac{\alpha^{2n+2}(x)+\alpha^{n+1}(x)\beta^{n+1}(x)-\alpha^{n+1}(x)\beta^{n+1}(x)-\beta^{2n+2}(x)+\alpha^{2n}(x)+\alpha^{n}(x)\beta^{n}(x)-\alpha^{n}(x)\beta^{n}(x)-\beta^{2n}(x)}{\alpha(x)-\beta(x)}\\
&\quad=\frac{\alpha^{2n+2}(x)-\beta^{2n+2}(x)}{\alpha(x)-\beta(x)}+\frac{\alpha^{2n}(x)-\beta^{2n}(x)}{\alpha(x)-\beta(x)}\\
&\quad=P_{2n+2}(x)+P_{2n}(x). \qquad (62)
\end{aligned}
$$

Since (7), we obtain

$$P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x). \qquad (63)$$

By using (62) and (63), we have

$$P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=Q_{2n+1}(x).$$

Thus, $P_{n+1}(x)Q_{n+1}(x)+P_n(x)Q_n(x)=P_{2n+2}(x)+P_{2n}(x)=Q_{2n+1}(x)$.

The similar proof of (xi) is used for (xii) and (xiii).

Therefore, the identities (i), (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii), and (xiii) are immediately seen.

Also, we find the relation between these matrix sequences by applying Lemma 10 and Lemma 11. They correspond to the following theorem and corollary.

Theorem 12 For $m,n\in\mathbb{N}$, the following results hold.

(i) $MGP_m(x)(MP_1(x))^n = MGP_{m+n}(x)$,

(ii) $MGq_m(x)(MP_1(x))^n = MGq_{m+n}(x)$,

(iii) $MGP_{n+1}(x)+MGP_{n-1}(x)=2MGq_n(x)$.

Proof. By Lemma 10 (i), we can write

$$MGP_m(x)(MP_1(x))^n = MGP_m(x)MP_n(x). \tag{64}$$

Since (41), (46), and matrix multiplication, then

$$\begin{aligned} MGP_m(x)MP_n(x) &= \begin{pmatrix} GP_{m+1}(x) & GP_m(x) \\ GP_m(x) & GP_{m-1}(x) \end{pmatrix}\begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{m+1}(x)P_{n+1}(x)+GP_m(x)P_n(x) & GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x) \\ GP_m(x)P_{n+1}(x)+GP_{m-1}(x)P_n(x) & GP_m(x)P_n(x)+GP_{m-1}(x)P_{n-1}(x) \end{pmatrix}. \end{aligned} \tag{65}$$

By using Lemma 11 (i) in (65), we have

$$\begin{aligned} MGP_m(x)MP_n(x) &= \begin{pmatrix} GP_{m+1}(x)P_{n+1}(x)+GP_m(x)P_n(x) & GP_{m+1}(x)P_n(x)+GP_m(x)P_{n-1}(x) \\ GP_m(x)P_{n+1}(x)+GP_{m-1}(x)P_n(x) & GP_m(x)P_n(x)+GP_{m-1}(x)P_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{m+n+1}(x) & GP_{m+n}(x) \\ GP_{m+n}(x) & GP_{m+n-1}(x) \end{pmatrix} \\ &= MGP_{m+n}(x) \end{aligned} \tag{66}$$

By using (66) in (64), we obtain

$$MGP_m(x)(MP_1(x))^n = MGP_{m+n}(x).$$

The similar proof of (i) is used for (ii).

Next, we will show that $MGP_{n+1}(x)+MGP_{n-1}(x)=2MGq_n(x)$.

By (41) and matrix addition, we can write

$$\begin{aligned} MGP_{n+1}(x)+MGP_{n-1}(x) &= \begin{pmatrix} GP_{n+2}(x) & GP_{n+1}(x) \\ GP_{n+1}(x) & GP_n(x) \end{pmatrix} + \begin{pmatrix} GP_n(x) & GP_{n-1}(x) \\ GP_{n-1}(x) & GP_{n-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} GP_{n+2}(x)+GP_n(x) & GP_{n+1}(x)+GP_{n-1}(x) \\ GP_{n+1}(x)+GP_{n-1}(x) & GP_n(x)+GP_{n-2}(x) \end{pmatrix} \end{aligned} \tag{67}$$

By using Lemma 11 (ix) in (67), we get

$$\begin{aligned} MGP_{n+1}(x)+MGP_{n-1}(x) &= \begin{pmatrix} GP_{n+2}(x)+GP_n(x) & GP_{n+1}(x)+GP_{n-1}(x) \\ GP_{n+1}(x)+GP_{n-1}(x) & GP_n(x)+GP_{n-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} 2Gq_{n+1}(x) & 2Gq_n(x) \\ 2Gq_n(x) & 2Gq_{n-1}(x) \end{pmatrix} \\ &= 2\begin{pmatrix} Gq_{n+1}(x) & Gq_n(x) \\ Gq_n(x) & Gq_{n-1}(x) \end{pmatrix} \\ &= 2MGq_n(x). \end{aligned}$$

Therefore, the identities (i), (ii), and (iii) are immediately seen.

Corollary 13 For $m,n \in \mathbb{N}$, the following results hold.

(i) $MGP_m\left(MP_1\right)^n = MGP_{m+n}$,

(ii) $MGQ_m\left(MP_1\right)^n = MGQ_{m+n}$,

(iii) $MGq_m\left(MP_1\right)^n = MGq_{m+n}$,

(iv) $MGP_{n+1} + MGP_{n-1} = 2MGq_n$.

Proof. Take $x=1$ in Theorem 12 (i), (ii), and (iii), we have (i), (iii), and (iv).

The similar proof of Theorem 12 is used for (ii).

Corollary 14 For $m,n \in \mathbb{N}$, the following results hold.

(i) $MP_m(x)\left(MP_1(x)\right)^n = MP_{m+n}(x)$,

(ii) $MQ_m(x)\left(MP_1(x)\right)^n = MQ_{m+n}(x)$,

(iii) $Mq_m(x)\left(MP_1(x)\right)^n = Mq_{m+n}(x)$.

(iv) $MP_{n+1}(x) + MP_{n-1}(x) = MQ_n(x)$,

(v) $MP_n(x)MQ_n(x) = MP_{2n+1}(x) + MP_{2n-1}(x) = MQ_{2n}(x)$.

Proof. The similar proof of Theorem 12 is used for (i), (ii), (iii), and (iv).

Next, we will show that $MP_n(x)MQ_n(x) = MP_{2n+1}(x) + MP_{2n-1}(x) = MQ_{2n}(x)$.

By (46), (47) and matrix multiplication, we have

$$\begin{aligned} MP_n(x)MQ_n(x) &= \begin{pmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{pmatrix} \begin{pmatrix} Q_{n+1}(x) & Q_n(x) \\ Q_n(x) & Q_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} P_{n+1}(x)Q_{n+1}(x) + P_n(x)Q_n(x) & P_{n+1}(x)Q_n(x) + P_n(x)Q_{n-1}(x) \\ P_n(x)Q_{n+1}(x) + P_{n-1}(x)Q_n(x) & P_n(x)Q_n(x) + P_{n-1}(x)Q_{n-1}(x) \end{pmatrix}. \end{aligned} \tag{68}$$

By using Lemma 11 (xi), (xii), and (xiii) in (68), we can write

$$\begin{aligned} MP_n(x)MQ_n(x) &= \begin{pmatrix} P_{n+1}(x)Q_{n+1}(x) + P_n(x)Q_n(x) & P_{n+1}(x)Q_n(x) + P_n(x)Q_{n-1}(x) \\ P_n(x)Q_{n+1}(x) + P_{n-1}(x)Q_n(x) & P_n(x)Q_n(x) + P_{n-1}(x)Q_{n-1}(x) \end{pmatrix} \\ &= \begin{pmatrix} P_{2n+2}(x) + P_{2n}(x) & P_{2n+1}(x) + P_{2n-1}(x) \\ P_{2n+1}(x) + P_{2n-1}(x) & P_{2n}(x) + P_{2n-2}(x) \end{pmatrix} \end{aligned} \tag{69}$$

By matrix addition, we have

$$\begin{pmatrix} P_{2n+2}(x) + P_{2n}(x) & P_{2n+1}(x) + P_{2n-1}(x) \\ P_{2n+1}(x) + P_{2n-1}(x) & P_{2n}(x) + P_{2n-2}(x) \end{pmatrix} = \begin{pmatrix} P_{2n+2}(x) & P_{2n+1}(x) \\ P_{2n+1}(x) & P_{2n}(x) \end{pmatrix} + \begin{pmatrix} P_{2n}(x) & P_{2n-1}(x) \\ P_{2n-1}(x) & P_{2n-2}(x) \end{pmatrix} = MP_{2n+1}(x) + MP_{2n-1}(x) \tag{70}$$

By using (69) and (70), we get that

$$MP_n(x)MQ_n(x) = MP_{2n+1}(x) + MP_{2n-1}(x) \tag{71}$$

By (iv), we obtain

$$MP_{2n+1}(x)+MP_{2n-1}(x)=MQ_{2n}(x). \quad (72)$$

By using (71) and (72), we get

$$MP_n(x)MQ_n(x)=MQ_{2n}(x).$$

Thus, $MP_n(x)MQ_n(x)=MP_{2n+1}(x)+MP_{2n-1}(x)=MQ_{2n}(x)$.

Therefore, the identities (i), (ii), (iii), (iv), and (v) are easily seen.

Moreover, we get a particular case, which corresponds to the following corollary.

Corollary 15 For $n\in\mathbb{N}$, the following results hold.

(i) $MGP_0(x)\left(MP_1(x)\right)^n=MGP_n(x)$,

(ii) $MGP_0\left(MP_1\right)^n=MGP_n$,

(iii) $MP_0(x)\left(MP_1(x)\right)^n=MP_n(x)$.

Proof. Take $m=0$ in Theorem 12 (i), Corollary 13 (i), and Corollary 14 (i), we obtain (i), (ii), and (iii).

Therefore, the identities (i), (ii), and (iii) are easily seen.

Note that matrix $MP_0(x)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is an identity matrix.

Now, we find that the matrix is a symmetric matrix, which is equal to its transposition as in the following theorem and corollary.

Theorem 16 For $m,n\in\mathbb{N}$, the following results hold.

(i) $\left(MGP_m(x)MGP_n(x)\right)^T=MGP_n(x)MGP_m(x)$,

(ii) $\left(MGq_m(x)MGq_n(x)\right)^T=MGq_n(x)MGq_m(x)$.

Proof. Since the transpose of the matrix, we obtain

$$\left(MGP_m(x)MGP_n(x)\right)^T=\left(MGP_n(x)\right)^T\left(MGP_m(x)\right)^T=MGP_n(x)MGP_m(x).$$

The similar proof of (i) is used for (ii).

Therefore, the identities (i) and (ii) are easily seen.

Corollary 17 For $m,n\in\mathbb{N}$, the following results hold.

(i) $\left(MGP_mMGP_n\right)^T=MGP_nMGP_m$,

(ii) $\left(MGQ_mMGQ_n\right)^T=MGQ_nMGQ_m$,

(iii) $\left(MGq_mMGq_n\right)^T=MGq_nMGq_m$.

Proof. Take $x=1$ in Theorem 16 (i) and (ii), we have (i) and (iii).

The similar proof of Theorem 16 is used for (ii).

Corollary 18 For $m,n\in\mathbb{N}$, the following results hold.

(i) $\left(MP_m(x)MP_n(x)\right)^T = MP_n(x)MP_m(x)$,

(ii) $\left(MQ_m(x)MQ_n(x)\right)^T = MQ_n(x)MQ_m(x)$,

(iii) $\left(Mq_m(x)Mq_n(x)\right)^T = Mq_n(x)Mq_m(x)$.

Proof. The similar proof of Theorem 16 is used for (i), (ii), and (iii).

Finally, we find some identity matrix sequences of summations by using Binet's formulas (49), (50), (56), (57), (58), (59), (60) and (61) as the following theorem and corollary.

Theorem 19 For $n\in\mathbb{N}$, $x>0$, and $x^2+1>0$, the following equalities hold.

(i) $\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x)=\frac{1}{t^2-2xt-1}\left(t^2MGP_0(x)+tMGP_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMGP_{n+1}(x)+MGP_n(x)\right)$,

(ii) $\sum_{k=0}^{n}\frac{1}{t^k}MGq_k(x)=\frac{1}{t^2-2xt-1}\left(t^2MGq_0(x)+tMGq_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMGq_{n+1}(x)+MGq_n(x)\right)$.

Proof. Let $MGP_0(x)$, $MGP_1(x)$ be initial conditions of 2x2 matrix sequence and $\alpha(x)=x+\sqrt{x^2+1}$, $\beta(x)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$.

Then we can write

$$\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x)$$
$$=\sum_{k=0}^{n}\left(\frac{\alpha^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right). \tag{73}$$

By definition of a geometric sequence, we have

$$\sum_{k=0}^{n}\left(\frac{\alpha^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right)$$
$$=\frac{\left(1-\left(\frac{\alpha(x)}{t}\right)^{n+1}\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\frac{\alpha(x)}{t}\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\left(1-\left(\frac{\beta(x)}{t}\right)^{n+1}\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\frac{\beta(x)}{t}\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{t\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\alpha(x)\right)\left(t-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{t\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\beta(x)\right)\left(t-\alpha(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \tag{74}$$

Since $\left(t-\alpha(x)\right)\left(t-\beta(x)\right)=t^2-2xt-1$, we can write

$$\frac{t\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\alpha(x)\right)\left(t-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{t\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n+1}\left(\alpha(x)-\beta(x)\right)\left(t-\beta(x)\right)\left(t-\alpha(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \tag{75}$$

By using (75) in (74), we obtain

$$\sum_{k=0}^{n}\left(\frac{\alpha^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^k(x)}{t^k\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right)$$
$$=\frac{\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \tag{76}$$

By using (76) in (73), we get that

$$\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x)=\frac{\left(t^{n+1}-\alpha^{n+1}(x)\right)\left(t-\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+1}-\beta^{n+1}(x)\right)\left(t-\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{\left(t^{n+2}-\beta(x)t^{n+1}-\alpha^{n+1}(x)t+\alpha^{n+1}(x)\beta(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$
$$-\frac{\left(t^{n+2}-\alpha(x)t^{n+1}-\beta^{n+1}(x)t+\beta^{n+1}(x)\alpha(x)\right)}{t^{n}\left(t^2-2xt-1\right)\left(\alpha(x)-\beta(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$
$$=\frac{1}{t^n\left(t^2-2xt-1\right)}\left(t^{n+2}MGP_0(x)+t^{n+1}MGP_1(x)-\left(\alpha(x)+\beta(x)\right)t^{n+1}MGP_0(x)-tMGP_{n+1}(x)-MGP_n(x)\right)$$
$$=\frac{1}{t^n\left(t^2-2xt-1\right)}\left(t^{n+2}MGP_0(x)+t^{n+1}\left(MGP_1(x)-2xMGP_0(x)\right)-tMGP_{n+1}(x)-MGP_n(x)\right)\cdot \tag{77}$$

Take $n=1$ in (33), we can write

$$MGP_1(x)-2xMGP_0(x)=MGP_{-1}(x). \tag{78}$$

By using (78) in (77), we obtain

$$\sum_{k=0}^{n}\frac{1}{t^k}MGP_k(x)=\frac{1}{t^n\left(t^2-2xt-1\right)}\left(t^{n+2}MGP_0(x)+t^{n+1}MGP_{-1}(x)-tMGP_{n+1}(x)-MGP_n(x)\right)$$

$$=\frac{1}{t^2-2xt-1}\left(t^2MGP_0(x)+tMGP_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMGP_{n+1}(x)+MGP_n(x)\right).$$

The similar proof of (i) is used for (ii).

Therefore, the identities (i) and (ii) are immediately seen.

Corollary 20 For $n\in\mathbb{N}$, $x>0$, and $x^2+1>0$, the following equalities hold.

(i) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MGP_k=\frac{1}{t^2-2t-1}\left(t^2MGP_0+tMGP_{-1}\right)-\frac{1}{t^n\left(t^2-2t-1\right)}\left(tMGP_{n+1}+MGP_n\right)$,

(iii) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MGQ_k=\frac{1}{t^2-2t-1}\left(t^2MGQ_0+tMGQ_{-1}\right)-\frac{1}{t^n\left(t^2-2t-1\right)}\left(tMGQ_{n+1}+MGQ_n\right)$,

(iii) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MGq_k=\frac{1}{t^2-2t-1}\left(t^2MGq_0+tMGq_{-1}\right)-\frac{1}{t^n\left(t^2-2t-1\right)}\left(tMGq_{n+1}+MGq_n\right)$.

Proof. Take $x=1$ in Theorem 19 (i) and (ii), we have (i) and (iii).

The similar proof of Theorem 19 is used for (ii).

Corollary 21 For $n\in\mathbb{N}$, $x>0$, and $x^2+1>0$, the following equalities hold.

(i) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MP_k(x)=\frac{1}{t^2-2xt-1}\left(t^2MP_0(x)+tMP_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMP_{n+1}(x)+MP_n(x)\right)$,

(ii) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}MQ_k(x)=\frac{1}{t^2-2xt-1}\left(t^2MQ_0(x)+tMQ_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMQ_{n+1}(x)+MQ_n(x)\right)$,

(iii) $\displaystyle\sum_{k=0}^{n}\frac{1}{t^k}Mq_k(x)=\frac{1}{t^2-2xt-1}\left(t^2Mq_0(x)+tMq_{-1}(x)\right)-\frac{1}{t^n\left(t^2-2xt-1\right)}\left(tMq_{n+1}(x)+Mq_n(x)\right)$.

Proof. The similar proof of Theorem 19 is used for (i), (ii), and (iii).

Lemma 22 For $m,\ j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i) $\displaystyle(-1)^m MGP_{j-m}(x)=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$,

(ii) $\displaystyle(-1)^m MGq_{j-m}(x)=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\left(MGq_1(x)-\beta(x)MGq_0(x)\right)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\left(MGq_1(x)-\alpha(x)MGq_0(x)\right)$,

(iii) $\displaystyle(-1)^m MGP_{j-m}=\frac{\alpha^j\beta^m}{\alpha-\beta}\left(MGP_1-\beta MGP_0\right)-\frac{\beta^j\alpha^m}{\alpha-\beta}\left(MGP_1-\alpha MGP_0\right)$,

(iv) $\displaystyle(-1)^m MGQ_{j-m}=\frac{\alpha^j\beta^m}{\alpha-\beta}\left(MGQ_1-\beta MGQ_0\right)-\frac{\beta^j\alpha^m}{\alpha-\beta}\left(MGQ_1-\alpha MGQ_0\right)$,

(v) $\displaystyle(-1)^m MGq_{j-m}=\frac{\alpha^j\beta^m}{\alpha-\beta}\left(MGq_1-\beta MGq_0\right)-\frac{\beta^j\alpha^m}{\alpha-\beta}\left(MGq_1-\alpha MGq_0\right)$,

(vi) $\displaystyle(-1)^m MP_{j-m}(x)=\frac{\alpha^j(x)\beta^m(x)}{\alpha(x)-\beta(x)}\left(MP_1(x)-\beta(x)MP_0(x)\right)-\frac{\beta^j(x)\alpha^m(x)}{\alpha(x)-\beta(x)}\left(MP_1(x)-\alpha(x)MP_0(x)\right)$,

(vii) $\left(-1\right)^m MQ_{j-m}\left(x\right)=\dfrac{\alpha^j\left(x\right)\beta^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MQ_1\left(x\right)-\beta\left(x\right)MQ_0\left(x\right)\right)-\dfrac{\beta^j\left(x\right)\alpha^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MQ_1\left(x\right)-\alpha\left(x\right)MQ_0\left(x\right)\right)$,

(viii) $\left(-1\right)^m Mq_{j-m}\left(x\right)=\dfrac{\alpha^j\left(x\right)\beta^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(Mq_1\left(x\right)-\beta\left(x\right)Mq_0\left(x\right)\right)-\dfrac{\beta^j\left(x\right)\alpha^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(Mq_1\left(x\right)-\alpha\left(x\right)Mq_0\left(x\right)\right)$.

Proof. Since (49) and $\alpha\left(x\right)=x+\sqrt{x^2+1}$, $\beta\left(x\right)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$, we obtain

$$\begin{aligned}
\left(-1\right)^m MGP_{j-m}\left(x\right)&=\left(-1\right)^m\frac{\alpha^{j-m}\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\beta\left(x\right)MGP_0\left(x\right)\right)-\left(-1\right)^m\frac{\beta^{j-m}\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\alpha\left(x\right)MGP_0\left(x\right)\right)\\
&=\left(-1\right)^m\frac{\alpha^j\left(x\right)\beta^m\left(x\right)}{\left(\alpha\left(x\right)-\beta\left(x\right)\right)\alpha^m\left(x\right)\beta^m\left(x\right)}\left(MGP_1\left(x\right)-\beta\left(x\right)MGP_0\left(x\right)\right)\\
&\quad-\left(-1\right)^m\frac{\beta^j\left(x\right)\alpha^m\left(x\right)}{\left(\alpha\left(x\right)-\beta\left(x\right)\right)\beta^m\left(x\right)\alpha^m\left(x\right)}\left(MGP_1\left(x\right)-\alpha\left(x\right)MGP_0\left(x\right)\right)\\
&=\frac{\alpha^j\left(x\right)\beta^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\beta\left(x\right)MGP_0\left(x\right)\right)-\frac{\beta^j\left(x\right)\alpha^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\alpha\left(x\right)MGP_0\left(x\right)\right).
\end{aligned}$$

Thus, $\left(-1\right)^m MGP_{j-m}\left(x\right)=\dfrac{\alpha^j\left(x\right)\beta^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\beta\left(x\right)MGP_0\left(x\right)\right)-\dfrac{\beta^j\left(x\right)\alpha^m\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\alpha\left(x\right)MGP_0\left(x\right)\right)$.

The similar proof of (i) is used for (ii), (iii), (iv), (v), (vi), (vii), and (viii).

Therefore, the identities (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii) are immediately seen.

Theorem 23 For $m,n,j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i)

$$\sum_{k=0}^{n}MGP_{mk+j}\left(x\right)=\frac{1}{\left(1-\alpha^m\left(x\right)\right)\left(1-\beta^m\left(x\right)\right)}\left(MGP_j\left(x\right)+\left(-1\right)^{m+1}MGP_{j-m}\left(x\right)-MGP_{m\,n+m+j}\left(x\right)+\left(-1\right)^m MGP_{m\,n+j}\left(x\right)\right),$$

(ii)

$$\sum_{k=0}^{n}MGq_{mk+j}\left(x\right)=\frac{1}{\left(1-\alpha^m\left(x\right)\right)\left(1-\beta^m\left(x\right)\right)}\left(MGq_j\left(x\right)+\left(-1\right)^{m+1}MGq_{j-m}\left(x\right)-MGq_{m\,n+m+j}\left(x\right)+\left(-1\right)^m MGq_{m\,n+j}\left(x\right)\right).$$

Proof. Let $\alpha\left(x\right)=x+\sqrt{x^2+1}$, $\beta\left(x\right)=x-\sqrt{x^2+1}$ be the roots of $t^2-2xt-1=0$. Then we have

$$\sum_{k=0}^{n}MGP_{m\,k+j}\left(x\right)$$
$$=\sum_{k=0}^{n}\left(\frac{\alpha^{m\,k+j}\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\beta\left(x\right)MGP_0\left(x\right)\right)-\frac{\beta^{m\,k+j}\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\alpha\left(x\right)MGP_0\left(x\right)\right)\right). \tag{79}$$

Since definition of a geometric sequence, we have

$$\sum_{k=0}^{n}\left(\frac{\alpha^{m\,k+j}\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\beta\left(x\right)MGP_0\left(x\right)\right)-\frac{\beta^{m\,k+j}\left(x\right)}{\alpha\left(x\right)-\beta\left(x\right)}\left(MGP_1\left(x\right)-\alpha\left(x\right)MGP_0\left(x\right)\right)\right)$$

$$= \frac{\alpha^{j}(x)\left(1-\alpha^{m\,n+m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\alpha^{m}(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^{j}(x)\left(1-\beta^{m\,n+m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$

$$= \frac{\alpha^{j}(x)\left(1-\alpha^{m\,n+m}(x)\right)\left(1-\beta^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$

$$-\frac{\beta^{j}(x)\left(1-\beta^{m\,n+m}(x)\right)\left(1-\alpha^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\beta^{m}(x)\right)\left(1-\alpha^{m}(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right). \quad (80)$$

By using Lemma 22 (i) in (80), we have

$$\sum_{k=0}^{n}\left(\frac{\alpha^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)-\frac{\beta^{m\,k+j}(x)}{\alpha(x)-\beta(x)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)\right)$$

$$= \frac{\alpha^{j}(x)\left(1-\alpha^{m\,n+m}(x)\right)\left(1-\beta^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_1(x)-\beta(x)MGP_0(x)\right)$$

$$-\frac{\beta^{j}(x)\left(1-\beta^{m\,n+m}(x)\right)\left(1-\alpha^{m}(x)\right)}{\left(\alpha(x)-\beta(x)\right)\left(1-\beta^{m}(x)\right)\left(1-\alpha^{m}(x)\right)}\left(MGP_1(x)-\alpha(x)MGP_0(x)\right)$$

$$= \frac{1}{\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_j(x)+(-1)^{m+1}MGP_{j-m}(x)-MGP_{m\,n+m+j}(x)+(-1)^{m}MGP_{m\,n+j}(x)\right).$$

Thus,

$$\sum_{k=0}^{n}MGP_{mk+j}(x)=\frac{1}{\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MGP_j(x)+(-1)^{m+1}MGP_{j-m}(x)-MGP_{m\,n+m+j}(x)+(-1)^{m}MGP_{m\,n+j}(x)\right).$$

The similar proof of (i) is used for (ii).

Therefore, the proof is complete.

Corollary 24 For $m,n,j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i) $\displaystyle\sum_{k=0}^{n}MGP_{mk+j}=\frac{1}{\left(1-\alpha^{m}\right)\left(1-\beta^{m}\right)}\left(MGP_j+(-1)^{m+1}MGP_{j-m}-MGP_{m\,n+m+j}+(-1)^{m}MGP_{m\,n+j}\right)$,

(ii) $\displaystyle\sum_{k=0}^{n}MGQ_{mk+j}=\frac{1}{\left(1-\alpha^{m}\right)\left(1-\beta^{m}\right)}\left(MGQ_j+(-1)^{m+1}MGQ_{j-m}-MGQ_{m\,n+m+j}+(-1)^{m}MGQ_{m\,n+j}\right)$,

(iii) $\displaystyle\sum_{k=0}^{n}MGq_{mk+j}=\frac{1}{\left(1-\alpha^{m}\right)\left(1-\beta^{m}\right)}\left(MGq_j+(-1)^{m+1}MGq_{j-m}-MGq_{m\,n+m+j}+(-1)^{m}MGq_{m\,n+j}\right)$.

Proof. Take $x=1$ in Theorem 23 (i) and (ii), we have (i) and (iii).

The similar proof of Theorem 23 is used for (ii).

Corollary 25 For $m,n,j\in\mathbb{N}$ and $j\geq m$, the following results hold.

(i) $\displaystyle\sum_{k=0}^{n}MP_{mk+j}(x)=\frac{1}{\left(1-\alpha^{m}(x)\right)\left(1-\beta^{m}(x)\right)}\left(MP_j(x)+(-1)^{m+1}MP_{j-m}(x)-MP_{m\,n+m+j}(x)+(-1)^{m}MP_{m\,n+j}(x)\right)$,

(ii) $\sum_{k=0}^{n} MQ_{mk+j}(x) = \frac{1}{(1-\alpha^m(x))(1-\beta^m(x))}\left(MQ_j(x)+(-1)^{m+1}MQ_{j-m}(x)-MQ_{m\,n+m+j}(x)+(-1)^m MQ_{m\,n+j}(x)\right)$,

(iii) $\sum_{k=0}^{n} Mq_{mk+j}(x) = \frac{1}{(1-\alpha^m(x))(1-\beta^m(x))}\left(Mq_j(x)+(-1)^{m+1}Mq_{j-m}(x)-Mq_{m\,n+m+j}(x)+(-1)^m Mq_{m\,n+j}(x)\right)$.

Proof. The similar proof of Theorem 23 is used for (i), (ii), and (iii).

Discussion

In this article, we get some identities of the relation between matrix sequences and summations by applying some properties of matrix operation, the relation between numbers and polynomials, and Binet's formulas of matrix sequences.

Conclusions

In this paper, some identities of matrix sequences prove by some properties of matrix operation, the relation between numbers and polynomials, and Binet's formulas of 2 x 2 matrix representation. We obtained especially some identities of the relationships between matrix sequences. Moreover, we conjecture which this concept extends to the matrix sequence in terms of other recurrence relations and present the n x n matrix for $n \geq 3$.

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Generalized Identities for third order Pell Number, Pell-Lucas Number and Modified Pell Number

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Abstract

In this paper, we first presented the generalized Pell Number, Pell-Lucas Number and modified Pell Number, which are the recurrence relation by from the previous three terms. We have the Binet's formula generating functions and generating functions of all three sequences. We establish some of the interesting properties involving of sequences those sequences.

Keywords: Pell sequence Pell-Lucas sequence, Modified Pell sequence, Binet's formula

1. Introduction

We will refer to the sequence of occurrences starting in the recurring relationship from the previous second terms: Fibonacci and Lucas number. Because of their general characteristics, there are many interesting properties and application to almost every fields of science and art.

Previously, the sequence mentioned above is a sequence of positive integers that have been studied for many years. Many researchers have therefore examined about these sequences and also some properties that are excellent research topics. These sequences are examples of a sequences defined by a recurrence relation of second terms. It is well known that the Fibonacci sequence $\{F_n\}$, Lucas sequence $\{L_n\}$, Fibonacci-

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like $\{S_n\}$ and Generalized Fibonacci-Like $\{T_n\}$ are defined by the following recurring relationship $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$, $S_n = S_{n-1} + S_{n-2}$, $S_0 = 2$, $S_1 = 2$ and $T_n = T_{n-1} + T_{n-2}$, $T_0 = m$, $T_1 = m$, for all $n \geq 2$, where m is a positive integer, respectively [1-2], [8].

In a similar way, other recurrence sequences of important positive integers as well are the sequence of Pell, Pell-Lucas and modified Pell sequence. which those sequences are represented by $\{P_n\}$, $\{Q_n\}$ and $\{q_n\}$ defined by the following recurrence $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$, $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$, $q_0 = 1$, $q_1 = 1$ for all $n \geq 2$, respectively [3-5].

Also, the Pell, Pell-Lucas and modified Pell sequence expand to the negative subscript, which are defined by [7], [9].

$$P_{-n} = \frac{P_n}{(-1)^{n+1}}, \text{ for all } n \geq 1, \tag{1.1}$$

and

$$Q_{-n} = \frac{Q_n}{(-1)^{n}}, \text{ for all } n \geq 1, \tag{1.2}$$

and

$$q_{-n} = \frac{q_n}{(-1)^{n}}, \text{ for all } n \geq 1. \tag{1.3}$$

The properties of the sequence have received a lot of attention. Many sequences appear in literature, including Pell, Pell-Lucas and modified Pell. It is well-known that the proof uses Binet's formula [6].

Moreover, for the reasons mentioned above, the sequence has more interest and can be used with other work and has an interesting direction at present. Therefore, the researchers were inspired by the study of Pell, Pell-Lucas and modified Pell sequence.

2. Main Results

In this section, we formulate some third terms sum identities for Pell sequence $\{P_n\}$, Pell-Lucas sequence $\{Q_n\}$ and modified Pell sequence $\{q_n\}$ are present Catalan's identity, Cassini's identity, d'Ocagne's identity, Binet's formula and Generating function.

Definition 2.1: The Pell sequence $\{P_n\}$, The Pell – Lucas $\{Q_n\}$ and Modified Pell number $\{q_n\}$ are defined by

$$P_n = P_{n-1} + 3P_{n-2} + P_{n-3}, \text{ for all } n \geq 3, \tag{2.1}$$

with initial conditions $P_o = 0$, $P_1 = 1$ and $P_2 = 2$, $Q_n = Q_{n-1} + 3Q_{n-2} + Q_{n-3}$, for all

$$n \geq 3, \tag{2.2}$$

with initial conditions $Q_o = 2$, $Q_1 = 2$ and $Q_2 = 6$, and $q_n = q_{n-1} + 3q_{n-2} + q_{n-3}$ for all

$$n \geq 3, \tag{2.3}$$

with initial conditions $q_o = 1$, $q_1 = 1$ and $q_2 = 3$.

The first few terms of $\{P_n\}$ are $0, 1, 2, 5, 12, 29, 70$ and so on, and $\{Q_n\}$ are $2, 2, 6, 14, 34, 82, 198, 478$ and so on, and $\{q_n\}$ are $1, 1, 3, 7, 17, 41, 99, 239, 577$ and so on. Similarly, the first few terms of $\{P_{-n}\}$, $\{Q_{-n}\}$ and $\{q_{-n}\}$ can be obtained from the equation (1.1), (1.2) and (1.3), $\{P_{-n}\}$ are $1,-2,$ $5,-12,-29,70$ and so on, $\{Q_{-n}\}$ are $-2,6,$ $-14,34,-82,198,-478$ and so on, and $\{q_{-n}\}$ are $-1,3,-7,17,-41,99,-239,$ 577 and so on, respectively. Each Pell sequence, Pell-Lucas sequence and modified Pell sequence are called Pell numbers, Pell-Lucas numbers and modified Pell number.

Furthermore, we will find Binet's formula to allow us to show the Pell number, Pell-Lucas number, and Modified Pell number, which has the following characteristic equation:

$$x^3 - x^2 - 3x - 1 = 0, \tag{2.4}$$

where α, β and γ are the root of the equation, $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, $\gamma = -1$ and $\alpha >$. $\beta > \gamma$ Note that $\alpha+\beta = 2$, $\alpha-\beta = 2\sqrt{2}$ and $\alpha\beta = \gamma$, respectively.

Next, we will say the equation is related to the repetitive relationship of (2.1), (2.2) and (2.3) defined by Theorem 2.2.

Theorem 2.2: (Binet's formula) The n^{th} Pell number, the n^{th} Pell – Lucas number and the n^{th} Modified Pell number are given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.5}$$

and

$$Q_n = \alpha^n + \beta^n, \tag{2.6}$$

and

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}, \tag{2.7}$$

where n is not a negative integer and α,β,γ are the roots of the characteristic equation (2.4), which $\alpha > \beta > \gamma$.

Proof. Since equation (2.4) has three different roots, the number of P_n is defined by

$$P_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Let $n=0, n=1$ and $n=2$, then solve the system of linear equations, we will $c_1 = \frac{1}{\alpha-\beta}$, $c_2 = -\frac{1}{\alpha-\beta}$ and $c_3 = 0$, therefore

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Similarly, the number of Q_n is given by

$$Q_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Use the same method as above, then solve this linear equation, we obtain $c_1 = 1 = c_2$ and $c_3 = 0$, thence

$$Q_n = \alpha^n + \beta^n.$$

Similarly, the number $\{q_n\}$ is given by

$$q_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Let $n=0, n=1$ and $n=2$, we obtain $c_1 = \frac{1}{\alpha+\beta} = c_2$ and $c_3 = 0$, thence

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}.$$

The proof completed. □

Theorem 2.3: (Catalan's identity)

Let n is not a negative integer. Then

$$P_{n+r}P_{n-r} - P_n^2 = \gamma^{n-r+1}P_r^2, \tag{2.8}$$

and

$$Q_{n+r}Q_{n-r} - Q_n^2 = \gamma^{n-r}Q_r^2 - 2\gamma^n, \quad (2.9)$$

and

$$q_{n+r}q_{n-r} - q_n^2 = \frac{\gamma^{n-r}q_{2r} - \gamma^n}{2}. \quad (2.10)$$

Proof. Since Binet's formula (2.5), we obtain

$$\begin{aligned} P_{n+r}P_{n-r} - P_n^2 &= \frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\cdot\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta} - \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2 \\ &= -\frac{\alpha^{n-r}\beta^{n-r}\left(\alpha^r-\beta^r\right)^2}{\left(\alpha-\beta\right)^2} \\ &= \gamma^{n-r+1}P_r^2. \end{aligned}$$

Similarly, by Binet's formula (2.6), we obtain

$$\begin{aligned} Q_{n+r}Q_{n-r} - Q_n^2 &= \left(\alpha^{n+r}+\beta^{n+r}\right)\cdot\left(\alpha^{n-r}+\beta^{n-r}\right) - \left(\alpha^n+\beta^n\right)^2 \\ &= \alpha^{n-r}\beta^{n-r}\left(\alpha^r-\beta^r\right)^2 \\ &= \alpha^{n-r}\beta^{n-r}\left(\alpha^{2r}+\beta^{2r}\right) - 2\alpha^{n-r}\beta^{n-r}\alpha^r\beta^r \\ &= \gamma^{n-r}Q_{2r} - 2\gamma^n. \end{aligned}$$

Similarly, by Binet's formula (2.7), we obtain

$$\begin{aligned} q_{n+r}q_{n-r} - q_n^2 &= \frac{\alpha^{n+r}+\beta^{n+r}}{\alpha+\beta}\cdot\frac{\alpha^{n-r}+\beta^{n-r}}{\alpha+\beta} - \left(\frac{\alpha^n+\beta^n}{\alpha+\beta}\right)^2 \\ &= \frac{\alpha^{n+r}\beta^{n-r}}{\left(\alpha+\beta\right)^2} + \frac{\alpha^{n-r}\beta^{n+r}}{\left(\alpha+\beta\right)^2} - \frac{2\alpha^n\beta^n}{\left(\alpha+\beta\right)^2} \\ &= \frac{\alpha^{n-r}\beta^{n-r}}{\alpha+\beta}\cdot\frac{\alpha^{2r}+\beta^{2r}}{\alpha+\beta} - \frac{\alpha^n\beta^n}{\alpha+\beta} \\ &= \frac{\gamma^{n-r}q_{2r}-\gamma^n}{2}. \end{aligned}$$

The proof completed. □

Theorem 2.4: (Catalan's identity or Simpson's identity) Let n is not a negative integer. Then

$$P_{n+1}P_{n-1} - P_n^2 = \gamma^n, \quad (2.11)$$

and

$$Q_{n+1}Q_{n-1} - Q_n^2 = 8\gamma^{n-1}, \quad (2.12)$$

and

$$q_{n+1}q_{n-1} - q_n^2 = 2\gamma^{n-1}. \quad (2.13)$$

Proof. Taking $r=1$ in Catalan's identity (2.8), (2.9) and (2.10), the proof completed. □

Theorem 2.5: (d'Ocagne's identity)

Let m, n are not a negative integer and $m > n$. Then

$$P_mP_{n+1} - P_{m+1}P_n = \gamma^nP_{m-n}, \quad (2.14)$$

and

$$Q_mQ_{n+1} - Q_{m+1}Q_n = 2\sqrt{2}\gamma^m\left(Q_{n-m} + 2\gamma\beta^{n-m}\right), \quad (2.15)$$

and

$$q_mq_{n+1} - q_{m+1}q_n = \sqrt{2}\gamma^m\left(q_{n-m} - \beta^{n-m}\right). \quad (2.16)$$

Proof. By Binet's formula (2.5), we have

$$\begin{aligned} P_mP_{n+1} - P_{m+1}P_n &= \frac{\alpha^m-\beta^m}{\alpha-\beta}\cdot\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\cdot\frac{\alpha^n-\beta^n}{\alpha-\beta} \\ &= \frac{\alpha^m\beta^n-\alpha^n\beta^m}{\alpha-\beta} \end{aligned}$$

$$= \alpha^n \beta^n \frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}$$
$$= \gamma^n P_{m-n}.$$

Similarly, by Binet's formula (2.6), we obtain

$$Q_m Q_{n+1} - Q_{m+1} Q_n = \left(\alpha^m + \beta^m\right) \cdot \left(\alpha^{n+1} + \beta^{n+1}\right) - \left(\alpha^{m+1} + \beta^{m+1}\right) \cdot \left(\alpha^n + \beta^n\right)$$
$$= \left(\alpha - \beta\right) \left(\alpha^n \beta^m - \alpha^m \beta^n\right)$$
$$= \left(\alpha - \beta\right) \alpha^m \beta^m \left(\alpha^{n-m} - \beta^{n-m}\right)$$
$$= 2\sqrt{2}\gamma^m \left(Q_{n-m} + 2\gamma\beta^{n-m}\right).$$

Similarly, by Binet's formula (2.7), we obtain

$$q_m q_{n+1} - q_{m+1} q_n = \frac{\alpha^m + \beta^m}{\alpha + \beta} \cdot \frac{\alpha^{n+1} + \beta^{n+1}}{\alpha + \beta} - \frac{\alpha^{m+1} + \beta^{m+1}}{\alpha + \beta} \cdot \frac{\alpha^n + \beta^n}{\alpha + \beta}$$
$$= \frac{\left(\alpha - \beta\right)\left(\alpha^n \beta^m - \alpha^m \beta^n\right)}{\left(\alpha + \beta\right)^2}$$
$$= \left(\alpha - \beta\right) \alpha^m \beta^m \frac{\left(\alpha^{n-m} - \beta^{n-m}\right)}{\left(\alpha + \beta\right)^2}$$
$$= \sqrt{2}\gamma^m \left(q_{n-m} - \beta^{n-m}\right).$$

The proof completed. □

Lemma 2.6 Let m, n are not a negative integer and $m > n$. Then

$$P_m P_{n+1} - P_{m+1} P_n = \frac{\gamma^n}{2\sqrt{2}}, \left(Q_{m-n} + 2\gamma\beta^{m-n}\right) \quad (2.17)$$

and

$$P_m P_{n+1} - P_{m+1} P_n = \frac{\gamma^n}{\sqrt{2}} \left(q_{m-n} + \gamma\beta^{m-n}\right). \quad (2.18)$$

Proof. The Proof same as Theorem 2.5.

Lemma 2.7 Let m, n are not a negative integer and $m > n$. Then

$$Q_m Q_{n+1} - Q_{m+1} Q_n = 8\gamma^m P_{n-m}, \quad (2.19)$$

and

$$Q_m Q_{n+1} - Q_{m+1} Q_n = 4\sqrt{2}\gamma^m \left(q_{n-m} + \gamma\beta^{n-m}\right). \quad (2.20)$$

Proof. The Proof same as Theorem 2.5.

Lemma 2.8 Let m, n are not a negative integer and $m > n$. Then

$$q_m q_{n+1} - q_{m+1} q_n = 2\gamma^m P_{n-m}, \quad (2.21)$$

and

$$q_m q_{n+1} - q_{m+1} q_n = \sqrt{2}\gamma^m. \left(q_{n-m} + \gamma\beta^{n-m}\right) \quad (2.22)$$

Proof. The Proof same as Theorem 2.5.

Theorem 2.9: Let $\{P_n\}, \{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences, m and n are not a negative integer and $m > n$. Then

$$\lim_{n \to \infty} \frac{P_n}{P_{n-1}} = \alpha, \quad (2.23)$$

and

$$\lim_{n \to \infty} \frac{Q_n}{Q_{n-1}} = \alpha, \quad (2.24)$$

and

$$\lim_{n \to \infty} \frac{q_n}{q_{n-1}} = \alpha. \quad (2.25)$$

Proof. By Binet's formula (2.5), we have

$$\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n-\beta^n}{\alpha^{n-1}-\beta^{n-1}}$$

$$=\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}-\frac{1}{\beta}\cdot\left(\frac{\beta}{\alpha}\right)^n}.$$

But $\alpha>\beta$, then $\frac{\beta}{\alpha}<1$ and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0.$

Therefor $\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\alpha$.

Similarly, $\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}}$

$$=\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}=\alpha.$$

Similarly, $\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}}$

$$=\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}=\alpha.$$

This completes the proof. □

Lemma 2.10: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences and n is not a negative integer. Then

$$\lim_{n\to\infty}\frac{P_n}{Q_{n-1}}=\frac{\alpha}{\alpha-\beta}, \tag{2.26}$$

and

$$\lim_{n\to\infty}\frac{P_n}{q_{n-1}}=\frac{\alpha+\beta}{\alpha-\beta}, \tag{2.27}$$

and

$$\lim_{n\to\infty}\frac{Q_n}{q_{n-1}}=\frac{\alpha-\beta}{\alpha}. \tag{2.28}$$

Proof. The Proof same as Theorem 2.9.

In this paper, the generating function for Pell, Pell-Lucas and modified Pell sequences are given as a result, these sequence are seen and the coefficients of the power series of the corresponding generating function.

The generating function for Pell, Pell-Lucas and modified Pell sequences. We can also find the generating function for all three sequences by suppose that the Pell, Pell-Lucas and modified Pell sequences are the coefficients of a potential series center at the origin, and let us consider the corresponding analytic $\{P_n\},\{Q_n\}$ and $\{q_n\}$ of the function, which the function is as follows Theorem.

Theorem 2.11: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences and n is not a negative integer. Then the generating function defined by

$$P_n(x)=\frac{x}{1-2x-x^2}, \tag{2.29}$$

and

$$Q_n(x)=\frac{2-2x}{1-2x-x^2}, \tag{2.30}$$

and

$$q_n(x)=\frac{1-x}{1-2x-x^2}. \tag{2.31}$$

Proof. Let n is a not negative integer and

$$P_n(x)=P_0+P_1x+P_2x^2+P_3x^3+P_4x^4+\ldots+P_nx^n+\ldots.$$

Then

$$2xP_n(x) = 2P_0x + 2P_1x^2 + 2P_2x^3 + \ldots + 2P_nx^{n+1} + \ldots$$

$$x^2P_n(x) = P_0x^2 + P_1x^3 + P_2x^4 + \ldots + P_nx^{n+2} + \ldots$$

$$P_n(x) - 2xP_n(x) - x^2P_n(x) = x$$

$$\left(1-2x-x^2\right)P_n(x) = x.$$

Thus $P_n(x) = \sum_{n=0}^{\infty} P_nx^n = \dfrac{x}{1-2x-x^2}$.

Similarly, we have

$$Q_n(x) = Q_0 + Q_1x + Q_2x^2 + Q_3x^3 + Q_4x^4 + \ldots + Q_nx^n + \ldots.$$

Then, we obtain

$$2xQ_n(x) = 2Q_0x + 2Q_1x^2 + 2Q_2x^3 + \ldots + 2Q_nx^{n+1} + \ldots$$

$$x^2Q_n(x) = Q_0x^2 + Q_1x^3 + Q_2x^4 + \ldots + Q_nx^{n+2} + \ldots$$

$$Q_n(x) - 2xQ_n(x) - x^2Q_n(x) = 2-2x$$

$$\left(1-2x1-x^2\right)Q_n(x) = 2-2x.$$

Thus $Q_n(x) = \sum_{n=0}^{\infty} Q_nx^n = \dfrac{2-2x}{1-2x-x^2}$.

Similarly, we have

$$q_n(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4 + \ldots + q_nx^n + \ldots.$$

Then, we write

$$2xq_n(x) = 2q_0x + 2q_1x^2 + 2q_2x^3 + \ldots + 2q_nx^{n+1} + \ldots$$

$$x^2q_n(x) = q_0x^2 + q_1x^3 + q_2x^4 + \ldots + q_nx^{n+2} + \ldots$$

$$q_n(x) - 2xq_n(x) - x^2q_n(x) = 1-x$$

$$\left(1-2x-x^2\right)q_n(x) = 1-x.$$

Thus $q_n(x) = \sum_{n=0}^{\infty} q_nx^n = \dfrac{1-x}{1-2x-x^2}$.

This completes the proof. □

From the Theorem 2.11 used to find the generating function. Next will be the polynomial of Pell, Pell–Lucas and Modified Pell sequences from the generating function, which using Maclaurin series helps to find the following theorem**.**

Theorem 2.12: The equality

$$\frac{x}{1-2x-x^2} = P_0 + P_1x + P_2x^2 + P_3x^3 + P_4x^4 + \ldots, \tag{2.32}$$

and

$$\frac{2-2x}{1-2x-x^2} = Q_0 + Q_1x + Q_2x^2 + Q_3x^3, + Q_4x^4 + \ldots, \tag{2.33}$$

and

$$\frac{1-x}{1-2x-x^2} = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4 + \ldots. \tag{2.34}$$

Proof. Since Maclaurin series, $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$. Let $f(x) = \dfrac{x}{1-2x-x^2}$ for all x, we obtain

$$f'(x) = -\frac{(-2-2x)x}{\left(1-2x-x^2\right)^2} + \frac{1}{1-2x-x^2}$$

$$f''(x) = -\frac{2(-2-2x)}{\left(1-2x-x^2\right)^2} + \frac{2x(-2-2x)^2}{\left(1-2x-x^2\right)^3} + \frac{2x}{\left(1-2x-x^2\right)^2}$$

$$f'''(x) = -\frac{6x(-2-2x)^3}{\left(1-2x-x^2\right)^4} - \frac{12x(-2-2x)}{\left(1-2x-x^2\right)^3}$$

$$+\frac{6\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^3}$$

$$+\frac{6}{\left(1-2x-x^2\right)^2}$$

$$f^{(4)}=\frac{24x\left(-2-2x\right)^4}{\left(1-2x-x^2\right)^5}+\frac{72x\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^4}$$

$$+\frac{24x}{\left(1-2x-x^2\right)^3}-\frac{24\left(-2-2x\right)^3}{\left(1-2x-x^2\right)^4}$$

$$-\frac{48\left(-2-2x\right)}{\left(1-2x-x^2\right)^3}$$

and so on.

Then

$f(0)=0,\ f'(1)=1,\ f''(0)=4,$

$f'''(0)=30,\ f^{(4)}(0)=288$ and so on,

$$\frac{x}{1-2x-x^2}=0+\frac{1}{1!}x+\frac{4}{2!}x^2+\frac{30}{3!}x^3$$

$$+\frac{288}{4!}x^4+\ldots.$$

Thus $\dfrac{x}{1-2x-x^2}=P_0+P_1x+P_2x^2+P_3x^3$

$$+P_4x^4+\ldots.$$

Similarly, $f(x)=\dfrac{2-2x}{1-2x-x^2}$ for all x, we obtain

$$f'(x)=-\frac{\left(-2-2x\right)\left(2-2x\right)}{\left(1-2x-x^2\right)^2}-\frac{2}{1-2x-x^2}$$

$$f''(x)=\frac{4\left(-2-2x\right)}{\left(1-2x-x^2\right)^2}$$

$$+\frac{2\left(2-2x\right)\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^3}$$

$$+\frac{2\left(2-2x\right)}{\left(1-2x-x^2\right)^2}$$

$$f'''(x)=-\frac{6\left(2-2x\right)\left(-2-2x\right)^3}{\left(1-2x-x^2\right)^4}$$

$$-\frac{12\left(2-2x\right)\left(-2-2x\right)}{\left(1-2x-x^2\right)^3}$$

$$-\frac{12\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^3}-\frac{12}{\left(1-2x-x^2\right)^2}$$

$$f^{(4)}=\frac{24\left(2-2x\right)\left(-2-2x\right)^4}{\left(1-2x-x^2\right)^5}$$

$$+\frac{72\left(2-2x\right)\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^4}$$

$$+\frac{24\left(2-2x\right)}{\left(1-2x-x^2\right)^3}+\frac{48\left(-2-2x\right)^3}{\left(1-2x-x^2\right)^4}$$

$$+\frac{96\left(-2-2x\right)}{\left(1-2x-x^2\right)^3},$$

and so on.

Then

$f(0)=2,\ f'(1)=2,\ f''(0)=12,$

$f'''(0)=84,\ f^{(4)}(0)=816$ and so on,

$$\frac{2-2x}{1-2x-x^2}=2+\frac{2}{1!}x+\frac{12}{2!}x^2+\frac{84}{3!}x^3$$

$$+\frac{816}{4!}x^4+\ldots.$$

Thus

$$\frac{2-2x}{1-2x-x^2}=Q_0+Q_1x+Q_2x^2+Q_3x^3$$

$$+Q_4x^4+\ldots.$$

Similarly, $f(x)=\dfrac{1-x}{1-2x-x^2}$ for all x, we obtain

$$f'(x)=-\frac{(-2-2x)(1-x)}{(1-2x-x^2)^2}-\frac{1}{1-2x-x^2}$$

$$f''(x)=\frac{2(-2-2x)}{(1-2x-x^2)^2}+\frac{2(1-x)(-2-2x)^2}{(1-2x-x^2)^3}+\frac{2(1-x)}{(1-2x-x^2)^2}$$

$$f'''(x)=-\frac{6(1-x)(-2-2x)^3}{(1-2x-x^2)^4}-\frac{12(1-x)(-2-2x)}{(1-2x-x^2)^3}-\frac{6(-2-2x)^2}{(1-2x-x^2)^3}-\frac{6}{(1-2x-x^2)^2}$$

$$f^{(4)}=\frac{24(1-x)(-2-2x)^4}{(1-2x-x^2)^5}+\frac{72(1-x)(-2-2x)^2}{(1-2x-x^2)^4}+\frac{24(1-x)}{(1-2x-x^2)^3}+\frac{24(-2-2x)^3}{(1-2x-x^2)^4}+\frac{48(-2-2x)}{(1-2x-x^2)^3},$$

and so on.

Then

$f(0)=1,\ f'(1)=1,\ f''(0)=6,$

$f'''(0)=42,\ f^{(4)}(0)=408$ and so on,

$$\frac{1-x}{1-2x-x^2}=1+\frac{1}{1!}x+\frac{6}{2!}x^2+\frac{42}{3!}x^3+\frac{408}{4!}x^4+\ldots.$$

Thus $$\frac{1-x}{1-2x-x^2}=q_0+q_1x+q_2x^2+q_3x^3+q_4x^4+\ldots.$$

This completes the proof. □

Next, we will discuss the power series, including $\sum_{n=0}^{\infty}P_nx^n, \sum_{n=0}^{\infty}Q_nx^n$ and $\sum_{n=0}^{\infty}q_nx^n$ in $x-x_0$ but $x_0=0$ converges is always an interval center at $x=0$. We test convergence of such series by complete convergence and series converges if $|x|<\dfrac{1}{\alpha}$ and series diverges if $|x|>\dfrac{1}{\alpha}$. The series convergences absolutely open interval $\left(-\dfrac{1}{\alpha},\dfrac{1}{\alpha}\right)$, making the convergence radius equal $\dfrac{1}{\alpha}$, as the following Theorem.

Theorem 2.13: Let $\sum_{n=0}^{\infty}P_nx^n, \sum_{n=0}^{\infty}Q_nx^n$ and $\sum_{n=0}^{\infty}q_nx^n$ are power series. Then interval of convergence for the given series is $\left(-\dfrac{1}{\alpha},\dfrac{1}{\alpha}\right)$ and the radius of convergence is $\dfrac{1}{\alpha}$.

Proof. By absolute convergence and Theorem 2.9, we have

$$\lim_{n\to\infty}\left|\frac{P_nx^n}{P_{n-1}x^{n-1}}\right|=\lim_{n\to\infty}|\alpha x|=\alpha|x|.$$

So the $\sum_{n=0}^{\infty}P_nx^n$ converges absolute if

$\lim_{n\to\infty}\left|\frac{P_n x^n}{P_{n-1}x^{n-1}}\right|<1$, then $\alpha|x|<1$ and $|x|<\frac{1}{\alpha}$.

The test value $x=\frac{1}{\alpha}$ or $x=-\frac{1}{\alpha}$, when replaced in series, will be

$$\sum_{n=0}^{\infty}P_n\left(\frac{1}{\alpha}\right)^n = P_0+\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}+\ldots \text{ and}$$

$$\sum_{n=0}^{\infty}P_n\left(-\frac{1}{\alpha}\right)^n = \sum_{n=0}^{\infty}(-1)^n P_n\left(\frac{1}{\alpha}\right)^n = P_0-\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}-\ldots.$$

So both of diverge, therefore the interval of convergence for the given power series is $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence is $\frac{1}{\alpha}$. The $\sum_{n=0}^{\infty}Q_n x^n$ and $\sum_{n=0}^{\infty}q_n x^n$ will prove similarly, then the interval of convergence for the given power series are $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence are $\frac{1}{\alpha}$, this completes the proof. □

From the theorem 2.13, found that the polynomial of Pell, Pell–Lucas and Modified Pell sequences are convergence to $\frac{1}{\alpha}$ and there is scope for convergence during the opening period $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$.

Lemma 2.14: The equality

$$P_n(x)=P_0+P_1x+P_2x^2+P_3x^3 +P_4x^4+\ldots, \tag{2.35}$$

and

$$Q_n(x)=Q_0+Q_1x+Q_2x^2+Q_3x^3 +Q_4x^4+\ldots, \tag{2.36}$$

and

$$q_n(x)=q_0+q_1x+q_2x^2+q_3x^3 +q_4x^4+\ldots. \tag{2.37}$$

for all $x\in\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$.

Then $P_n(0)=P_0$, $Q_n(0)=Q_0$ and $q_n(0)=q_0$.

Proof. Taking $x=0$ in (2.35), (2.36) and (2.37). The proof completed. □

Furthermore, how to find $P_n(0)$, $Q_n(0)$ and $q_n(0)$. We also use the equation (2.29), (2.30) and (2.31) in the Theorem 2.11 by giving x = 0, then $P_n(0)=P_0$, $Q_n(0)=Q_0$ and $q_n(0)=q_0$, respectively.

3. Conclusion

In this article, first of all, we consider the generality of Pell, Pell-Lucas and modified Pell sequence by the result of the previous three terms. Then we introduced the Pell, Pell-Lucas and modified Pell number. Until finally, we got the Binet's formula and the generating function of

Pell, Pell-Lucas and the modified Pell sequence. In addition, we also received the information some important identities involving the terms of these sequences.

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Generalized Identities for third order Pell Number, Pell-Lucas Number and Modified Pell Number

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Abstract

In this paper, we first presented the generalized Pell Number, Pell-Lucas Number and modified Pell Number, which are the recurrence relation by from the previous three terms. We have the Binet's formula generating functions and generating functions of all three sequences. We establish some of the interesting properties involving of sequences those sequences.

Keywords: Pell sequence Pell-Lucas sequence, Modified Pell sequence, Binet's formula

1. Introduction

We will refer to the sequence of occurrences starting in the recurring relationship from the previous second terms: Fibonacci and Lucas number. Because of their general characteristics, there are many interesting properties and application to almost every fields of science and art.

Previously, the sequence mentioned above is a sequence of positive integers that have been studied for many years. Many researchers have therefore examined about these sequences and also some properties that are excellent research topics. These sequences are examples of a sequences defined by a recurrence relation of second terms. It is well known that the Fibonacci sequence $\{F_n\}$, Lucas sequence $\{L_n\}$, Fibonacci-

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like $\{S_n\}$ and Generalized Fibonacci-Like $\{T_n\}$ are defined by the following recurring relationship
$F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$,
$L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$,
$S_n = S_{n-1} + S_{n-2}$, $S_0 = 2$, $S_1 = 2$ and
$T_n = T_{n-1} + T_{n-2}$, $T_0 = m$, $T_1 = m$, for all $n \geq 2$, where m is a positive integer, respectively [1-2], [8].

In a similar way, other recurrence sequences of important positive integers as well are the sequence of Pell, Pell-Lucas and modified Pell sequence. which those sequences are represented by $\{P_n\}$, $\{Q_n\}$ and $\{q_n\}$ defined by the following recurrence $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$, $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$, $q_0 = 1$, $q_1 = 1$ for all $n \geq 2$, respectively [3-5].

Also, the Pell, Pell-Lucas and modified Pell sequence expand to the negative subscript, which are defined by [7], [9].

$$P_{-n} = \frac{P_n}{(-1)^{n+1}}, \text{ for all } n \geq 1, \tag{1.1}$$

and

$$Q_{-n} = \frac{Q_n}{(-1)^{n}}, \text{ for all } n \geq 1, \tag{1.2}$$

and

$$q_{-n} = \frac{q_n}{(-1)^{n}}, \text{ for all } n \geq 1. \tag{1.3}$$

The properties of the sequence have received a lot of attention. Many sequences appear in literature, including Pell, Pell-Lucas and modified Pell. It is well-known that the proof uses Binet's formula [6].

Moreover, for the reasons mentioned above, the sequence has more interest and can be used with other work and has an interesting direction at present. Therefore, the researchers were inspired by the study of Pell, Pell-Lucas and modified Pell sequence.

2. Main Results

In this section, we formulate some third terms sum identities for Pell sequence $\{P_n\}$, Pell-Lucas sequence $\{Q_n\}$ and modified Pell sequence $\{q_n\}$ are present Catalan's identity, Cassini's identity, d'Ocagne's identity, Binet's formula and Generating function.

Definition 2.1: The Pell sequence $\{P_n\}$, The Pell – Lucas $\{Q_n\}$ and Modified Pell number $\{q_n\}$ are defined by

$$P_n = P_{n-1} + 3P_{n-2} + P_{n-3}, \text{ for all } n \geq 3, \tag{2.1}$$

with initial conditions $P_o = 0$, $P_1 = 1$ and $P_2 = 2$, $Q_n = Q_{n-1} + 3Q_{n-2} + Q_{n-3}$, for all

$$n \geq 3, \tag{2.2}$$

with initial conditions $Q_o = 2$, $Q_1 = 2$ and $Q_2 = 6$, and $q_n = q_{n-1} + 3q_{n-2} + q_{n-3}$ for

$$\text{all } n \geq 3, \tag{2.3}$$

with initial conditions $q_o = 1$, $q_1 = 1$ and $q_2 = 3$.

The first few terms of $\{P_n\}$ are $0,1,2,5,12,29,70$ and so on, and $\{Q_n\}$ are $2,2,6,14,34,82,198,478$ and so on, and $\{q_n\}$ are $1,1,3,7,17,41,99,239,577$ and so on. Similarly, the first few terms of $\{P_{-n}\}$, $\{Q_{-n}\}$ and $\{q_{-n}\}$ can be obtained from the equation (1.1), (1.2) and (1.3), $\{P_{-n}\}$ are $1,-2,5,-12,-29,70$ and so on, $\{Q_{-n}\}$ are $-2,6,-14,34,-82,198,-478$ and so on, and $\{q_{-n}\}$ are $-1,3,-7,17,-41,99,-239,577$ and so on, respectively. Each Pell sequence, Pell-Lucas sequence and modified Pell sequence are called Pell numbers, Pell-Lucas numbers and modified Pell number.

Furthermore, we will find Binet's formula to allow us to show the Pell number, Pell-Lucas number, and Modified Pell number, which has the following characteristic equation:

$$x^3 - x^2 - 3x - 1 = 0, \tag{2.4}$$

where α, β and γ are the root of the equation, $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, $\gamma = -1$ and $\alpha >$. $\beta > \gamma$ Note that $\alpha+\beta = 2$, $\alpha-\beta = 2\sqrt{2}$ and $\alpha\beta = \gamma$, respectively.

Next, we will say the equation is related to the repetitive relationship of (2.1), (2.2) and (2.3) defined by Theorem 2.2.

Theorem 2.2: (Binet's formula) The n^{th} Pell number, the n^{th} Pell – Lucas number and the n^{th} Modified Pell number are given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.5}$$

and

$$Q_n = \alpha^n + \beta^n, \tag{2.6}$$

and

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}, \tag{2.7}$$

where n is not a negative integer and α, β, γ are the roots of the characteristic equation (2.4), which $\alpha > \beta > \gamma$.

Proof. Since equation (2.4) has three different roots, the number of P_n is defined by

$$P_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Let $n=0, n=1$ and $n=2$, then solve the system of linear equations, we will $c_1 = \frac{1}{\alpha-\beta}$, $c_2 = -\frac{1}{\alpha-\beta}$ and $c_3 = 0$, therefore

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Similarly, the number of Q_n is given by

$$Q_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Use the same method as above, then solve this linear equation, we obtain $c_1 = 1 = c_2$ and $c_3 = 0$, thence

$$Q_n = \alpha^n + \beta^n.$$

Similarly, the number $\{q_n\}$ is given by

$$q_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Let $n=0, n=1$ and $n=2$, we obtain $c_1 = \frac{1}{\alpha+\beta} = c_2$ and $c_3 = 0$, thence

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}.$$

The proof completed. □

Theorem 2.3: (Catalan's identity)

Let n is not a negative integer. Then

$$P_{n+r}P_{n-r} - P_n^2 = \gamma^{n-r+1}P_r^2, \tag{2.8}$$

and

$$Q_{n+r}Q_{n-r} - Q_n^2 = \gamma^{n-r}Q_r^2 - 2\gamma^n, \quad (2.9)$$

and

$$q_{n+r}q_{n-r} - q_n^2 = \frac{\gamma^{n-r}q_{2r} - \gamma^n}{2}. \quad (2.10)$$

Proof. Since Binet's formula (2.5), we obtain

$$\begin{aligned} P_{n+r}P_{n-r} - P_n^2 &= \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha-\beta}\cdot\frac{\alpha^{n-r} - \beta^{n-r}}{\alpha-\beta} \\ &\quad -\left(\frac{\alpha^n - \beta^n}{\alpha-\beta}\right)^2 \\ &= -\frac{\alpha^{n-r}\beta^{n-r}\left(\alpha^r - \beta^r\right)^2}{\left(\alpha-\beta\right)^2} \\ &= \gamma^{n-r+1}P_r^2. \end{aligned}$$

Similarly, by Binet's formula (2.6), we obtain

$$\begin{aligned} Q_{n+r}Q_{n-r} - Q_n^2 &= \left(\alpha^{n+r} + \beta^{n+r}\right)\cdot\left(\alpha^{n-r} + \beta^{n-r}\right) \\ &\quad -\left(\alpha^n + \beta^n\right)^2 \\ &= \alpha^{n-r}\beta^{n-r}\left(\alpha^r - \beta^r\right)^2 \\ &= \alpha^{n-r}\beta^{n-r}\left(\alpha^{2r} + \beta^{2r}\right) \\ &\quad -2\alpha^{n-r}\beta^{n-r}\alpha^r\beta^r \\ &= \gamma^{n-r}Q_{2r} - 2\gamma^n. \end{aligned}$$

Similarly, by Binet's formula (2.7), we obtain

$$\begin{aligned} q_{n+r}q_{n-r} - q_n^2 &= \frac{\alpha^{n+r} + \beta^{n+r}}{\alpha+\beta}\cdot\frac{\alpha^{n-r} + \beta^{n-r}}{\alpha+\beta} \\ &\quad -\left(\frac{\alpha^n + \beta^n}{\alpha+\beta}\right)^2 \\ &= \frac{\alpha^{n+r}\beta^{n-r}}{\left(\alpha+\beta\right)^2} + \frac{\alpha^{n-r}\beta^{n+r}}{\left(\alpha+\beta\right)^2} \\ &\quad -\frac{2\alpha^n\beta^n}{\left(\alpha+\beta\right)^2} \\ &= \frac{\alpha^{n-r}\beta^{n-r}}{\alpha+\beta}\cdot\frac{\alpha^{2r} + \beta^{2r}}{\alpha+\beta} \\ &\quad -\frac{\alpha^n\beta^n}{\alpha+\beta} \\ &= \frac{\gamma^{n-r}q_{2r} - \gamma^n}{2}. \end{aligned}$$

The proof completed. □

Theorem 2.4: (Catalan's identity or Simpson's identity) Let n is not a negative integer. Then

$$P_{n+1}P_{n-1} - P_n^2 = \gamma^n, \quad (2.11)$$

and

$$Q_{n+1}Q_{n-1} - Q_n^2 = 8\gamma^{n-1}, \quad (2.12)$$

and

$$q_{n+1}q_{n-1} - q_n^2 = 2\gamma^{n-1}. \quad (2.13)$$

Proof. Taking $\mathrm{r} = 1$ in Catalan's identity (2.8), (2.9) and (2.10), the proof completed. □

Theorem 2.5: (d'Ocagne's identity)

Let m, n are not a negative integer and $m > n$. Then

$$P_mP_{n+1} - P_{m+1}P_n = \gamma^n P_{m-n}, \quad (2.14)$$

and

$$Q_mQ_{n+1} - Q_{m+1}Q_n = 2\sqrt{2}\gamma^m \left(Q_{n-m} + 2\gamma\beta^{n-m}\right), \quad (2.15)$$

and

$$q_mq_{n+1} - q_{m+1}q_n = \sqrt{2}\gamma^m\left(q_{n-m} - \beta^{n-m}\right). \quad (2.16)$$

Proof. By Binet's formula (2.5), we have

$$\begin{aligned} P_mP_{n+1} - P_{m+1}P_n &= \frac{\alpha^m - \beta^m}{\alpha-\beta}\cdot\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta} \\ &\quad -\frac{\alpha^{m+1} - \beta^{m+1}}{\alpha-\beta}\cdot\frac{\alpha^n - \beta^n}{\alpha-\beta} \\ &= \frac{\alpha^m\beta^n - \alpha^n\beta^m}{\alpha-\beta} \end{aligned}$$

$$= \alpha^n \beta^n \frac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta}$$

$$= \gamma^n P_{m-n}.$$

Similarly, by Binet's formula (2.6), we obtain

$$Q_m Q_{n+1} - Q_{m+1} Q_n = \left(\alpha^m + \beta^m\right) \cdot \left(\alpha^{n+1} + \beta^{n+1}\right) - \left(\alpha^{m+1} + \beta^{m+1}\right) \cdot \left(\alpha^n + \beta^n\right)$$

$$= \left(\alpha - \beta\right) \left(\alpha^n \beta^m - \alpha^m \beta^n\right)$$

$$= \left(\alpha - \beta\right) \alpha^m \beta^m \left(\alpha^{n-m} - \beta^{n-m}\right)$$

$$= 2\sqrt{2}\gamma^m \left(Q_{n-m} + 2\gamma\beta^{n-m}\right).$$

Similarly, by Binet's formula (2.7), we obtain

$$q_m q_{n+1} - q_{m+1} q_n = \frac{\alpha^m + \beta^m}{\alpha + \beta} \cdot \frac{\alpha^{n+1} + \beta^{n+1}}{\alpha + \beta} - \frac{\alpha^{m+1} + \beta^{m+1}}{\alpha + \beta} \cdot \frac{\alpha^n + \beta^n}{\alpha + \beta}$$

$$= \frac{\left(\alpha - \beta\right)\left(\alpha^n \beta^m - \alpha^m \beta^n\right)}{\left(\alpha + \beta\right)^2}$$

$$= \left(\alpha - \beta\right) \alpha^m \beta^m \frac{\left(\alpha^{n-m} - \beta^{n-m}\right)}{\left(\alpha + \beta\right)^2}$$

$$= \sqrt{2}\gamma^m \left(q_{n-m} - \beta^{n-m}\right).$$

The proof completed. □

Lemma 2.6 Let m, n are not a negative integer and $m > n$. Then

$$P_m P_{n+1} - P_{m+1} P_n = \frac{\gamma^n}{2\sqrt{2}} \left(Q_{m-n} + 2\gamma\beta^{m-n}\right) \tag{2.17}$$

and

$$P_m P_{n+1} - P_{m+1} P_n = \frac{\gamma^n}{\sqrt{2}} \left(q_{m-n} + \gamma\beta^{m-n}\right). \tag{2.18}$$

Proof. The Proof same as Theorem 2.5.

Lemma 2.7 Let m, n are not a negative integer and $m > n$. Then

$$Q_m Q_{n+1} - Q_{m+1} Q_n = 8\gamma^m P_{n-m}, \tag{2.19}$$

and

$$Q_m Q_{n+1} - Q_{m+1} Q_n = 4\sqrt{2}\gamma^m \left(q_{n-m} + \gamma\beta^{n-m}\right). \tag{2.20}$$

Proof. The Proof same as Theorem 2.5.

Lemma 2.8 Let m, n are not a negative integer and $m > n$. Then

$$q_m q_{n+1} - q_{m+1} q_n = 2\gamma^m P_{n-m}, \tag{2.21}$$

and

$$q_m q_{n+1} - q_{m+1} q_n = \sqrt{2}\gamma^m \cdot \left(q_{n-m} + \gamma\beta^{n-m}\right) \tag{2.22}$$

Proof. The Proof same as Theorem 2.5.

Theorem 2.9: Let $\{P_n\}, \{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences, m and n are not a negative integer and $m > n$. Then

$$\lim_{n \to \infty} \frac{P_n}{P_{n-1}} = \alpha, \tag{2.23}$$

and

$$\lim_{n \to \infty} \frac{Q_n}{Q_{n-1}} = \alpha, \tag{2.24}$$

and

$$\lim_{n \to \infty} \frac{q_n}{q_{n-1}} = \alpha. \tag{2.25}$$

Proof. By Binet's formula (2.5), we have

$$\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n-\beta^n}{\alpha^{n-1}-\beta^{n-1}}$$

$$=\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}-\frac{1}{\beta}\cdot\left(\frac{\beta}{\alpha}\right)^n}.$$

But $\alpha>\beta$, then $\frac{\beta}{\alpha}<1$ and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0.$

Therefor $\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\alpha$.

Similarly, $\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}}$

$$=\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}=\alpha.$$

Similarly, $\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}}$

$$=\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}=\alpha.$$

This completes the proof. □

Lemma 2.10: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences and n is not a negative integer. Then

$$\lim_{n\to\infty}\frac{P_n}{Q_{n-1}}=\frac{\alpha}{\alpha-\beta}, \tag{2.26}$$

and

$$\lim_{n\to\infty}\frac{P_n}{q_{n-1}}=\frac{\alpha+\beta}{\alpha-\beta}, \tag{2.27}$$

and

$$\lim_{n\to\infty}\frac{Q_n}{q_{n-1}}=\frac{\alpha-\beta}{\alpha}. \tag{2.28}$$

Proof. The Proof same as Theorem 2.9.

In this paper, the generating function for Pell, Pell-Lucas and modified Pell sequences are given as a result, these sequence are seen and the coefficients of the power series of the corresponding generating function.

The generating function for Pell, Pell-Lucas and modified Pell sequences. We can also find the generating function for all three sequences by suppose that the Pell, Pell-Lucas and modified Pell sequences are the coefficients of a potential series center at the origin, and let us consider the corresponding analytic $\{P_n\},\{Q_n\}$ and $\{q_n\}$ of the function, which the function is as follows Theorem.

Theorem 2.11: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences and n is not a negative integer. Then the generating function defined by

$$P_n(x)=\frac{x}{1-2x-x^2}, \tag{2.29}$$

and

$$Q_n(x)=\frac{2-2x}{1-2x-x^2}, \tag{2.30}$$

and

$$q_n(x)=\frac{1-x}{1-2x-x^2}. \tag{2.31}$$

Proof. Let n is a not negative integer and

$$P_n(x)=P_0+P_1x+P_2x^2+P_3x^3+P_4x^4 +\ldots+P_nx^n+\ldots.$$

Then

$$2xP_n(x) = 2P_0x + 2P_1x^2 + 2P_2x^3 + \ldots + 2P_nx^{n+1} + \ldots$$

$$x^2P_n(x) = P_0x^2 + P_1x^3 + P_2x^4 + \ldots + P_nx^{n+2} + \ldots$$

$$P_n(x) - 2xP_n(x) - x^2P_n(x) = x$$

$$(1-2x-x^2)P_n(x) = x.$$

Thus $P_n(x) = \sum_{n=0}^{\infty} P_nx^n = \frac{x}{1-2x-x^2}$.

Similarly, we have

$$Q_n(x) = Q_0 + Q_1x + Q_2x^2 + Q_3x^3 + Q_4x^4 + \ldots + Q_nx^n + \ldots.$$

Then, we obtain

$$2xQ_n(x) = 2Q_0x + 2Q_1x^2 + 2Q_2x^3 + \ldots + 2Q_nx^{n+1} + \ldots$$

$$x^2Q_n(x) = Q_0x^2 + Q_1x^3 + Q_2x^4 + \ldots + Q_nx^{n+2} + \ldots$$

$$Q_n(x) - 2xQ_n(x) - x^2Q_n(x) = 2 - 2x$$

$$(1-2x1-x^2)Q_n(x) = 2-2x.$$

Thus $Q_n(x) = \sum_{n=0}^{\infty} Q_nx^n = \frac{2-2x}{1-2x-x^2}$.

Similarly, we have

$$q_n(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4 + \ldots + q_nx^n + \ldots.$$

Then, we write

$$2xq_n(x) = 2q_0x + 2q_1x^2 + 2q_2x^3 + \ldots + 2q_nx^{n+1} + \ldots$$

$$x^2q_n(x) = q_0x^2 + q_1x^3 + q_2x^4 + \ldots + q_nx^{n+2} + \ldots$$

$$q_n(x) - 2xq_n(x) - x^2q_n(x) = 1 - x$$

$$(1-2x-x^2)q_n(x) = 1-x.$$

Thus $q_n(x) = \sum_{n=0}^{\infty} q_nx^n = \frac{1-x}{1-2x-x^2}$.

This completes the proof. □

From the Theorem 2.11 used to find the generating function. Next will be the polynomial of Pell, Pell–Lucas and Modified Pell sequences from the generating function, which using Maclaurin series helps to find the following theorem.

Theorem 2.12: The equality

$$\frac{x}{1-2x-x^2} = P_0 + P_1x + P_2x^2 + P_3x^3 + P_4x^4 + \ldots, \tag{2.32}$$

and

$$\frac{2-2x}{1-2x-x^2} = Q_0 + Q_1x + Q_2x^2 + Q_3x^3, + Q_4x^4 + \ldots, \tag{2.33}$$

and

$$\frac{1-x}{1-2x-x^2} = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4 + \ldots. \tag{2.34}$$

Proof. Since Maclaurin series, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$. Let $f(x) = \frac{x}{1-2x-x^2}$ for all x, we obtain

$$f'(x) = -\frac{(-2-2x)x}{(1-2x-x^2)^2} + \frac{1}{1-2x-x^2}$$

$$f''(x) = -\frac{2(-2-2x)}{(1-2x-x^2)^2} + \frac{2x(-2-2x)^2}{(1-2x-x^2)^3} + \frac{2x}{(1-2x-x^2)^2}$$

$$f'''(x) = -\frac{6x(-2-2x)^3}{(1-2x-x^2)^4} - \frac{12x(-2-2x)}{(1-2x-x^2)^3}$$

$$+\frac{6(-2-2x)^2}{(1-2x-x^2)^3}$$

$$+\frac{6}{(1-2x-x^2)^2}$$

$$f^{(4)}=\frac{24x(-2-2x)^4}{(1-2x-x^2)^5}+\frac{72x(-2-2x)^2}{(1-2x-x^2)^4}$$

$$+\frac{24x}{(1-2x-x^2)^3}-\frac{24(-2-2x)^3}{(1-2x-x^2)^4}$$

$$-\frac{48(-2-2x)}{(1-2x-x^2)^3}$$

and so on.

Then

$f(0)=0,\ f'(1)=1,\ f''(0)=4,$

$f'''(0)=30,\ f^{(4)}(0)=288$ and so on,

$$\frac{x}{1-2x-x^2}=0+\frac{1}{1!}x+\frac{4}{2!}x^2+\frac{30}{3!}x^3+\frac{288}{4!}x^4+\ldots.$$

Thus $\frac{x}{1-2x-x^2}=P_0+P_1x+P_2x^2+P_3x^3+P_4x^4+\ldots.$

Similarly, $f(x)=\frac{2-2x}{1-2x-x^2}$ for all x, we obtain

$$f'(x)=-\frac{(-2-2x)(2-2x)}{(1-2x-x^2)^2}-\frac{2}{1-2x-x^2}$$

$$f''(x)=\frac{4(-2-2x)}{(1-2x-x^2)^2}$$

$$+\frac{2(2-2x)(-2-2x)^2}{(1-2x-x^2)^3}$$

$$+\frac{2(2-2x)}{(1-2x-x^2)^2}$$

$$f'''(x)=-\frac{6(2-2x)(-2-2x)^3}{(1-2x-x^2)^4}$$

$$-\frac{12(2-2x)(-2-2x)}{(1-2x-x^2)^3}$$

$$-\frac{12(-2-2x)^2}{(1-2x-x^2)^3}-\frac{12}{(1-2x-x^2)^2}$$

$$f^{(4)}=\frac{24(2-2x)(-2-2x)^4}{(1-2x-x^2)^5}$$

$$+\frac{72(2-2x)(-2-2x)^2}{(1-2x-x^2)^4}$$

$$+\frac{24(2-2x)}{(1-2x-x^2)^3}+\frac{48(-2-2x)^3}{(1-2x-x^2)^4}$$

$$+\frac{96(-2-2x)}{(1-2x-x^2)^3},$$

and so on.

Then

$f(0)=2,\ f'(1)=2,\ f''(0)=12,$

$f'''(0)=84,\ f^{(4)}(0)=816$ and so on,

$$\frac{2-2x}{1-2x-x^2}=2+\frac{2}{1!}x+\frac{12}{2!}x^2+\frac{84}{3!}x^3+\frac{816}{4!}x^4+\ldots.$$

Thus

$$\frac{2-2x}{1-2x-x^2}=Q_0+Q_1x+Q_2x^2+Q_3x^3+Q_4x^4+\ldots.$$

Similarly, $f(x)=\dfrac{1-x}{1-2x-x^2}$ for all x, we obtain

$$f'(x)=-\frac{(-2-2x)(1-x)}{(1-2x-x^2)^2}-\frac{1}{1-2x-x^2}$$

$$f''(x)=\frac{2(-2-2x)}{(1-2x-x^2)^2}+\frac{2(1-x)(-2-2x)^2}{(1-2x-x^2)^3}+\frac{2(1-x)}{(1-2x-x^2)^2}$$

$$f'''(x)=-\frac{6(1-x)(-2-2x)^3}{(1-2x-x^2)^4}-\frac{12(1-x)(-2-2x)}{(1-2x-x^2)^3}-\frac{6(-2-2x)^2}{(1-2x-x^2)^3}-\frac{6}{(1-2x-x^2)^2}$$

$$f^{(4)}=\frac{24(1-x)(-2-2x)^4}{(1-2x-x^2)^5}+\frac{72(1-x)(-2-2x)^2}{(1-2x-x^2)^4}+\frac{24(1-x)}{(1-2x-x^2)^3}+\frac{24(-2-2x)^3}{(1-2x-x^2)^4}+\frac{48(-2-2x)}{(1-2x-x^2)^3},$$

and so on.

Then

$f(0)=1,\ f'(1)=1,\ f''(0)=6,$

$f'''(0)=42,\ f^{(4)}(0)=408$ and so on,

$$\frac{1-x}{1-2x-x^2}=1+\frac{1}{1!}x+\frac{6}{2!}x^2+\frac{42}{3!}x^3+\frac{408}{4!}x^4+\ldots.$$

Thus $$\frac{1-x}{1-2x-x^2}=q_0+q_1x+q_2x^2+q_3x^3+q_4x^4+\ldots.$$

This completes the proof. □

Next, we will discuss the power series, including $\sum_{n=0}^{\infty}P_nx^n,\sum_{n=0}^{\infty}Q_nx^n$ and $\sum_{n=0}^{\infty}q_nx^n$ in $x-x_0$ but $x_0=0$ converges is always an interval center at $x=0$. We test convergence of such series by complete convergence and series converges if $|x|<\dfrac{1}{\alpha}$ and series diverges if $|x|>\dfrac{1}{\alpha}$. The series convergences absolutely open interval $\left(-\dfrac{1}{\alpha},\dfrac{1}{\alpha}\right)$, making the convergence radius equal $\dfrac{1}{\alpha}$, as the following Theorem.

Theorem 2.13: Let $\sum_{n=0}^{\infty}P_nx^n,\sum_{n=0}^{\infty}Q_nx^n$ and $\sum_{n=0}^{\infty}q_nx^n$ are power series. Then interval of convergence for the given series is $\left(-\dfrac{1}{\alpha},\dfrac{1}{\alpha}\right)$ and the radius of convergence is $\dfrac{1}{\alpha}$.

Proof. By absolute convergence and Theorem 2.9, we have

$$\lim_{n\to\infty}\left|\frac{P_nx^n}{P_{n-1}x^{n-1}}\right|=\lim_{n\to\infty}|\alpha x|=\alpha|x|.$$

So the $\sum_{n=0}^{\infty}P_nx^n$ converges absolute if

$\lim_{n\to\infty}\left|\frac{P_n x^n}{P_{n-1}x^{n-1}}\right|<1$, then $\alpha|x|<1$ and $|x|<\frac{1}{\alpha}$.

The test value $x=\frac{1}{\alpha}$ or $x=-\frac{1}{\alpha}$, when replaced in series, will be

$$\sum_{n=0}^{\infty}P_n\left(\frac{1}{\alpha}\right)^n=P_0+\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}+\dots \text{ and}$$

$$\sum_{n=0}^{\infty}P_n\left(-\frac{1}{\alpha}\right)^n=\sum_{n=0}^{\infty}(-1)^n P_n\left(\frac{1}{\alpha}\right)^n$$
$$=P_0-\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}-\dots.$$

So both of diverge, therefore the interval of convergence for the given power series is $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence is $\frac{1}{\alpha}$. The $\sum_{n=0}^{\infty}Q_n x^n$ and $\sum_{n=0}^{\infty}q_n x^n$ will prove similarly, then the interval of convergence for the given power series are $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence are $\frac{1}{\alpha}$, this completes the proof. □

From the theorem 2.13, found that the polynomial of Pell, Pell–Lucas and Modified Pell sequences are convergence to $\frac{1}{\alpha}$ and there is scope for convergence during the opening period $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$.

Lemma 2.14: The equality

$$P_n(x)=P_0+P_1x+P_2x^2+P_3x^3+P_4x^4+\dots, \tag{2.35}$$

and

$$Q_n(x)=Q_0+Q_1x+Q_2x^2+Q_3x^3+Q_4x^4+\dots, \tag{2.36}$$

and

$$q_n(x)=q_0+q_1x+q_2x^2+q_3x^3+q_4x^4+\dots. \tag{2.37}$$

for all $x\in\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$.

Then $P_n(0)=P_0$, $Q_n(0)=Q_0$ and $q_n(0)=q_0$.

Proof. Taking $x=0$ in (2.35), (2.36) and (2.37). The proof completed. □

Furthermore, how to find $P_n(0)$, $Q_n(0)$ and $q_n(0)$. We also use the equation (2.29), (2.30) and (2.31) in the Theorem 2.11 by giving x = 0, then $P_n(0)=P_0$, $Q_n(0)=Q_0$ and $q_n(0)=q_0$, respectively.

3. Conclusion

In this article, first of all, we consider the generality of Pell, Pell-Lucas and modified Pell sequence by the result of the previous three terms. Then we introduced the Pell, Pell-Lucas and modified Pell number. Until finally, we got the Binet's formula and the generating function of

Pell, Pell-Lucas and the modified Pell sequence. In addition, we also received the information some important identities involving the terms of these sequences.

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Generalized Identities Related for the Fibonacci Number, Lucas Number, Fibonacci-Like Number and Generalized Fibonacci-Like Number By Matrix Method

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Abstract

In this paper, we present generalized identities for Fibonacci, Lucas, Fibonacci-Like and Generalized Fibonacci-Like sequence. We obtain some identity relations by using the matrix method and Binet's formula.

Keywords: Fibonacci number, Lucas number, Fibonacci-Like number, Generalized Fibonacci-Like number Binet's formula, Matrix method

1. Introduction

The Fibonacci sequence is a very important research in the study and research in the past decade. This sequence can be applied to engineering business as well as to science.

Researcher study about the generalized Fibonacci sequence by changing the initials conditions $F_0 = a$ and $F_1 = b$. Moreover the coefficients p and q of the recurrence sequence are changing by $F_n = pF_{n-1} + qF_{n-2}$ for $n \geq 2$ (see [1]-[15]).

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In addition to studying the Fibonacci numbers from the recurrence sequence, we also want to study the Fibonacci numbers using matrix operations to find some properties (see [16]-[18]).

Sequence can be used with many aspects and this direction is very interesting. For this reason, researchers are motivated to study of Fibonacci, Lucas, and Fibonacci-Like sequences. In addition, the sequence can be proved using the matrix method and Binet's formula. In this paper, we use matrix method to show some identity.

2. Preliminaries

In this section, we will introduce the previous article, which is well-known for use in our research.

The Fibonacci sequence $\{F_n\}$ [3] is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 3 \tag{2.1}$$

with initial conditions $F_1 = F_2 = 1$. The first few terms of the sequence $\{F_n\}$ are $1, 1, 2, 3, 5, 8, 13, 21$ and so on.

Binet's formula allows us to show the Fibonacci numbers in the function of the root $R_1 \& R_2$ from the recurrence relation (2.1), which is related to the following characteristic equations

$$x^2 - x - 1 = 0, \tag{2.2}$$

and $R_1 = \dfrac{1+\sqrt{5}}{2}$, $R_2 = \dfrac{1-\sqrt{5}}{2}$ so that $R_1 + R_2 = 1$, $R_1^2 - 1 = R_1$, $R_2^2 - 1 = R_2$, $R_1 R_2 = -1$.

Thus the Binet's formula of Fibonacci numbers is given by

$$F_n = \frac{R_1^n - R_2^n}{R_1 - R_2}, \tag{2.3}$$

where $R_1 \& R_2$ are the root of the characteristic equation and $R_1 > R_2$.

The Lucas sequence $\{L_n\}$ [1] is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2 \tag{2.4}$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. The first few terms of the sequence $\{L_n\}$ are $2, 1, 3, 4, 7, 11, 18, 29$ and so on.

The Binet's formula allows us to express the Lucas numbers in function of the roots $R_1 \& R_2$ of the following characteristic equation as in (2.2).

Thus the Binet's formula of Lucas numbers is given by

$$L_n = R_1^n + R_2^n, \tag{2.5}$$

where $R_1 \& R_2$ are the root of the characteristic equation and $R_1 > R_2$.

The Generalized Fibonacci-Like sequence $\{T_n\}$ [4] is defined by the recurrence relation

$$T_n = T_{n-1} + T_{n-2}, \text{ for } n \geq 2. \tag{2.6}$$

with initial conditions $T_0 = m$ and $T_1 = m$, where m is positive integer. The first few terms of the sequence $\{T_n\}$ are $m, m, 2m, 3m, 5m, 8m, 13m, 21m$ and so on.

The Binet's formula allows us to express the Generalized Fibonacci-Like numbers in function of the roots R_1 & R_2 of the following characteristic equation as in (2.2).

Thus the Binet's formula of Fibonacci-Like numbers is

given by

$$T_n = m\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \qquad (2.7)$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

Particular cases of (2.6), if $m = 2$ then we call the Fibonacci-Like sequence $\{S_n\}$ [12] is defined by

$$S_n = S_{n-1} + S_{n-2}, \text{ for } n \geq 2 \qquad (2.8)$$

with initial conditions $S_0 = 2$ and $S_1 = 2$. The first few terms of the sequence $\{S_n\}$ are $2, 2, 4, 6, 10, 16, 26$ and so on.

The Binet's formula allows us to express the Fibonacci-Like numbers in function of the roots R_1 & R_2 of the following characteristic equation as in (2.2).

Thus the Binet's formula of Fibonacci-Like sequence is given by

$$S_n = 2\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \qquad (2.9)$$

where R_1 & R_2 are the root of the characteristic equation and $R_1 > R_2$.

In 1960, Charles H. King [22] studied on the following Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

He showed that $\det(Q) = -1$ and

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \text{ for } n \geq 1$$

Moreover, it is clearly shown below that $\det(Q^n) = (-1)^n$ then

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

3. Main Results

In this section, we establish some identity relations of the Fibonacci sequence $\{F_n\}$, the Lucas sequence $\{L_n\}$, the Fibonacci-Like sequence $\{S_n\}$, the Generalized Fibonacci-Like sequence $\{T_n\}$ by using matrix methods. We begin with the following Lemma.

Lemma 3.1: $T_{2n+1} = 2T_{2n-1} + T_{2n-2}$, (3.1)

where n is positive integer.

Proof. By Binet's formula (2.7), we have

$$\begin{aligned} T_{2n+1} &= m\frac{R_1^{2n+2} - R_2^{2n+2}}{R_1 - R_2} \\ &= m\frac{R_1^{2n}R_1^2 - R_2^{2n}R_2^2}{R_1 - R_2} \\ &= m\frac{R_1^{2n}\left(2 + R_1^{-1}\right) - R_2^{2n}\left(2 + R_2^{-1}\right)}{R_1 - R_2} \\ &= m\frac{2R_1^{2n} + R_1^{2n-1} - 2R_2^{2n} - R_2^{2n-1}}{R_1 - R_2} \\ &= m\frac{2R_1^{2n} - 2R_2^{2n} + R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\ &= 2m\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} + m\frac{R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\ &= 2T_{2n-1} + T_{2n-2}. \end{aligned}$$

Thus, this completes the Proof. $\square$

Theorem 3.2: Let $Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$. Then

$$Q_T^n = m^{n-1}\begin{pmatrix} T_{2n} & T_{2n-1} \\ T_{2n-1} & T_{2n-2} \end{pmatrix}, \tag{3.2}$$

where m and n are positive integer.

Proof. We proof that $Q_T^n = m^{n-1}\begin{pmatrix} T_{2n} & T_{2n-1} \\ T_{2n-1} & T_{2n-2} \end{pmatrix}$ for every $n \in N$, by using the Principle of Mathematical Induction on n. Because

$$Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix} = \begin{pmatrix} T_2 & T_1 \\ T_1 & T_0 \end{pmatrix}.$$

Thus $n = 1$ is true. We assume the result is true for a positive integer $n = k$, then

$$Q_T^k = m^{k-1}\begin{pmatrix} T_{2k} & T_{2k-1} \\ T_{2k-1} & T_{2k-2} \end{pmatrix}.$$

Since Lemma 3.1, we consider the positive integer $n = k+1$. Then

$$\begin{aligned} Q_T^{k+1} &= Q_T^k Q_T \\ &= m^{k-1}\begin{pmatrix} T_{2k} & T_{2k-1} \\ T_{2k-1} & T_{2k-2} \end{pmatrix}\begin{pmatrix} 2m & m \\ m & m \end{pmatrix} \\ &= m^{k-1}\begin{pmatrix} 2mT_{2k} + mT_{2k-1} & mT_{2k} + mT_{2k-1} \\ 2mT_{2k-1} + mT_{2k-2} & mT_{2k-1} + mT_{2k-2} \end{pmatrix} \\ &= m^{k}\begin{pmatrix} 2T_{2k} + T_{2k-1} & T_{2k} + T_{2k-1} \\ 2T_{2k-1} + T_{2k-2} & T_{2k-1} + T_{2k-2} \end{pmatrix} \\ &= m^{k}\begin{pmatrix} T_{2k+2} & T_{2k+1} \\ T_{2k+1} & T_{2k} \end{pmatrix}. \end{aligned}$$

Thus $n = k+1$ is true, this completes the Proof. □

Corollary 3.3: Let $Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$. Then

$$Q_T^n = \frac{m^{n-1}}{2}\begin{pmatrix} S_{2n} & S_{2n-1} \\ S_{2n-1} & S_{2n-2} \end{pmatrix}, \tag{3.3}$$

where m and n are positive integer.

Corollary 3.4: Let $Q_T = \begin{pmatrix} 2m & m \\ m & m \end{pmatrix}$. Then

$$Q_T^n = m^{n-1}\begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}, \tag{3.4}$$

where m and n are positive integer.

Corollary 3.5: Let $Q_L = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Then

$$Q_L^n = \begin{cases} \dfrac{5^{\frac{n}{2}}}{m}\begin{pmatrix} T_n & T_{n-1} \\ T_{n-1} & T_{n-2} \end{pmatrix}, & \text{for } n \text{ is even}, \\ 5^{\frac{n-1}{2}}\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}, & \text{for } n \text{ is odd}, \end{cases} \tag{3.5}$$

where m and n are positive integer.

From Corollary 3.5, so that if $Q_L = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ then

$$Q_L^n = \begin{cases} \dfrac{5^{\frac{n}{2}}}{2}\begin{pmatrix} S_n & S_{n-1} \\ S_{n-1} & S_{n-2} \end{pmatrix}, & \text{for } n \text{ is even}, \\ 5^{\frac{n-1}{2}}\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}, & \text{for } n \text{ is odd}, \end{cases}$$

where m and n are positive integer.

Theorem 3.6: For a positive integer n, following equalities hold:

i) $\det\left(Q_T^n\right) = m^{2n}$,

ii) $T_{2n}T_{2n-2} - T_{2n-1}^2 = m^2$. (3.6)

Proof. By $\det\left(Q_T\right) = m^2$. Thus $\det\left(Q_T^n\right) = \left(\det\left(Q_T\right)\right)^n = \left(m^2\right)^n = m^{2n}$, and the determinant for Q_T^n in (3.2) will be ii), we get $m^{2n-2}\left(T_{2n}T_{2n-2} - T_{2n-1}^2\right) = m^{2n}$, thus $T_{2n}T_{2n-2} - T_{2n-1}^2 = m^2$. □

Corollary 3.7: For a positive integer n, following equalities hold:

i) $\det\left(Q_T^n\right)=m^{2n}$,

ii) $S_{2n}S_{2n-2}-S_{2n-1}^2=4.$ (3.7)

Corollary 3.8: For a positive integer n, following equalities hold:

i) $\det\left(Q_T^n\right)=m^{2n}$,

ii) $F_{2n+1}F_{2n-1}-F_{2n}^2=1.$ (3.8)

Corollary 3.9: For a positive integer n, following equalities hold:

i) $\det\left(Q_L^n\right)=5^n$,

ii) $T_nT_{n-2}-T_{n-1}^2=m^2$,

iii) $S_nS_{n-2}-S_{n-1}^2=4$,

iv) $L_{n+1}L_{n-1}-L_n^2=5.$ (3.9)

Theorem 3.10: Let n is positive integer. Then the Binet's formula of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$T_n=m\frac{R_1^{n+1}-R_2^{n+1}}{R_1-R_2}, \tag{3.10}$$

where $R_1=\frac{1+\sqrt{5}}{2}$ and $R_2=\frac{1-\sqrt{5}}{2}$.

Proof. Let Q_T is matrix in Theorem 3.2, $\lambda_1=\frac{3m+\sqrt{5}m}{2}$ and $\lambda_2=\frac{3m-\sqrt{5}m}{2}$ are the eigenvalues of matrix Q_T, $v_1=\begin{pmatrix}R_1 & 1\end{pmatrix}$ and $v_2=\begin{pmatrix}R_2 & 1\end{pmatrix}$ are eigenvectors that is correspond to a eigenvalues. Then we find diagonalizable of matrix Q_T by

$$D=P^{-1}Q_TP=\begin{pmatrix}\frac{3m+\sqrt{5}m}{2} & 0\\ 0 & \frac{3m-\sqrt{5}m}{2}\end{pmatrix},$$

where

$$P=\left(v_1^T,v_2^T\right)=\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}=\begin{pmatrix}R_1 & R_2\\ 1 & 1\end{pmatrix}$$

and

$$D=\operatorname{diag}\left(\lambda_1,\lambda_2\right)=\begin{pmatrix}\frac{3m+\sqrt{5}m}{2} & 0\\ 0 & \frac{3m-\sqrt{5}m}{2}\end{pmatrix}.$$

Thus $Q_T=PDP^{-1}=\begin{pmatrix}2m & m\\ m & m\end{pmatrix}$.

Since properties of the Diagonal Matrix, we obtain

$$Q_T^n=PD^nP^{-1}=\begin{pmatrix}m^n\frac{R_1^{2n+1}-R_2^{2n+1}}{R_1-R_2} & m^n\frac{R_1^{2n}-R_2^{2n}}{R_1-R_2}\\ m^n\frac{R_1^{2n}-R_2^{2n}}{R_1-R_2} & m^n\frac{R_1^{2n-1}-R_2^{2n-1}}{R_1-R_2}\end{pmatrix},$$

where n is positive integer. We get

$$m^{n-1}\begin{pmatrix}T_{2n} & T_{2n-1}\\ T_{2n-1} & T_{2n-2}\end{pmatrix}=m^{n-1}\begin{pmatrix}m\frac{R_1^{2n+1}-R_2^{2n+1}}{R_1-R_2} & m\frac{R_1^{2n}-R_2^{2n}}{R_1-R_2}\\ m\frac{R_1^{2n}-R_2^{2n}}{R_1-R_2} & m\frac{R_1^{2n-1}-R_2^{2n-1}}{R_1-R_2}\end{pmatrix}.$$

This completes the Proof. □

Corollary 3.11: Let m and n are positive integer. Then the Binet's formula of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$T_n=\frac{m}{2}S_n=m\frac{R_1^{n+1}-R_2^{n+1}}{R_1-R_2}, \tag{3.11}$$

where $R_1=\frac{1+\sqrt{5}}{2}$ and $R_2=\frac{1-\sqrt{5}}{2}$.

Corollary 3.12: Let m and n are positive integer. Then the Binet's formula of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$T_n = mF_{n+1} = m\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2}, \qquad (3.12)$$

where $R_1 = \frac{1+\sqrt{5}}{2}$ and $R_2 = \frac{1-\sqrt{5}}{2}$.

Lemma 3.13: $T_{2n} + T_{2n-2} = mL_{2n}$, (3.13)

where m and n are positive integer.

Proof. By Binet's formula (2.5) and (2.7), we have

$$\begin{aligned}
&T_{2n} + T_{2n-2} \\
&= m\frac{R_1^{2n+1} - R_2^{2n+1}}{R_1 - R_2} + m\frac{R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\
&= m\frac{R_1^{2n+1} - R_2^{2n+1} + R_1^{2n-1} - R_2^{2n-1}}{R_1 - R_2} \\
&= m\frac{R_1^{2n+1} + R_1^{2n-1} - R_2^{2n+1} - R_2^{2n-1}}{R_1 - R_2} \\
&= m\frac{R_1^{2n}\left(R_1 + R_1^{-1}\right) - R_2^{2n}\left(R_2 + R_2^{-1}\right)}{R_1 - R_2} \\
&= m\sqrt{5}\frac{R_1^{2n} + R_2^{2n}}{R_1 - R_2} \\
&= m\left(R_1^{2n} + R_2^{2n}\right) \\
&= mL_{2n}.
\end{aligned}$$

Thus, this completes the Proof. □

Lemma 3.14: $L_{2n}^2 - 4 = 5F_{2n}^2$, (3.14)

where m and n are positive integer.

Proof. By Binet's formula (2.3) and (2.5), we have

$$\begin{aligned}
&L_{2n}^2 - 4 \\
&= \left(R_1^{2n} + R_2^{2n}\right)^2 - 4 \\
&= R_1^{4n} + 2R_1^{2n}R_2^{2n} + R_2^{4n} - 4 \\
&= R_1^{4n} + 2\left(R_1R_2\right)^{2n} + R_2^{4n} - 4 \\
&= R_1^{4n} + 2 + R_2^{4n} - 4 \\
&= R_1^{4n} - 2 + R_2^{4n} \\
&= R_1^{4n} - 2R_1^{2n}R_2^{2n} + R_2^{4n} \\
&= \left(R_1^{2n} - R_2^{2n}\right)^2 \\
&= \frac{5\left(R_1^{2n} - R_2^{2n}\right)^2}{\left(R_1 - R_2\right)^2} \\
&= 5F_{2n}^2.
\end{aligned}$$

Thus, this completes the Proof. □

Theorem 3.15: The generalized two roots of Q_T^n are

$$\lambda_1 = m\frac{\left(L_{2n} + \sqrt{5}F_{2n}\right)}{2} \text{ and}$$

$$\lambda_2 = m\frac{\left(L_{2n} - \sqrt{5}F_{2n}\right)}{2}. \qquad (3.15)$$

Where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $F_{2n} = \frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Proof. The characteristic equation of Q_T^n. By Lemma 3.13 and Theorem 3.6, we get

$$\begin{aligned}
\det\left(Q_T^n - \lambda I\right) &= \begin{vmatrix} m^{n-1}T_{2n} - \lambda & m^{n-1}T_{2n-1} \\ m^{n-1}T_{2n-1} & m^{n-1}T_{2n-2} - \lambda \end{vmatrix} \\
&= m^{2n-2}\left(T_{2n} - \lambda\right)\left(T_{2n-2} - \lambda\right) - m^{2n-2}T_{2n-1}^2 \\
&= m^{2n-2}\lambda^2 - m^{2n-2}\left(T_{2n} + T_{2n-2}\right)\lambda \\
&\quad - m^{2n-2}\left(T_{2n}T_{2n-2} - T_{2n-1}^2\right) \\
&= m^{2n-2}\lambda^2 - m^{2n-2}mL_{2n}\lambda - m^{2n-2}m^2 \\
&= m^{2n-2}\lambda^2 - m^{2n-1}L_{2n}\lambda - m^{2n}.
\end{aligned}$$

Thus, the characteristic equation of Q_T^n is

$$\lambda^2 - mL_{2n}\lambda - m^2 = 0,$$

and we get the generalized characteristic roots as following:

$$\lambda_1, \lambda_1 = \frac{mL_{2n} \pm m\sqrt{L_{2n}^2 - 4}}{2}.$$

By Lemma 3.14, it can be writing

$$\lambda_1, \lambda_2 = m\frac{\left(L_{2n} \pm \sqrt{5}F_{2n}\right)}{2}.$$

Therefore,

$R_1^{2n} = \frac{L_{2n} + \sqrt{5}F_{2n}}{2}$ and $R_2^{2n} = \frac{L_{2n} - \sqrt{5}F_{2n}}{2}$.

Thus, we give the Binet's formula by matrix method for the Fibonacci numbers and Lucas numbers given in (2.3) and (2.5) by

$$F_{2n} = \frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2} \text{ and } L_{2n} = R_1^{2n} + R_2^{2n}.$$

This completes the Proof. □

Corollary 3.16: The generalized two roots of Q_T^n are

$$\lambda_1 = \frac{mL_{2n} + \sqrt{\left(\sqrt{5}T_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}$$

and

$$\lambda_2 = \frac{mL_{2n} - \sqrt{\left(\sqrt{5}T_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}, \tag{3.16}$$

where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $T_{2n-1} = m\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Corollary 3.17: The generalized two roots of Q_T^n are

$$\lambda_1 = \frac{mL_{2n} + \sqrt{\left(\frac{\sqrt{5}m}{2}S_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}$$

and

$$\lambda_2 = \frac{mL_{2n} - \sqrt{\left(\frac{\sqrt{5}m}{2}S_{2n-1} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}, \tag{3.17}$$

where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $S_{2n-1} = 2\frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Corollary 3.18: The generalized two roots of Q_T^n are

$$\lambda_1 = \frac{mL_{2n} + \sqrt{\left(\sqrt{5}mF_{2n} + 2mR_2^{2n}\right)^2 + 4m^2}}{2}$$

and

$$\lambda_2 = \frac{mL_{2n} - \sqrt{\left(\sqrt{5}mF_{2n} + 2mR_2^{2n}\right)^2 + 4m^2}}{2},$$

(3.18)

where λ_1 and λ_2 are roots of Q_T^n. Then $L_{2n} = R_1^{2n} + R_2^{2n}$ and $F_{2n} = \frac{R_1^{2n} - R_2^{2n}}{R_1 - R_2}$.

Lemma 3.19: $\lim_{n\to\infty}\frac{T_{2n-1}}{T_{2n-2}} = R_1$. (3.19)

Proof. By Binet's formula (2.7), we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{T_{2n-1}}{T_{2n-2}} &= \lim_{n\to\infty}\frac{R_1^{2n} - R_2^{2n}}{R_1^{2n-1} - R_2^{2n-1}} \\
&= \lim_{n\to\infty}\frac{1-\left(\frac{R_2}{R_1}\right)^{2n}}{\frac{1}{R_1} - \left(\frac{R_2}{R_1}\right)^{2n}\frac{1}{R_2}} \\
&= \frac{\lim_{n\to\infty}1 - \lim_{n\to\infty}\left(\frac{R_2}{R_1}\right)^{2n}}{\lim_{n\to\infty}\frac{1}{R_1} - \lim_{n\to\infty}\left(\left(\frac{R_2}{R_1}\right)^{2n}\frac{1}{R_2}\right)} \\
&= \frac{\lim_{n\to\infty}1 - \lim_{n\to\infty}\left(\frac{R_2}{R_1}\right)^{2n}}{\lim_{n\to\infty}\frac{1}{R_1} - \left(\lim_{n\to\infty}\left(\frac{R_2}{R_1}\right)^{2n}\lim_{n\to\infty}\frac{1}{R_2}\right)} \\
&= R_1.
\end{aligned}$$

Thus, this completes the proof. □

Theorem 3.20: Let

$$\frac{Q_T^n}{m^{n-1}T_{2n-2}}=\begin{pmatrix}\frac{T_{2n}}{T_{2n-2}} & \frac{T_{2n-1}}{T_{2n-2}}\\ \frac{T_{2n-1}}{T_{2n-2}} & 1\end{pmatrix}.$$ Then the

determinant of $\lim_{n\to\infty}\frac{Q_T^n}{m^{n-1}T_{2n-2}}$ as follows characteristic equations of the Generalized Fibonacci-Like sequence $\{T_n\}$ is given by

$$R_1^2-R_1-1=0. \tag{3.20}$$

Proof. Since the matrix Q_T^n in (3.2), we can write

$$\frac{Q_T^n}{m^{n-1}T_{2n-2}}=\begin{pmatrix}\frac{T_{2n}}{T_{2n-2}} & \frac{T_{2n-1}}{T_{2n-2}}\\ \frac{T_{2n-1}}{T_{2n-2}} & 1\end{pmatrix},$$

and by Lemma 3.19 , thus

$$\lim_{n\to\infty}\frac{Q_T^n}{m^{n-1}T_{2n-2}}=\begin{pmatrix}\lim_{n\to\infty}\frac{T_{2n}}{T_{2n-2}} & \lim_{n\to\infty}\frac{T_{2n-1}}{T_{2n-2}}\\ \lim_{n\to\infty}\frac{T_{2n-1}}{T_{2n-2}} & \lim_{n\to\infty}1\end{pmatrix}=\begin{pmatrix}R_1^2 & R_1\\ R_1 & 1\end{pmatrix}.$$

But $R_1^2=R_1+1$, thus

$$\lim_{n\to\infty}\frac{Q_T^n}{m^{n-1}T_{2n-2}}=\begin{pmatrix}R_1+1 & R_1\\ R_1 & 1\end{pmatrix}.$$

Therefore

$$\det\left(\lim_{n\to\infty}\frac{Q_T^n}{m^{n-1}T_{2n-2}}\right)=\begin{vmatrix}R_1+1 & R_1\\ R_1 & 1\end{vmatrix}$$
$$=R_1^2-R_1-1,$$

and thus $R_1^2-R_1-1=0$. This completes the Proof. □

Corollary 3.21: Let

$$\frac{2Q_T^n}{m^nS_{2n-2}}=\begin{pmatrix}\frac{S_{2n}}{S_{2n-2}} & \frac{S_{2n-1}}{S_{2n-2}}\\ \frac{S_{2n-1}}{S_{2n-2}} & 1\end{pmatrix}.$$ Then the

determinant of $\lim_{n\to\infty}\frac{2Q_T^n}{m^nS_{2n-2}}$ as follows characteristic equations of the Fibonacci-Like sequence $\{S_n\}$ is given by

$$R_1^2-R_1-1=0.$$

Corollary 3.22: Let $\frac{Q_T^n}{m^nF_{2n-1}}=\begin{pmatrix}\frac{F_{2n+1}}{F_{2n-1}} & \frac{F_{2n}}{F_{2n-1}}\\ \frac{F_{2n}}{F_{2n-1}} & 1\end{pmatrix}$.

Then the determinant of $\lim_{n\to\infty}\frac{Q_T^n}{m^nF_{2n-1}}$ as follows characteristic equations of Fibonacci sequence $\{F_n\}$ is given by

$$R_1^2-R_1-1=0.$$

Theorem 3.23: Let n and k are positive integer. Then the following relation between $\{S_n\}$ and $\{T_n\}$ is given by

$$mS_{2n+2k}=T_{2k+1}S_{2n-1}+T_{2k}S_{2n-2}. \tag{3.21}$$

Proof. By relation between Fibonacci-Like sequence $\{S_n\}$ and Fibonacci-Like sequence $\{T_n\}$

$$m\begin{pmatrix}S_{2n+1}\\ S_{2n}\end{pmatrix}=Q_T\begin{pmatrix}S_{2n-1}\\ S_{2n-2}\end{pmatrix}.$$

And we multiply with Q_T^k, we get

$$mQ_T^k\begin{pmatrix}S_{2n+1}\\ S_{2n}\end{pmatrix}=Q_T^{k+1}\begin{pmatrix}S_{2n-1}\\ S_{2n-2}\end{pmatrix}.$$

Thus

$$m\begin{pmatrix}S_{2n+2k+1}\\ S_{2n+2k}\end{pmatrix}=\begin{pmatrix}T_{2k+2}S_{2n-1}+T_{2k+1}S_{2n-2}\\ T_{2k+1}S_{2n-1}+T_{2k}S_{2n-2}\end{pmatrix}.$$

This completes the Proof. □

Corollary 3.24: Let n and k are positive integer. Then the following relation between $\{F_n\}$ and $\{T_n\}$ is given by

$$mF_{2n+2k+1} = T_{2k-1}F_{2n+2} + T_{2k-2}F_{2n+1}. \quad (3.22)$$

Theorem 3.25: Let n and r are positive integers and $n \geq r$. Then the following equalities are hold:

i) $mT_{2n+2r} = T_{2n}T_{2r} + T_{2n-1}T_{2r-1}$,

ii) $mT_{4n} = T_{2n}^2 + T_{2n-1}^2$,

iii) $mT_{4n+1} = T_{2n}\left(T_{2n+1} + T_{2n-1}\right)$,

iv) $m\, T_{2n-2r} = T_{2n}T_{2r-2} - T_{2n-1}T_{2r-1}$.

(3.23)

Proof. Let the Q_T^n-matrix in (3.2). By $Q_T^{n+r} = Q_T^n Q_T^r$, we get

$$m^{n+r-1}\begin{pmatrix} T_{2n+2r} & T_{2n+2r-1} \\ T_{2n+2r-1} & T_{2n+2r-2} \end{pmatrix}$$
$$= m^{n+r-2}\begin{pmatrix} T_{2n}T_{2r} + T_{2n-1}T_{2r-1} & T_{2n}T_{2r-1} + T_{2n-1}T_{2r-2} \\ T_{2n-1}T_{2r} + T_{2n-2}T_{2r-1} & T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r-2} \end{pmatrix}$$

Thus

$$m\begin{pmatrix} T_{2n+2r} & T_{2n+2r-1} \\ T_{2n+2r-1} & T_{2n+2r-2} \end{pmatrix}$$
$$= \begin{pmatrix} T_{2n}T_{2r} + T_{2n-1}T_{2r-1} & T_{2n}T_{2r-1} + T_{2n-1}T_{2r-2} \\ T_{2n-1}T_{2r} + T_{2n-2}T_{2r-1} & T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r-2} \end{pmatrix}$$

Therefore, equalities i), ii), and iii). And we get

$$Q_T^{-r} = m^{-r-1}\begin{pmatrix} T_{2r-2} & -T_{2r-1} \\ -T_{2r-1} & T_{2r} \end{pmatrix}.$$

Since $Q_T^{n-r} = Q_T^n Q_T^{-r}$. We have

$$m^{n-r-1}\begin{pmatrix} T_{2n-2r} & T_{2n-2r-1} \\ T_{2n-2r-1} & T_{2n-2r-2} \end{pmatrix}$$
$$= m^{n-r-2}\begin{pmatrix} T_{2n}T_{2r-2} - T_{2n-1}T_{2r-1} & -T_{2n}T_{2r-1} + T_{2n-1}T_{2r} \\ T_{2n-1}T_{2r-2} - T_{2n-2}T_{2r-1} & -T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r} \end{pmatrix}$$

Thus

$$m\begin{pmatrix} T_{2n-2r} & T_{2n-2r-1} \\ T_{2n-2r-1} & T_{2n-2r-2} \end{pmatrix}$$
$$= \begin{pmatrix} T_{2n}T_{2r-2} - T_{2n-1}T_{2r-1} & -T_{2n}T_{2r-1} + T_{2n-1}T_{2r} \\ T_{2n-1}T_{2r-2} - T_{2n-2}T_{2r-1} & -T_{2n-1}T_{2r-1} + T_{2n-2}T_{2r} \end{pmatrix},$$

and iv) immediately seen. This completes the Proof. □

Corollary 3.26: Let n and r are positive integers and $n \geq r$. Then the following equalities are hold:

i) $2T_{2n+2r} = T_{2n}S_{2r} + T_{2n-1}S_{2r-1}$,

ii) $2T_{4n} = T_{2n}S_{2n} + T_{2n-1}S_{2n-1}$,

iii) $2T_{4n+2} = T_{2n}S_{2n+2} + T_{2n-1}S_{2n+1}$,

iv) $2T_{2n-2r} = T_{2n}S_{2r-2} - T_{2n-1}S_{2r-1}$. (3.24)

Corollary 3.27: Let n and r are positive integers and $n \geq r$. Then the following equalities are hold:

i) $T_{2n+2r} = T_{2n}F_{2r+1} + T_{2n-1}F_{2r}$,

ii) $T_{4n} = T_{2n}F_{2n+1} + T_{2n-1}F_{2n}$,

iii) $T_{4n+2} = T_{2n}F_{2n+3} + T_{2n-1}F_{2n+2}$,

iv) $T_{2n-2r} = T_{2n}F_{2r-1} - T_{2n-1}F_{2r}$. (3.25)

4. Conclusions

In this paper, the properties of number are proved by Binet's formula and matrix representation. We obtain some properties and related some identities for Fibonacci sequence $\{F_n\}$, Lucas sequence $\{L_n\}$, Fibonacci-Like sequence $\{S_n\}$ and Generalized Fibonacci-Like sequence $\{T_n\}$.

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