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THE METHOD FOR SOLVING FIXED POINT PROBLEM OF G-NONEXPANSIVE MAPPING IN HILBERT SPACES ENDOWED WITH GRAPHS AND NUMERICAL EXAMPLE

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The main aim of this paper is to study a strong convergence theorem of viscosity approximation method for G-nonexpansive mapping defined on a Hilbert space endowed with a directed graph. By using our main result, we give a numerical expample to approximate the value of π.

Key words : G-nonexpansive mappings; viscosity approximation; edge-preserving.

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1. Introduction

The fixed point theory plays an important role in nonlinear functional analysis and is a very useful tool in various fields. In particular, fixed point theorem has been applied in many branches of sciences. For a recent trend of fixed point problem, one of the most interesting problems is the combination of fixed point theory and graph theory. In the past few years, many researchers have studied fixed point theorems in a metric space endowed with a graphs; see [1-4] and references cited therein.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be contraction if there is $0 < k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

The set of all fixed points of a mapping T is denoted by $F(T)$, i.e., $x \in F(T)$ if and only if $x = Tx$.

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ be a set of its edges, assume that G has no parallel edges, we denote G^{-1} as the directed graph obtained from G by reversing the direction of edges. That is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=1}^{n}$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a (directed) path between any two vertices of G.

For studying contractive-type mappings, the Banach contraction mapping principle, which was firstly introduced by Banach [5] in 1922, has been an important source for solving existence problems in fixed point theory. Some of the contractive-type mapping were studied in many directions, see [6, 7]. In 2008, Jachymski [8] combined the concept of fixed point theory and graph theory in a metric space to generalized Banach contraction mapping principle in a metric space endowed with a directed graph. He also introduced a contractive-type mapping with a directed graph as follows.

Definition 1.1 — [8]. Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$.

We say that a mapping $f : X \to X$ is a G-contraction if f preserves edges of G, *i.e.*,

$$x, y \in X, (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)$$

and there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

In the past few years, many authors have studied a concept of G-contraction in order to improve and extend the above definition, see for instance [9-12] and references cited therein. Let C be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \to C$, then T is said to be G-nonexpansive if the following conditions hold:

(1) T is edge-preserving, *i.e.*, for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;

(2) $\|Tx - Ty\| \leq \|x - y\|$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.

This mapping was introduced by Tiammee *et al.* [13] in 2015.

We know that Halpern iteration process is an important tool in fixed point problem and it can generate a strongly convergent sequence provided that the underlying space is smooth enough. So, in order to prove a strong convergence of the Halpern iteration process in a Hilbert space endowed with a directed graph , Tiammee *et al*. [13] introduced Property G and proved strong convergence of the Halpern iteration process for finding the set of fixed point of G-nonexpansive mappings in a Hilbert space endowed with a directed graph as the following theorem.

***Theorem* 1.2** — *Let C be a nonempty closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence satisfying*

$$x_0 \in C, x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \geq 0. \tag{1}$$

Let $\{x_n\}$ be a sequence defined by Halpern iteration, where $u = x_0$. If $\{x_n\}$ is dominated by Px_0 and $\{x_n\}$ dominates x_0, then $\{x_n\}$ converges strongly to Px_0, where P is the metric projection on $F(T)$.

One of the most interesting iteration processes is the viscosity approximation method introduced by Moudafi [15]. In 2004, Xu [14] studied the such method for a nonexpansive mapping in a Hilbert space and introduced an iterative scheme for finding the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$x_0 \in C, x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0, \tag{2}$$

where $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, $f : C \to C$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Then, they proved a strong convergence theorem under suitable conditions of the parameters $\{\alpha_n\}$.

In this paper, motivated by [13] and [14], we prove a strong convergence theorem for finding the set of fixed point of G-nonexpansive mapping in a Hilbert space endowed with a directed graph. By using our main result, we give a numerical expample to approximate the value of π.

2. Preliminaries

In this paper, we denote " weak and strong convergence " by notations " $\rightharpoonup$" and " $\to$", respectively. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition* [19], *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas are needed to prove the main theorem.

Definition 2.1 — A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to $x \in H$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. In this case, we write $x_n \rightharpoonup x$.

***Theorem* 2.2** — *[16]. Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset C of X is weakly compact,* i.e.*, every bounded sequence in C has a weakly convergent subsequence.*

Lemma 2.3 — [17]. Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) : $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) : $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\displaystyle\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 — [18]. Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.5 — Let H be a real Hilbert space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x, y \in H$.

Property G : [13]. Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$, where $V(G) = C$, be a directed graph. Then C is said to have Property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

The following basic definitions of domination in graphs [20, 21] are needed to prove the main theorem.

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $z \in V(G) \setminus X$ there exists $x \in X$ such that $(x, z) \in E(G)$ and we say that x dominates z or z is dominated by x. Let $z \in V$, a set $X \subseteq V$ is dominated by z if $(z, x) \in E(G)$ for any $x \in X$ and we say that X dominates z if $(x, z) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops.

***Theorem* 2.6** — *[13]. Let X be a normed space and $G = (V(G), E(G))$ a directed graph with $V(G) = X$. Suppose $T : X \to X$ is a G-nonexpansive mapping. If X has a Property G, then T is continuous.*

***Theorem* 2.7** — *[13]. Let X be a Hilbert space and C be a subset of X having Property G. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose $T : C \to C$ is a G-nonexpansive mapping and $F(T) \times F(T) \subseteq E(G)$. Then $F(T)$ is closed and convex.*

Definition 2.8 — [13]. Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

3. Main Result

In this section, we prove a strong convergence theorem of viscosity approximation methods for G-nonexpansive mapping in Hilbert spaces endowed with a directed graph.

The following Proposition is needed to prove the main theorem.

Proposition 3.1 — Let C be a convex subset of a vector space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let G be transitive, $T : C \to C$ be edge-preserving, and $f : C \to C$ be a G-contraction mapping. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, n \geq 0, \end{cases}$$

where $(f(x_0), Tx_0)$ and $(x_0, f(x_0))$ are in $E(G)$. If $\{x_n\}$ dominates x_0, then (x_n, x_{n+1}), (x_0, x_n), (x_0, Tx_n), and (x_n, Tx_n) are in $E(G)$ for all $n \in \mathbb{N}$.

PROOF : We prove by induction. By transitivity of G and since $(x_0, f(x_0))$ and $(f(x_0), Tx_0)$ are in $E(G)$, we have $(x_0, Tx_0) \in E(G)$. Since $E(G)$ is convex, $(x_0, f(x_0))$ and (x_0, Tx_0) are in $E(G)$, we obtain

$$(\alpha_0 x_0 + (1-\alpha_0)x_0, \alpha_0 f(x_0) + (1-\alpha_0)Tx_0) = (x_0, x_1) \in E(G).$$

Since T is edge-preserving, f is G-contraction mapping and $(x_0, x_1) \in E(G)$, then $(Tx_0, Tx_1) \in E(G)$ and $((f(x_0), f(x_1))) \in E(G)$, respectively. By transitivity of G and since (x_0, Tx_0) and (Tx_0, Tx_1) are in $E(G)$, we obtain $(x_0, Tx_1) \in E(G)$. By assumption, $(x_1, x_0) \in E(G)$. Then, by transitivity of G and $(x_0, Tx_1) \in E(G)$, we get $(x_1, Tx_1) \in E(G)$. By transitivity of G and since $(f(x_0), Tx_0)$ and (Tx_0, Tx_1) are in $E(G)$, we obtain $(f(x_0), Tx_1) \in E(G)$. Since $E(G)$ is convex, $(f(x_0), Tx_1)$ and $(f(x_0), f(x_1))$ are in $E(G)$, we obtain

$$(\alpha_1 f(x_0) + (1-\alpha_1)f(x_0), \alpha_1 f(x_1) + (1-\alpha_1)Tx_1) = (f(x_0), x_2) \in E(G).$$

By transitivity of G and since (x_1, x_0) and $(x_0, f(x_0))$ are in $E(G)$, we obtain $(x_1, f(x_0)) \in E(G)$. Again, by transitivity of G and since $(x_1, f(x_0))$ and $(f(x_0), x_2)$ are in $E(G)$, we obtain $(x_1, x_2) \in E(G)$.

Next, assume that (x_k, x_{k+1}), (x_0, x_k), (x_0, Tx_k), and (x_k, Tx_k) are in $E(G)$. Since T is edge-preserving and $(x_k, x_{k+1}) \in E(G)$, then $(Tx_k, Tx_{k+1}) \in E(G)$. By transitivity of G, and (x_0, Tx_k), (Tx_k, Tx_{k+1}) are in $E(G)$, we have $(x_0, Tx_{k+1}) \in E(G)$. Since $\{x_n\}$ dominates x_0, we have $(x_{k+1}, x_0) \in E(G)$. By transitivity of G, and (x_{k+1}, x_0), (x_0, Tx_{k+1}) are in $E(G)$, we have $(x_{k+1}, Tx_{k+1}) \in E(G)$. By transitivity of G, and (x_0, x_k), (x_k, x_{k+1}) are in $E(G)$, we get $(x_0, x_{k+1}) \in E(G)$. Since T is edge-preserving, f is G-contraction mapping, and $(x_0, x_{k+1}) \in E(G)$, we have (Tx_0, Tx_{k+1}), $(f(x_0), f(x_{k+1}))$ are in $E(G)$, respectively. By transitivity of G and since $(f(x_0), Tx_0)$ and (Tx_0, Tx_{k+1}) are in $E(G)$, we obtain $(f(x_0), Tx_{k+1}) \in E(G)$. Since $E(G)$ is convex, $(f(x_0), Tx_{k+1})$ and $(f(x_0), f(x_{k+1}))$ are in $E(G)$, we obtain

$$\begin{aligned}(\alpha_{k+1} f(x_0) + (1-\alpha_{k+1})f(x_0), \alpha_{k+1} f(x_{k+1}) + (1-\alpha_{k+1})Tx_{k+1})\\ = (f(x_0), x_{k+2}) \in E(G).\end{aligned}$$

By transitivity of G and since (x_{k+1}, x_0) and $(x_0, f(x_0))$ are in $E(G)$, we obtain $(x_{k+1}, f(x_0)) \in E(G)$. Again, by transitivity of G and since $(x_{k+1}, f(x_0))$ and $(f(x_0), x_{k+2})$ are in $E(G)$, we obtain $(x_{k+1}, x_{k+2}) \in E(G)$.

So, by induction, we can conclude that (x_n, x_{n+1}), (x_0, x_n), and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$. $\square$

***Theorem* 3.2** — *Let C be a nonempty closed convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Let $f : C \to C$ be a G-contraction mapping with coefficient $\alpha \in (0, 1)$. Assume that there exists $x_0 \in C$ such that $(f(x_0), Tx_0)$ and $(x_0, f(x_0))$ are in $E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\} \subseteq (0, 1)$ satisfies

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0 \text{ , (ii)} \sum_{n=0}^{\infty} \alpha_n = \infty \text{ , (iii)} \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

If $\{x_n\}$ dominates $P_{F(T)}f(x_0)$ and $\{x_n\}$ dominates x_0, then the sequence $\{x_n\}$ converge strongly to $x_0 = P_{F(T)}f(x_0)$.

PROOF : We divide the proof into five steps:

Step 1 : We show that the sequence $\{x_n\}$ is bounded. Let $x^* = P_{F(T)}f(x_0)$. From Proposition 3.1, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Since $x^* \in F(T)$ and $x^* = P_{F(T)}f(x_0)$ is dominated by $\{x_n\}$, we have $(x_n, x^*) \in E(G)$. From the definition of x_n, we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n \|f(x_n) - x^*\| + (1 - \alpha_n)\|Tx_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n \alpha \|x_n - x^*\| + (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\ &= (1 - \alpha_n(1 - \alpha))\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\|. \end{aligned}$$

By mathematical induction, we obtain that

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\right\}, \forall n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is bounded and so are $\{Tx_n\}$ and $\{f(x_n)\}$.

Step 2 : We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From the definition of x_n and $(x_n,x_{n+1})\in E(G)$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n f(x_n)+(1-\alpha_n)Tx_n-\alpha_{n-1}f(x_{n-1})\\
&\quad -\ (1-\alpha_{n-1})Tx_{n-1}\|\\
&= \|\alpha_n f(x_n)-\alpha_n f(x_{n-1})+\alpha_n f(x_{n-1})-\alpha_{n-1}f(x_{n-1})\\
&\quad +\ (1-\alpha_n)Tx_n-(1-\alpha_n)Tx_{n-1}+(1-\alpha_n)Tx_{n-1}\\
&\quad -\ (1-\alpha_{n-1})Tx_{n-1}\|\\
&= \|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1})\\
&\quad +\ (1-\alpha_n)(Tx_n-Tx_{n-1})+(\alpha_{n-1}-\alpha_n)Tx_{n-1}\|\\
&\leq \alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&\quad +\ (1-\alpha_n)\|Tx_n-Tx_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\|\\
&\leq \alpha_n\alpha\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&\quad +\ (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\|\\
&= (1-\alpha_n(1-\alpha))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&\quad +\ |\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\|. \qquad (4)
\end{aligned}$$

Applying Lemma 2.3, (4), and the conditions (i), (ii), (iii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (5)$$

Step 3 : We show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. For each $n\in\mathbb{N}$, we have

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|\\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-Tx_n\|.
\end{aligned}$$

Because $\{Tx_n\}$ and $\{f(x_n)\}$ are bounded, from the condition (i), (ii), and (5), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \qquad (6)$$

Step 4 : We show that $\lim_{n\to\infty}\sup\langle f(z_0)-z_0,x_n-z_0\rangle\leq 0$ where $z_0=P_{F(T)}f(z_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0,x_n-z_0\rangle=\lim_{k\to\infty}\langle f(z_0)-z_0,x_{n_k}-z_0\rangle. \qquad (7)$$

Because all the $\{x_{n_k}\}$ lie in the weakly compact set C and C has Property G, we may assume without loss of generality that $\{x_{n_k}\} \rightharpoonup \omega$ for some $\omega \in C$ and $(x_{n_k}, \omega) \in E(G)$. Suppose $\omega \notin F(T)$, then $\omega \neq T\omega$. By G-nonexpansiveness of T, (6), and the Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - T\omega\| \\
&\leq \liminf_{k\to\infty} \big(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - T\omega\|\big) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\| .
\end{aligned}$$

This is a contradiction. Then $\omega \in F(T)$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in F(T)$. By (7) and Lemma 2.4, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle &= \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle \\
&= \langle f(z_0) - z_0, \omega - z_0\rangle \\
&\leq 0. \qquad (8)
\end{aligned}$$

Step 5 : Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{F(T)}f(z_0)$. By G-nonexpansiveness of T and $(z_0, x_n) \in E(G)$, and Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + (1-\alpha_n)(Tx_n - z_0)\|^2 \\
&\leq \|(1-\alpha_n)(Tx_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&= (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\|f(x_n) - f(z_0)\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\alpha\|x_n - z_0\|^2 + \alpha_n\alpha\|x_{n+1} - z_0\|^2 \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

It implies that

$$\|x_{n+1} - z_0\|^2 \leq \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle$$

$$= \left(1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)\|x_n-z_0\|^2+\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\Bigg(\frac{\alpha_n}{2(1-\alpha)}\|x_n-z_0\|^2.$$
$$+\frac{1}{1-\alpha}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\Bigg).$$

From the conditions (i), (ii), (8), and Lemma 2.3, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}f(z_0)$. This completes the proof. □

4. Numerical Results

The purpose of this section we give a numerical example to support our some result. The following example is given for supporting Theorem 3.2.

Example 4.1 : Let $H = \mathbb{R}$ and $C = [0,1]$ with the usual norm $\|x-y\| = |x-y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x,y) : x, y \in [0, \frac{3}{5}]$ such that $|x-y| \le \frac{1}{5}\}$. Let $f : C \to C$ be defined by $f(x) = \frac{x}{9}$, for all $x \in [0,1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{10}x & \text{if } x \in [0,1), \\ \frac{8}{5} & \text{if } x = 1. \end{cases}$$

Solution : We observe that $F(T) = \{0\}$. Choose $x_0 = \frac{1}{5}$, then $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x,y) \in E(G)$. Then $x, y \in [0, \frac{3}{5}]$ and $|x-y| \le \frac{1}{5}$. It implies that

$$|Tx-Ty| \le \frac{1}{10}|x-y| \le |x-y| \le \frac{1}{5}.$$

Then, we have $(Tx, Ty) \in E(G)$ and $\|Tx-Ty\| \le \|x-y\|$. Thus T is G-nonexpansive. For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{2(n+1)}$. We rewrite (3) as follows:

$$x_{n+1} = \left(\frac{1}{2(n+1)}\right)\left(\frac{x_n}{9}\right)+\left(1-\frac{1}{2(n+1)}\right)\left(\frac{x_n}{10}\right). \tag{9}$$

Since $x_0 = \frac{1}{5} \in [0, \frac{1}{5}]$, from (9), we have

$$x_1 = \left(\frac{1}{2(1)}\right)\left(\frac{x_0}{9}\right)+\left(1-\frac{1}{2(1)}\right)\left(\frac{x_0}{10}\right).$$

By the convexity, we have $x_1 \in [0, \frac{1}{5}]$. Since $x_1 \in [0, \frac{1}{5}]$ and (9), we have

$$x_2 = \left(\frac{1}{2(2)}\right)\left(\frac{x_2}{9}\right) + \left(1 - \frac{1}{2(2)}\right)\left(\frac{x_2}{10}\right).$$

By the convexity, we have $x_2 \in [0, \frac{1}{5}]$. By continuing in this way, we have $x_n \in [0, \frac{1}{5}]$, for all $n \in \mathbb{N}$. It implies that $x_n \leq \frac{1}{5}$ for all $n \in \mathbb{N}$. It follows that $(x_n, P_{F(T)}f(x_0)) = (x_n, 0) \in E(G)$. That is. $P_{F(T)}f(x_0)$ is dominated by $\{x_n\}$. It can be observed that parameters satisfy all the conditions of Theorem 3.2 and $C = [0, 1]$ satisfy Property G. Hence, the sequence $\{x_n\}$ converges strongly to 0.

Next, we show that T is not a nonexpansive mapping. Choose $x = 1$ and $y = \frac{3}{5}$, we have

$$\left|T(1) - T\left(\frac{3}{5}\right)\right| = \left|\frac{8}{5} - \frac{3}{50}\right| = \frac{77}{50} > \frac{2}{5} = \left|1 - \frac{3}{5}\right|.$$

Mathematicians know that the number π is an important mathematical constant. For the previous decades, many researcher have been trying to approximate the value of π; see [22, 23] and the references therein. By using our main result, we introduce the new method to approximate the value of π as shown in the following example.

Example 4.2 : Let $H = \mathbb{R}$ and $C = [3, 4]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x, y) : x, y \in [3, \frac{18}{5}]$ such that $|x - y| \leq \frac{16}{5}\}$. Let $f : C \to C$ be defined by $f(x) = \frac{1}{5}x + \frac{4}{5}(\pi)$, for all $x \in [3, 4]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{3}x + \frac{2}{3}(\pi) & \text{if } x \in [3, 4), \\ \frac{56}{35} & \text{if } x = 4. \end{cases}$$

Solution : We observe that $F(T) = \{\pi\}$. Choose $x_0 = \frac{16}{5}$, then $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x, y) \in E(G)$. Then $x, y \in [3, \frac{18}{5}]$ and $|x - y| \leq \frac{16}{5}$. It implies that

$$|Tx - Ty| = \left|\frac{1}{3}x + \frac{2}{3}(\pi) - \frac{1}{3}y - \frac{2}{3}(\pi)\right| \leq \frac{1}{3}|x - y| \leq |x - y| \leq \frac{16}{5}.$$

Then, we have $(Tx, Ty) \in E(G)$ and $\|Tx - Ty\| \leq \|x - y\|$. Thus T is G-nonexpansive. For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{2(n+1)}$. We rewrite (3) as follows:

$$x_{n+1} = \left(\frac{1}{2(n+1)}\right)\left(\frac{1}{5}x_n + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(n+1)}\right)\left(\frac{1}{3}x_n + \frac{2}{3}(\pi)\right). \quad (10)$$

Since $x_0 = \frac{16}{5} \in [3, \frac{16}{5}]$, from (10), we have

$$x_1 = \left(\frac{1}{2(1)}\right)\left(\frac{1}{5}x_0 + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(1)}\right)\left(\frac{1}{3}x_0 + \frac{2}{3}(\pi)\right).$$

By the convexity, we have $x_1 \in [3, \frac{16}{5}]$. Since $x_1 \in [3, \frac{16}{5}]$ and (10), we have

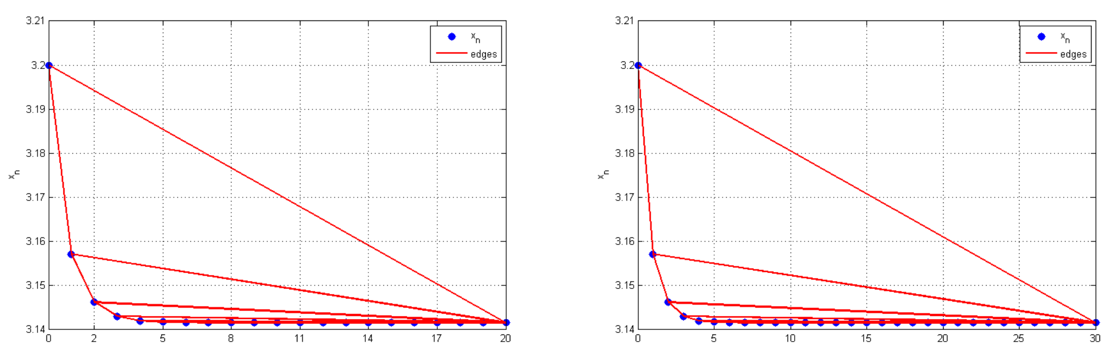
$$x_2 = \left(\frac{1}{2(2)}\right)\left(\frac{1}{5}x_1 + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(2)}\right)\left(\frac{1}{3}x_1 + \frac{2}{3}(\pi)\right).$$

By the convexity, we have $x_2 \in (3, \frac{16}{5}]$. By continuing in this way, we have $x_n \in [3, \frac{16}{5}]$, for all $n \in \mathbb{N}$. It implies that $3 \leq x_n \leq \frac{16}{5}$ for all $n \in \mathbb{N}$. Then $|x_n - \pi| \leq \frac{16}{5}$ for all $n \in \mathbb{N}$. It follows that $(x_n, P_{F(T)}f(\pi)) = (x_n, \pi) \in E(G)$. That is. $P_{F(T)}f(\pi)$ is dominated by $\{x_n\}$. It can be observed that parameters satisfy all the conditions of Theorem 3.2 and $C = [3, 4]$ satisfy Property G. Since $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to π.

Next, we show that T is not a nonexpansive mapping. Choose $x = 4$ and $y = \frac{18}{5}$, we have

$$\begin{aligned}
\left|T(4) - T\left(\frac{18}{5}\right)\right| &= \left|\frac{56}{35} - \left(\frac{1}{3}\left(\frac{18}{5}\right) + \frac{2}{3}(\pi)\right)\right| \\
&\approx \left|\frac{56}{35} - \left(\frac{1}{3}\left(\frac{18}{5}\right) + \frac{2}{3}\left(\frac{22}{7}\right)\right)\right| \\
&= \frac{178}{105} \\
&> \frac{2}{5} \\
&= \left|4 - \frac{18}{5}\right|.
\end{aligned}$$

Using the algorithm (10) and choosing $x_0 = \frac{16}{5}$ with $n = 20$ and $n = 30$, we have the numerical result to approximate the value of π as shown in Table 1 and Figure 1.



(A) n=20 (B) n=30

Figure 1: The convergence of $\{x_n\}$ with initial values $x_0 = \frac{16}{5}$.

$n = 20$		$n = 30$	
n	x_n	n	x_n
0	3.200000000000000	0	3.200000000000000
1	3.157167945965848	1	3.157167945965848
2	3.146265241302610	2	3.146265241302610
3	3.143046347544892	3	3.143046347544892
4	3.142052990008907	4	3.142052990008907
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	3.141593185425522	15	3.141592659742157
$\vdots$	$\vdots$	$\vdots$	$\vdots$
16	3.141592654255843	26	3.141592653589803
17	3.141592653809197	27	3.141592653589797
18	3.141592653662115	28	3.141592653589794
19	3.141592653613647	29	3.141592653589793
20	3.141592653597665	30	3.141592653589793

Table 1: The values of the sequences $\{x_n\}$ with initial value $x_0 = \frac{16}{5}$.

5. Conclusion

In this work, we introduce a viscosity approximation method of G-nonexpansive mapping defined on a Hilbert space endowed with a directed graph. We obtain a strong convergence theorem for the sequence generated by the proposed method under suitable conditions. However, we should like remark the following:

1. In Theorem 3.2, we use the concept of a viscosity approximation method and our result is proved with an assumption on a directed graph, which is a different result from Xu [14].
2. From Theorem 3.2, we can conclude that the sequence $\{x_n\}$, in Example 4.2, converges to π.
3. In Example 4.2, the sequence $\{x_n\}$ converges to π as shown in the Table 1 and Figure 1.
4. In order to gain more accuracy of π, the iterative approximation is depended on the number of n as shown in the Table 1.

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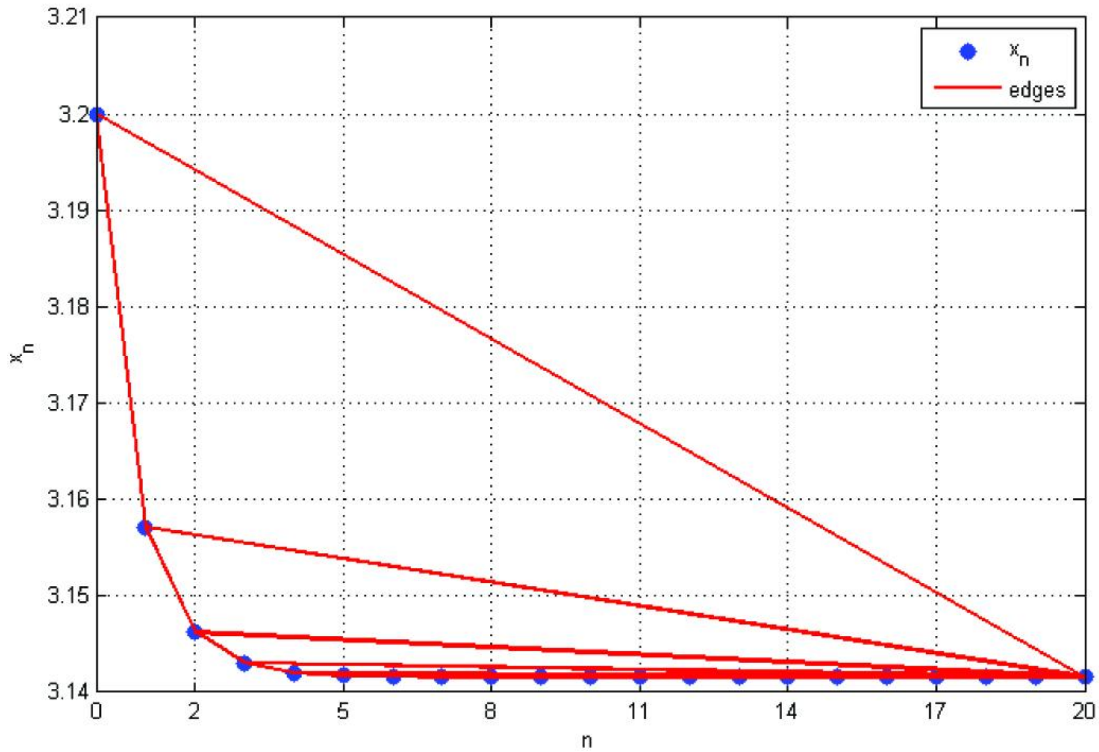
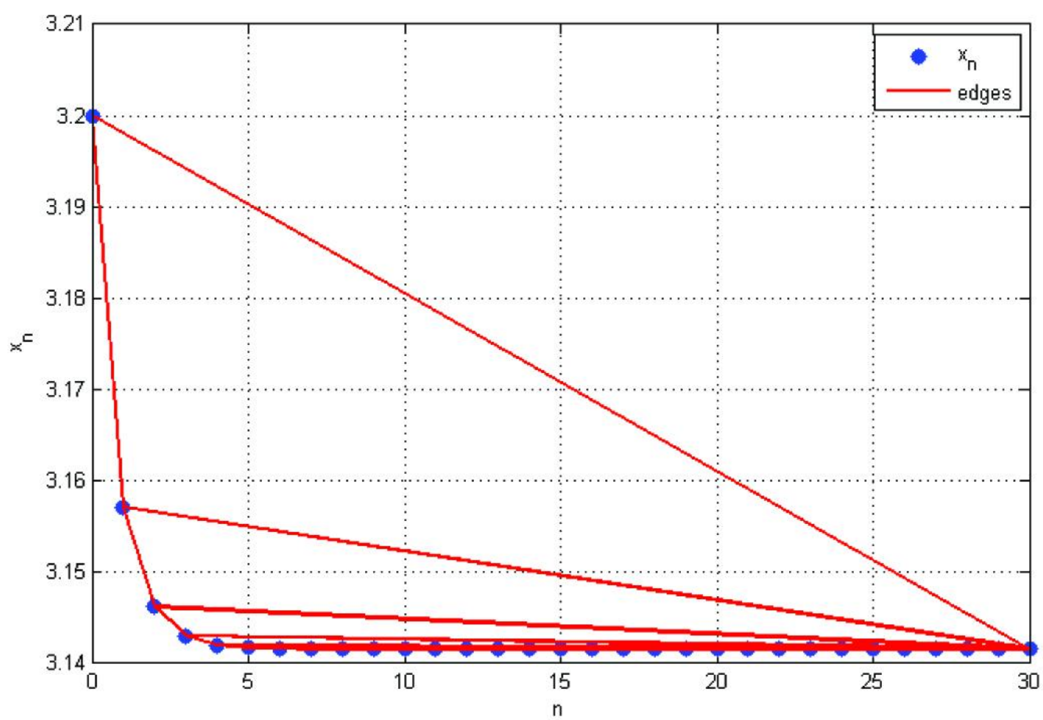

3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
x_n
0
2
5
8
11
14
17
20
n
x_n
edges


3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
x_n
0
5
10
15
20
25
30
n
x_n
edges

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Viscosity approximation method for the sum of two different types of finitely many accretive operators in Banach spaces

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***Abstract*—In this paper, we present a new viscosity iterative algorithm to solve the problem of finding zeros of the sum of finite families of m- accretive operators and finite families of α- inverse strongly accretive operators in a real q- uniformly smooth and strictly convex Banach spaces. Strong convergence theorems are established, which extend the corresponding works given by many others.**

Keywords*—*accretive operators, α- inverse strongly accretive operators, zero point, q- uniformly smooth Banach space.

I. Introduction

Let E be a real Banach space. Let $A: E \to E$ be a single-valued nonlinear mapping and $B: E \to 2^E$ be a set-valued mapping. We consider the following problem: find $u \in E$ such that

$$0 \in Au + Bu \qquad (1.1)$$

Many practical problems can reduced to the problem (1.1) and it is well-known that this problem provides a convenient framework for the unified study of optimal solutions in many optimization related areas including variational inequalities, complementarity, mathematical programming, mathematical economics, optimal control, equilibria, game theory, etc.

The classical method for solving problem (1.1) is the forward-backward splitting algorithm, which were proposed by Lions and Mercier [1], by Passty [2], and, in a dual form for convex programming, by Han and Lou [3]. The classical forward-backward splitting algorithm is given in the following way: $x_1 \in E$ and

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n , \quad n \geq 1. \qquad (1.2)$$

We see that for each step of iterate involves only with A as the forward step and B as the backward step, but not the sum of B and based on iterative algorithm (1.2) much work has been done for finding $x \in H$ such that $x \in (A+B)^{-1}0$, where A and B are α- inversely strongly monotone mapping and maximal monotone operator defined in the Hilbert space H, respectively. However, most of the existing work are under taken in the frame of Hilbert spaces; see [1-6], *etc*

Recently, Qin et al., [7] introduced the following iterative algorithm in $q-$ uniformly smooth Banach spaces E :

$x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n (I + r_n B)^{-1}\left[(I - r_n A)x_n + e_n\right] + \gamma_n f_n , \qquad (1.3)$$

for all $n \geq 0$, where C is closed convex subset of E, $\{e_n\}$ is the error sequence, f is contraction, A and B are α- inverse-strongly accretive operator and m- accretive operator, respectively. Then they proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a zero point of the sum of A and B under some appropriate conditions.

Motivated and inspired by Qin et al. [7] and the research going on in this direction, in this paper, we introduce a new viscosity iterative algorithm (3.1) to solve the problem of finding zeros of the sum of finite families of m- accretive operators and finite families of α- inverse strongly accretive operators in a real q- uniformly smooth and strictly convex Banach spaces. Under suitable conditions, some strong convergence theorems are proved. Our results extend and improve some corresponding results in the literature.

II. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. Let $\langle \cdot,\cdot \rangle$ denote the pairing between E and E^*. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also called uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$

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for any two sequences $\{x_n\},\{y_n\}$ in E such that $\|x_n\|=\|y_n\|=1$ and $\lim_{n\to\infty}\left\|\frac{x_n+y_n}{2}\right\|=1.$ Let $S(\mathrm{E})=\{x\in E:\|x\|=1\}$. A Banach space E is said to be Gateaux differentiable if the limit $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,y\in S(E)$. In this case E is smooth. Let $\rho_E:[0,\infty)\to[0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(\mathrm{t})=\sup\left\{\frac{1}{2}\left(\|x+y\|+\|x-y\|\right)-1:x\in S(\mathrm{E}),\ \|y\|\le t\right\}$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(\mathrm{t})}{t}\to 0$ as $t\to 0$. Let $q>1$ then E is said to be q-uniformly smooth if there exists $c>0$ such that $\rho_E(\mathrm{t})\le \mathrm{c}t^q$. It is easy to see that if E is q-uniformly smooth, then $q\le 2$ and E is uniformly smooth. Recall that the generalized duality mapping $J_q:E\to 2^{E^*}$ is denoted by

$$J_q(x)=\left\{x^*\in E^*:\langle x,x^*\rangle=\|x\|^q,\|x^*\|=\|x\|^{q-1}\right\},\ \forall x\in E.$$

In particular, $J_q=J_2$ is called the normalized duality mapping and $J_q(x)=\|x\|^{q-2}J_2(x)$ for $x\neq 0$. If $E:=H$ is a real Hilbert space, then $J=I$ where I is the identity mapping. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

Let C be a nonempty closed convex subset of E. Then C is called a retract of E if there is a continuous mapping Q from E onto C such that $Qx=x$, for all $x\in C$. We call such Q a retraction of E onto C. It follows that if Q is a retraction, then $Qy=y$, for all y in the range of Q. A retraction Q is said to be sunny if $Q(Qx+t(x-Qx))=Qx$, for all $x\in E$ and $t\ge 0$. If a sunny retraction Q is also nonexpansive, then C is said to be sunny nonexpansive retract of E. Let T be a mapping of C into itself. We denote by $F(T)$ the set of fixed points of T, i.e. $F(T):=\{x\in C\mid Tx=x\}$. A mapping $T:C\to C$ of C is said to be nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|,\text{ for all } x,y\in C.$$

A mapping $f:C\to C$ is said to be a contraction if there exists a constant $\beta\in(0,1)$ such that

$$\|f(x)-f(y)\|\le\beta\|x-y\| \text{ for all } x,y\in C.$$

Let $A:\mathrm{E}\to 2^E$ be a set-valued mapping. We denote $D(A)$ by domain of A, that is $D(A)=\{x\in E:Ax\neq\varnothing\}$ and also denote $R(A)$ by the range of A, that is $R(A)=\cup\{Ax:x\in D(A)\}$. A set-valued mapping $A:D(A)\to 2^E$ is said to be accretive if for each $x,y\in D(A)$ there exists $j_q(x-y)\in J_q(x-y)$ such that

$$\langle u-v,j_q(x-y)\rangle\ge 0 \text{ for all } u\in Ax \text{ and } v\in Ay$$

An operator A is called m-accretive if it is accretive and $R(I+rA)=E$ for all $r>0$. Let $\alpha>0$, an accretive operator A is called α-inverse-strongly accretive if for each $x,y\in D(A)$ there exists $j_q(x-y)\in J_q(x-y)$ such that

$$\langle u-v,j_q(x-y)\rangle\ge\alpha\|u-v\|^q$$

for all $u\in Ax$ and $v\in Ay$. We denote by J_r^A (for $r>0$) the resolvent of accretive operator A, that is, $J_r^A=(I+rA)^{-1}$, and also denote $A^{-1}0$ by the set of zeros of A, that is $A^{-1}0=\{x\in D(\mathrm{A}):\mathrm{Ax}=0\}$. It is well-known that J_r^A is nonexpansive and $F(J_r^A)=A^{-1}0.$

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [8] *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q>0$ such that*

$$\|x+y\|^q\le\|x\|^q+q\langle y,j_qx\rangle+C_q\|y\|^q,\text{ for all } x,y\in E.$$

Lemma 2.2 [9] *Let $q>1$, then the following inequality holds:*

$$ab\le\frac{1}{q}a^q+\frac{q-1}{q}b^{\frac{q}{q-1}}$$

for any positive real numbers a,b.

Lemma 2.3 [10] *Let E be a Banach space and let A be an m-accretive operator. For $\lambda>0$, $\mu>0$ and $x\in E$, we have*

$$J_\lambda x=J_\mu\left(\frac{\mu}{\lambda}x+\left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right),$$

where $J_\lambda=(I+\lambda A)^{-1}$ and $J_\mu=(I+\mu A)^{-1}$.

Lemma 2.4 [11] *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $A:C\to E$ be a single-valued operator, and let $B:E\to 2^E$ be an m-accretive operator. Then*

$$F(J_r(I-rA))=(A+B)^{-1}0,$$

where $J_r(I-rA)$ is the resolvent of B for $r>0$.

Lemma 2.5 [12] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, and $\{\beta_n\}$ be a sequence in $(0,1)$ with $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$. Suppose that*

$$x_{n+1}=(1-\beta_n)y_n+\beta_n x_n,\ \textit{for all } n\ge 1 \textit{ and}$$

$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\le 0.$ *Then* $\lim_{n\to\infty}\|y_n-x_n\|=0.$

Lemma 2.6 [13] *Assume that* $\{a_n\}$ *is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n, \text{ for all } n \geq 0,$$

where t_n *is a sequence in in* $(0,1)$ *such that* $\sum_{n=0}^{\infty} t_n = \infty$ and $\{b_n\}$ *is a sequence in* $\mathbb{R}$ *such that* $\limsup_{n\to\infty}\left(\frac{b_n}{t_n}\right) \leq 0$ or $\sum_{n=1}^{\infty}|b_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 [14] *Let* C *be a nonempty closed convex subset of a real uniformly smooth Banach space* E. *Let* $f : C \to C$ *be a contractive mapping, and let* $T : C \to C$ *be a nonexpansive mapping. For each* $t \in (0,1)$, *let* x_t *be the unique solution of the equation* $x = tf(x) + (1-t)Tx$. *Then* $\{x_t\}$ *converges strongly to a fixed point* $\bar{x} = Q_{F(T)} f(\bar{x})$.

Lemma 2.8 *Let* E *be a real strictly convex and* q-*uniformly smooth Banach space with constant* C_q. *Let* C *be a nonempty closed and convex subset of* E. *For* $i = 1,2,3,...,N$, *let* $A_i : E \to 2^E$ *be* m-*accretive operator such that* $\overline{D(A_i)} \subseteq C$ *and let* $B_i : C \to E$ *be* α-*inverse strongly accretive operator such that* $\bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 \neq \emptyset$. *Let* $a_0, a_1, ..., a_N$ *be real numbers in* $(0,1)$ *such that* $\sum_{i=0}^{N} a_i = 1$ *and* $W_n = a_0 I + \sum_{i=1}^{N} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$, *where* $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ *and* $0 < r_{n,i} \leq \left(\frac{q\alpha}{C_q}\right)^{\frac{1}{q-1}}$ *for all* $i = 1,2,3,...,N$ *.and* $n \geq 1$. *Then* $W_n : C \to C$ *is nonexpansive and* $F(W_n) = \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0$, *for all* $n \geq 1$.

Proof. First, we will show that W_n is nonexpansive for all $n \geq 1$. Let $x, y \in C$. Then for $i = 1,2,3,...,N$, it follows from Lemma 2.1 that

$$\begin{aligned}
&\|(I - r_{n,i}B_i)x - (I - r_{n,i}B_i)y\|^q \\
&\quad \leq \|x-y\|^q + qr_{n,i}\langle B_i x - B_i y, j_q(x-y)\rangle \\
&\qquad + C_q r_{n,i}^q \|B_i x - B_i y\|^q \\
&\quad \leq \|x-y\|^q - qr_{n,i}\alpha\|B_i x - B_i y\|^q + C_q r_{n,i}^q \|B_i x - B_i y\|^q \\
&\quad = \|x-y\|^q - (\alpha q - C_q r_{n,i}^{q-1}) r_{n,i}\|B_i x - B_i y\|^q \\
&\quad \leq \|x-y\|^q.
\end{aligned}$$

Thus $(I - r_{n,i}B_i)$ is nonexpansive for all $i = 1,2,3,...,N$. Since $J_{r_{n,i}}^{A_i}$ and $(I - r_{n,i}B_i)$ are nonexpansive for all $i = 1,2,3,...,N$, we get that

$$\begin{aligned}
&\|W_n x - W_n y\| \\
&\quad \leq a_0\|x-y\| + \sum_{i=1}^{N} a_i \left\|J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)x - J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y\right\| \\
&\quad \leq a_0\|x-y\| + \sum_{i=1}^{N} a_i \|(I - r_{n,i}B_i)x - (I - r_{n,i}B_i)y\| \\
&\quad \leq a_0\|x-y\| + \sum_{i=1}^{N} a_i \|x-y\| \\
&\quad \leq \|x-y\|.
\end{aligned}$$

Thus W_n is nonexpansive for all $n \geq 1$. Next, we will show that $F(W_n) = \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0$, for all $n \geq 1$. It is obvious that $\bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 \subseteq F(W_n)$. So, we are left to show that $F(W_n) \subseteq \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0$. Let $u \in F(W_n)$. Then $W_n u = u$ and for all $v \in \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 \subseteq F(W_n)$, we have

$$\begin{aligned}
\|u - v\| &\leq a_0\|u-v\| + a_1\left\|J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v\right\| + \ldots \\
&\quad + a_N\left\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\right\| \\
&\leq (a_0 + a_1 + \ldots + a_{N-1})\|u - v\| \\
&\quad + a_N\left\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\right\| \\
&\leq (1 - a_N)\|u - v\| + a_N\left\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\right\| \\
&\leq \|u - v\|.
\end{aligned}$$

Therefore

$$\|u - v\| = (1 - a_N)\|u - v\| + a_N\left\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\right\|,$$

which implies that

$\|u - v\| = \left\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\right\|$. Similarly,

$$\|u - v\| = \left\|J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v\right\| = \cdots = \left\|J_{r_{n,N-1}}^{A_{N-1}}(I - r_{n,N-1}B_{N-1})u - v\right\|.$$

Then

$$\begin{aligned}
\|u - v\| = \Bigg\| &\frac{a_1}{\sum_{i=1}^{N} a_i}\left(J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v\right) \\
&+ \frac{a_2}{\sum_{i=1}^{N} a_i}\left(J_{r_{n,2}}^{A_2}(I - r_{n,2}B_2)u - v\right) + \ldots \\
&+ \frac{a_N}{\sum_{i=1}^{N} a_i}\left(J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\right)\Bigg\|.
\end{aligned}$$

By the strictly convex city of E, we have that

$$\begin{aligned}
u - v &= J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v = J_{r_{n,2}}^{A_2}(I - r_{n,2}B_2)u - v \\
&= \cdots = J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v.
\end{aligned}$$

Then $J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)u = u$ for all $i = 1,2,3,...,N$.

Thus $u \in \bigcap_{i=1}^{N}(A_i+B_i)^{-1}0,$ and so $F(W_n) \subseteq \bigcap_{i=1}^{N}(A_i+B_i)^{-1}0.$ Which complete the proof. □.

III. MAIN RESULTS

Theorem 3.1 *Let E be a real strictly convex and q-uniformly smooth Banach space with constant C_q. Let C be a nonempty closed and convex subset of E. Let $f: C \to C$ be a contractive mapping with the constant $\beta \in (0,1)$. Let $A_i : C \to 2^E$ be m-accretive operator and let $B_i : C \to E$ be α-inverse strongly accretive operator, for $i = 1,2,3,...,N$. Assume that $\Omega = \bigcap_{i=1}^{N}(A_i+B_i)^{-1}0$ is nonempty. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 \in E; \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n; \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\left[a_0 y_n + \sum_{i=1}^{N} a_i J_{r_{n,i}}^{A_i}(I + r_{n,i}B_i)y_n\right], \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$, for $i = 1,2,3,...,N$, and $0 < a_m < 1$, for $m = 0,1,2,3,...,N$, $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$ and $\{r_{n,i}\} \subset (0,\infty)$. Suppose that the above control sequences satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty;$

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1;$

(iii) $0 < b \leq r_{n,i} \leq c < \left(\frac{q\alpha}{C_q}\right)^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty}|r_{n+1,i} - r_{n,i}| < \infty$

for $n \geq 1$ and $i = 1,2,3,...,N$; where b and c are real numbers.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$, where $\bar{x} = Q_\Omega f(\bar{x})$ and Q_Ω is the unique sunny nonexpansive retraction of C onto Ω

Proof We divide the proof into several step. First, we will show that $\{x_n\}$ is bounded. Let $p \in \Omega$.

Put $W_n = a_0 I + \sum_{i=1}^{N} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$ and $u_{n,i} = (I - r_{n,i}B_i)y_n$ for all $i = 1,2,3,...,N$ and $n \geq 1$. Then we get that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - p\| \\ &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq (1-\alpha_n(1-\beta))\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned} \tag{3.2}$$

By Lemma 2.8, we have W_n is nonexpansive and for all $n \geq 1$. Then it follows from (3.1) and (3.2) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n)W_n y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|W_n y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\alpha_n(1-\beta))\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n\|f(p) - p\| \\ &\leq (1-\alpha_n(1-\beta)(1-\beta_n))\|x_n - p\| \\ &\quad + \alpha_n(1-\beta_n)\|f(p) - p\| \\ &\leq \max\left\{\|x_n - p\|, \frac{\|f(p)-p\|}{1-\beta}\right\} \end{aligned} \tag{3.3}$$

By using the inductive method, we can easily get the following result from (3.3) that

$$\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p)-p\|}{1-\beta}\right\}.$$

Thus the sequence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{y_n\}$. Since $I - r_{n,i}B$ and $J_{r_{n,i}}^{A_i}$ are nonexpansive for all $i = 1,2,3,...,N$ and for all $n \geq 1$, we get that $\{u_{n,i}\}$ and $\{J_{r_{n,i}}^{A_i}u_{n,i}\}$ are bounded for all $i = 1,2,3,...,N$ and so $\{W_n y_n\}$ is also bounded. Put

$$M_1 = \sup\left\{\|x_n\|, \|B_i y_n\|, \|u_{n,i}\|, \|J_{r_{n,i}}^{A_i}u_{n,i}\|, \|W_n y_n\| : n \geq 1, i = 1,2,...,N\right\}.$$

Next, we will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Note that,

$$\|y_{n+1} - y_n\| \leq (1-\alpha_{n+1}(1-\beta))\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - x_n\|$$

and $n \geq 1$, then we have

$$\begin{aligned} \|u_{n+1,i} - u_{n,i}\| &\leq \|(I - r_{n+1,i}B_i)(y_{n+1} - y_n)\| + |r_{n+1,i} - r_{n,i}|\|B_i y_n\| \\ &\leq \|y_{n+1} - y_n\| + |r_{n+1,i} - r_{n,i}|\|B_i y_n\| \\ &\leq (1-\alpha_{n+1}(1-\beta))\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - x_n\| \\ &\quad + |r_{n+1,i} - r_{n,i}|\|B_i y_n\| \end{aligned} \tag{3.4}$$

Next, we consider $\left\|J_{r_{n+1,i}}^{A_i}(I - r_{n+1,i}B_i)y_{n+1} - J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n\right\|$.

If $r_{n,i} \leq r_{n+1,i}$, then it follows from Lemma 2.3 that

$$\begin{aligned} &\left\|J_{r_{n+1,i}}^{A_i}u_{n+1,i} - J_{r_{n,i}}^{A_i}u_{n,i}\right\| \\ &= \left\|J_{r_{n,i}}^{A_i}\left(\frac{r_{n,i}}{r_{n+1,i}}u_{n+1,i} + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right)J_{r_{n+1,i}}^{A_i}u_{n+1,i}\right) - J_{r_{n,i}}^{A_i}u_{n,i}\right\| \\ &\leq \left\|\frac{r_{n,i}}{r_{n+1,i}}u_{n+1,i} + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right)J_{n+1,i}^{A_i}u_{n+1,i} - u_{n,i}\right\| \\ &\leq \frac{r_{n,i}}{r_{n+1,i}}\|u_{n+1,i} - u_{n,i}\| + \left(1 - \frac{r_{n,i}}{r_{n+1,i}}\right)\left\|J_{n+1,i}^{A_i}u_{n+1,i} - u_{n,i}\right\| \\ &\leq \|u_{n+1,i} - u_{n,i}\| + \frac{r_{n+1,i} - r_{n,i}}{b}2M_1. \end{aligned} \tag{3.5}$$

If $r_{n+1,i} \leq r_{n,i}$, then imitating the proof of (3.5), we have

$$\left\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - J_{r_{n,i}}^{A_i} u_{n,i}\right\|$$
$$\le \left\|u_{n+1,i} - u_{n,i}\right\| + \frac{r_{n,i} - r_{n+1,i}}{b} 2M_1. \qquad (3.6)$$

Combining (3.5) and (3.6), we have, for $n \ge 1$,

$$\left\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - J_{r_{n,i}}^{A_i} u_{n,i}\right\|$$
$$\le \left\|u_{n+1,i} - u_{n,i}\right\| + \frac{2\left|r_{n,i} - r_{n+1,i}\right|}{b} M_1 \qquad (3.7)$$

Set $M_2 = \left(\frac{2}{b} + M_1\right)$ and using (3.7), we obtain

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_n y_n\| &\le a_0 \|y_{n+1} - y_n\| \\ &+ \sum_{i=1}^{N} \left\|J_{r_{n+1,i}}^{A_i}(I - r_{n+1,i}B_i)y_{n+1} - J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n\right\| \\ &\le \|y_{n+1} - y_n\| + M_2 \sum_{i=1}^{N} \left|r_{n,i} - r_{n+1,i}\right|. \qquad (3.8)\end{aligned}$$

Now, using (3.4) and (3.8), we get that

$$\begin{aligned}\|W_{n+1}y_{n+1} - W_n y_n\| &\le (1 - \alpha_{n+1}(1-\beta))\|x_{n+1} - x_n\| \\ &+ |\alpha_{n+1} - \alpha_n| \|f(x_n) - x_n\| + M_2 \sum_{i=1}^{N} \left|r_{n,i} - r_{n+1,i}\right| \\ &\le \|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n| M_1 \\ &+ M_2 \sum_{i=1}^{N} \left|r_{n,i} - r_{n+1,i}\right| \qquad (3.9)\end{aligned}$$

From the assumption on $\{\alpha_n\}$ and $\{r_{n,i}\}$, in view of (3.9) we obtain that

$$\limsup_{n\to\infty}\left(\|W_{n+1}y_{n+1} - W_n y_n\| - \|x_{n+1} - x_n\|\right) \le 0.$$

It follows from Lemma 2.5, that

$$\lim_{n\to\infty}\|W_n y_n - x_n\| = 0. \qquad (3.10)$$

Which implies that

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|W_n y_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Next, we will show that $\lim_{n\to\infty}\|W_n y_n - y_n\| = 0.$

We note that

$$\begin{aligned}\|W_n x_n - x_n\| &\le \|W_n x_n - W_n y_n\| + \|W_n y_n - x_n\| \\ &\le \|x_n - y_n\| + \|W_n y_n - x_n\| \qquad (3.11)\end{aligned}$$

Since $\|y_n - x_n\| = \alpha_n \|f(x_n) - x_n\| \to 0$ as $n \to \infty$, it follows from (3.10) and (3.11) that

$$\lim_{n\to\infty}\|W_n x_n - x_n\| = 0. \qquad (3.12)$$

Since $\|W_n y_n - y_n\| \le \|W_n y_n - x_n\| + \|x_n - y_n\|$, for all $n \ge 0$, we obtain that

$$\lim_{n\to\infty}\|W_n y_n - y_n\| = 0. \qquad (3.13)$$

Since f is contractive and W_n is nonexpansive for all $n \in \mathbb{N}$, we get that the mapping $tf + (1-t)W_n$ is contractive for all $t \in (0,1)$. Let x_t be a unique fixed point of mapping $tf + (1-t)W_n x_t$ for all $t \in (0,1)$. Then by Lemma 2.7, we have $x_t \to x^*$, where $x^* \in F(W_n) = \bigcap_{i=1}^{N}(A_i + B_i)^{-1}0 = \Omega.$ (i.e. $x^* = Q_\Omega f(x^*)$). Next, we will show that

$$\limsup_{n\to\infty}\left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle \le 0.$$

Since

$$\begin{aligned}\|x_t - y_n\|^q &\le t\left\langle f(x_t) - y_n, J_q(x_t - y_n)\right\rangle \\ &+ (1-t)\left\langle W_n x_t - y_n, J_q(x_t - y_n)\right\rangle \\ &\le t\left\langle f(x_t) - x_t, J_q(x_t - y_n)\right\rangle + t\left\langle x_t - y_n, J_q(x_t - y_n)\right\rangle \\ &+ (1-t)\left\langle W_n x_t - W_n y_n, J_q(x_t - y_n)\right\rangle \\ &+ (1-t)\left\langle W_n y_n - y_n, J_q(x_t - y_n)\right\rangle \\ &\le t\left\langle f(x_t) - x_t, J_q(x_t - y_n)\right\rangle + \|x_t - y_n\|^q \\ &+ \|W_n y_n - y_n\|\|x_t - y_n\|^{q-1},\end{aligned}$$

we get that

$$\left\langle f(x_t) - x_t, J_q(y_n - x_t)\right\rangle \le \frac{1}{t}\|W_n y_n - y_n\|\|x_t - y_n\|^{q-1}.$$

Fix t and let $n \to \infty$. Then, it follows from (3.13) that

$$\limsup_{n\to\infty}\left\langle f(x_t) - x_t, J_q(y_n - x_t)\right\rangle \le 0. \qquad (3.14)$$

Since the duality map J_q is single-valued and strong-weak* uniformly continuous, on bounded set of a Banach space E with uniformly Gâteaux differentiable norm, we get that

$$\begin{aligned}&\left|\left\langle f(x_t) - x_t, J_q(y_n - x_t)\right\rangle - \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle\right| \\ &= \left|\left\langle f(x^*) - x^*, J_q(y_n - x^*) - J_q(y_n - x_t)\right\rangle\right. \\ &\left.+ \left\langle f(x^*) - x^* - (f(x_t) - x_t), J_q(y_n - x_t)\right\rangle\right| \\ &\le \left|\left\langle f(x^*) - x^*, J_q(y_n - x^*) - J_q(y_n - x_t)\right\rangle\right| \\ &+ \left\|f(x^*) - x^* - (f(x_t) - x_t)\right\|\|y_n - x_t\|^{q-1}.\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in (0,\delta)$, the following holds.

$$\left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle \le \left\langle f(x_t) - x_t, J_q(y_n - x_t)\right\rangle + \varepsilon.$$

Using (3.14), we obtain that

$$\limsup_{n\to\infty}\left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle \le 0. \qquad (3.15)$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$.

Using Lemma 2.2, we obtain that

$$\begin{aligned}\|y_n - x^*\|^q &\le \alpha_n \left\langle f(x_n) - x^*, J_q(y_n - x^*)\right\rangle \\ &+ (1-\alpha_n)\left\langle x_n - x^*, J_q(y_n - x^*)\right\rangle \\ &\le \alpha_n \left\langle f(x_n) - f(x^*), J_q(y_n - x^*)\right\rangle \\ &+ \alpha_n \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle\end{aligned}$$

$$+(1-\alpha_n)\left\langle x_n - x^*, J_q(y_n - x^*)\right\rangle$$

$$\leq \alpha_n \beta \left\| x_n - x^* \right\| \left\| y_n - x^* \right\|^{q-1} + \alpha_n \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle$$
$$+(1-\alpha_n)\left\| x_n - x^* \right\| \left\| y_n - x^* \right\|^{q-1}$$
$$= (1-\alpha_n(1-\beta))\left\| x_n - x^* \right\| \left\| y_n - x^* \right\|^{q-1}$$
$$+\alpha_n \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle$$
$$= (1-\alpha_n(1-\beta))\left(\frac{1}{q}\left\| x_n - x^* \right\|^q + \frac{q-1}{q}\left\| y_n - x^* \right\|^q \right)$$
$$+\alpha_n \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle,$$

which implies

$$\left\| y_n - x^* \right\|^q \leq (1-\alpha_n(1-\beta))\left\| x_n - x^* \right\|^q$$
$$+\alpha_n q \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle.$$

It follows that

$$\left\| x_{n+1} - x^* \right\|^q \leq \beta_n \left\langle x_n - x^*, J_q(x_{n+1} - x^*)\right\rangle$$
$$+(1-\beta_n)\left\langle W_n y_n - x^*, J_q(x_{n+1} - x^*)\right\rangle$$
$$\leq \beta_n \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|^{q-1}$$
$$+(1-\beta_n)\left\| W_n y_n - x^* \right\| \left\| x_{n+1} - x^* \right\|^{q-1}$$
$$\leq \beta_n \left\| x_n - x^* \right\| \left\| x_{n+1} - x^* \right\|^{q-1}$$
$$+(1-\beta_n)\left\| y_n - x^* \right\| \left\| x_{n+1} - x^* \right\|^{q-1}$$
$$\leq \beta_n \left(\frac{1}{q}\left\| x_n - x^* \right\|^q + \frac{q-1}{q}\left\| x_{n+1} - x^* \right\|^q \right)$$
$$+(1-\beta_n)\left(\frac{1}{q}\left\| y_n - x^* \right\|^q + \frac{q-1}{q}\left\| x_{n+1} - x^* \right\|^q \right)$$
$$\leq (1-\alpha_n(1-\beta_n)(1-\beta))\left\| x_n - x^* \right\|^q$$
$$+(1-\beta_n)\alpha_n q \left\langle f(x^*) - x^*, J_q(y_n - x^*)\right\rangle.$$

By using assumption, (3.16) and Lemma 2.6, we get that $x_n \to x^*$ as $n \to \infty$. This complete the proof. □

If $N = 1$, then we get the following corollary.

Corollary 3.2 *Let E be a real strictly convex and q-uniformly smooth Banach space with constant C_q. Let C be a nonempty closed and convex subset of E. Let $f: C \to C$ be a contractive mapping with the constant $\beta \in (0,1)$. Let $A: C \to 2^E$ be m-accretive operator and let $B: C \to E$ be α-inverse strongly accretive operator. Assume that $\Omega = (A+B)^{-1}0$ is nonempty. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 \in E; \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n; \\ x_{n+1} = \beta_n x_n + (1-\beta_n)[a_0 y_n + (1-a_0)J_{r_n}^A (I - r_n B) y_n], \ n \geq 1, \end{cases}$$

where $J_{r_n}^A = (I + r_n A)^{-1}$, $a_0 \in (0,1)$, $\{\alpha_n\}$, $\{\beta_n\}$ are real number sequences in $(0,1)$ and $\{r_{n,i}\} \subset (0,\infty)$. Suppose that the above control sequences satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < b \leq r_n \leq c < \left(\frac{q\alpha}{C_q} \right)^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$

for $n \geq 1$, where b and c are real numbers.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$, where $\bar{x} = Q_\Omega f(\bar{x})$ and Q_Ω is the unique sunny nonexpansive retraction of C onto Ω.

If $A_i \equiv 0$ for all $i = 1,2,3,...,N.$, then we get the following corollary

Corollary 3.3 *Let E be a real strictly convex and q-uniformly smooth Banach space with constant C_q. Let C be a nonempty closed and convex subset of E. Let $f: C \to C$ be a contractive mapping with the constant $\beta \in (0,1)$. Let $B_i: C \to E$ be α-inverse strongly accretive operator, for $i = 1,2,3,...,N$. Assume that $\Omega = \bigcap_{i=1}^{N} (B_i)^{-1}0$ is nonempty. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 \in E; \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n; \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\left[a_0 y_n + \sum_{i=1}^{N} a_i (y_n + r_{n,i} B_i y_n) \right], \end{cases}$$

for all $n \geq 1$, where $0 < a_m < 1$, for $m = 0,1,2,3,...,N$, $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$ and $\{r_{n,i}\} \subset (0,\infty)$. Suppose that the above control sequences satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < b \leq r_{n,i} \leq c < \left(\frac{q\alpha}{C_q} \right)^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_{n+1,i} - r_{n,i}| < \infty$

for $n \geq 1$ and $i = 1,2,3,...,N$; where b and c are real numbers.

Then $\{x_n\}$ *converges strongly to a point* $\overline{x} \in \Omega$, *where* $\overline{x} = Q_\Omega f(\overline{x})$ *and* Q_Ω *is the unique sunny nonexpansive retraction of* C *onto* Ω

If $B_i \equiv 0$ for all $i = 1,2,3,...,N.$, then we get the following corollary

Corollary 3.4 *Let* E *be a real strictly convex and* q-*uniformly smooth Banach space with constant* C_q. *Let* C *be a nonempty closed and convex subset of* E. *Let* $f : C \to C$ *be a contractive mapping with the constant* $\beta \in (0,1)$. *Let* $A_i : C \to 2^E$ *be m-accretive operator for* $i = 1,2,3,...,N$. *Assume that* $\Omega = \bigcap_{i=1}^{N} (A_i)^{-1}0$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 \in E; \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n; \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\left[a_0 y_n + \sum_{i=1}^{N} a_i J_{r_{n,i}}^{A_i} y_n \right], \ n \geq 1, \end{cases}$$

where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ *,for* $i = 1,2,3,...,N$ *, and* $0 < a_m < 1$, *for* $m = 0,1,2,3,...,N$, $\{\alpha_n\}$, $\{\beta_n\}$ *are real number sequences in* $(0,1)$ and $\{r_{n,i}\} \subset (0,\infty)$. *Suppose that the above control sequences satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1;$

(iii) $0 < b \leq r_{n,i} \leq c < \left(\frac{q\alpha}{C_q} \right)^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} \left| r_{n+1,i} - r_{n,i} \right| < \infty$

for $n \geq 1$ *and* $i = 1,2,3,...,N$, *where* b *and* c *are real numbers.*

Then $\{x_n\}$ *converges strongly to a point* $\overline{x} \in \Omega$, *where* $\overline{x} = Q_\Omega f(\overline{x})$ *and* Q_Ω *is the unique sunny nonexpansive retraction of* C *onto* Ω

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Generalized Identities for third order Pell Number, Pell-Lucas Number and Modified Pell Number

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Abstract

In this paper, we first presented the generalized Pell Number, Pell-Lucas Number and modified Pell Number, which are the recurrence relation by from the previous three terms. We have the Binet's formula generating functions and generating functions of all three sequences. We establish some of the interesting properties involving of sequences those sequences.

Keywords: Pell sequence Pell-Lucas sequence, Modified Pell sequence, Binet's formula

1. Introduction

We will refer to the sequence of occurrences starting in the recurring relationship from the previous second terms: Fibonacci and Lucas number. Because of their general characteristics, there are many interesting properties and application to almost every fields of science and art.

Previously, the sequence mentioned above is a sequence of positive integers that have been studied for many years. Many researchers have therefore examined about these sequences and also some properties that are excellent research topics. These sequences are examples of a sequences defined by a recurrence relation of second terms. It is well known that the Fibonacci sequence $\{F_n\}$, Lucas sequence $\{L_n\}$, Fibonacci-

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like $\{S_n\}$ and Generalized Fibonacci-Like $\{T_n\}$ are defined by the following recurring relationship
$F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$,
$L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$,
$S_n = S_{n-1} + S_{n-2}$, $S_0 = 2$, $S_1 = 2$ and
$T_n = T_{n-1} + T_{n-2}$, $T_0 = m$, $T_1 = m$, for all $n \geq 2$, where m is a positive integer, respectively [1-2], [8].

In a similar way, other recurrence sequences of important positive integers as well are the sequence of Pell, Pell-Lucas and modified Pell sequence. which those sequences are represented by $\{P_n\}$, $\{Q_n\}$ and $\{q_n\}$ defined by the following recurrence $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$, $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$, $q_0 = 1$, $q_1 = 1$ for all $n \geq 2$, respectively [3-5].

Also, the Pell, Pell-Lucas and modified Pell sequence expand to the negative subscript, which are defined by [7], [9].

$$P_{-n} = \frac{P_n}{(-1)^{n+1}}, \text{ for all } n \geq 1, \tag{1.1}$$

and

$$Q_{-n} = \frac{Q_n}{(-1)^{n}}, \text{ for all } n \geq 1, \tag{1.2}$$

and

$$q_{-n} = \frac{q_n}{(-1)^{n}}, \text{ for all } n \geq 1. \tag{1.3}$$

The properties of the sequence have received a lot of attention. Many sequences appear in literature, including Pell, Pell-Lucas and modified Pell. It is well-known that the proof uses Binet's formula [6].

Moreover, for the reasons mentioned above, the sequence has more interest and can be used with other work and has an interesting direction at present. Therefore, the researchers were inspired by the study of Pell, Pell-Lucas and modified Pell sequence.

2. Main Results

In this section, we formulate some third terms sum identities for Pell sequence $\{P_n\}$, Pell-Lucas sequence $\{Q_n\}$ and modified Pell sequence $\{q_n\}$ are present Catalan's identity, Cassini's identity, d'Ocagne's identity, Binet's formula and Generating function.

Definition 2.1: The Pell sequence $\{P_n\}$, The Pell – Lucas $\{Q_n\}$ and Modified Pell number $\{q_n\}$ are defined by

$$P_n = P_{n-1} + 3P_{n-2} + P_{n-3}, \text{ for all } n \geq 3, \tag{2.1}$$

with initial conditions $P_o = 0$, $P_1 = 1$ and $P_2 = 2$, $Q_n = Q_{n-1} + 3Q_{n-2} + Q_{n-3}$, for all

$$n \geq 3, \tag{2.2}$$

with initial conditions $Q_o = 2$, $Q_1 = 2$ and $Q_2 = 6$, and $q_n = q_{n-1} + 3q_{n-2} + q_{n-3}$ for all

$$n \geq 3, \tag{2.3}$$

with initial conditions $q_o = 1$, $q_1 = 1$ and $q_2 = 3$.

The first few terms of $\{P_n\}$ are $0,1,2,5,12,29,70$ and so on, and $\{Q_n\}$ are $2,2,6,14,34,82,198,478$ and so on, and $\{q_n\}$ are $1,1,3,7,17,41,99,239,577$ and so on. Similarly, the first few terms of $\{P_{-n}\}$, $\{Q_{-n}\}$ and $\{q_{-n}\}$ can be obtained from the equation (1.1), (1.2) and (1.3), $\{P_{-n}\}$ are $1,-2,$ $5,-12,-29,70$ and so on, $\{Q_{-n}\}$ are $-2,6,$ $-14,34,-82,198,-478$ and so on, and $\{q_{-n}\}$ are $-1,3,-7,17,-41,99,-239,$ 577 and so on, respectively. Each Pell sequence, Pell-Lucas sequence and modified Pell sequence are called Pell numbers, Pell-Lucas numbers and modified Pell number.

Furthermore, we will find Binet's formula to allow us to show the Pell number, Pell-Lucas number, and Modified Pell number, which has the following characteristic equation:

$$x^3 - x^2 - 3x - 1 = 0, \qquad (2.4)$$

where α, β and γ are the root of the equation, $\alpha = 1+\sqrt{2},\ \beta = 1-\sqrt{2}$, $\gamma = -1$ and $\alpha >$. $\beta > \gamma$ Note that $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$ and $\alpha\beta = \gamma$, respectively.

Next, we will say the equation is related to the repetitive relationship of (2.1), (2.2) and (2.3) defined by Theorem 2.2.

Theorem 2.2: (Binet's formula) The n^{th} Pell number, the n^{th} Pell – Lucas number and the n^{th} Modified Pell number are given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad (2.5)$$

and

$$Q_n = \alpha^n + \beta^n, \qquad (2.6)$$

and

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}, \qquad (2.7)$$

where n is not a negative integer and α, β, γ are the roots of the characteristic equation (2.4), which $\alpha > \beta > \gamma$.

Proof. Since equation (2.4) has three different roots, the number of P_n is defined by

$$P_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Let $n=0, n=1$ and $n=2$, then solve the system of linear equations, we will $c_1 = \frac{1}{\alpha-\beta}$, $c_2 = -\frac{1}{\alpha-\beta}$ and $c_3 = 0$, therefore

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Similarly, the number of Q_n is given by

$$Q_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Use the same method as above, then solve this linear equation, we obtain $c_1 = 1 = c_2$ and $c_3 = 0$, thence

$$Q_n = \alpha^n + \beta^n.$$

Similarly, the number $\{q_n\}$ is given by

$$q_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

for some coefficients c_1, c_2 and c_3. Let $n=0, n=1$ and $n=2$, we obtain $c_1 = \frac{1}{\alpha+\beta} = c_2$ and $c_3 = 0$, thence

$$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta}.$$

The proof completed. $\square$

Theorem 2.3: (Catalan's identity)

Let n is not a negative integer. Then

$$P_{n+r}P_{n-r} - P_n^2 = \gamma^{n-r+1}P_r^2, \qquad (2.8)$$

and

$$Q_{n+r}Q_{n-r} - Q_n^2 = \gamma^{n-r}Q_r^2 - 2\gamma^n, \quad (2.9)$$

and

$$q_{n+r}q_{n-r} - q_n^2 = \frac{\gamma^{n-r}q_{2r} - \gamma^n}{2}. \quad (2.10)$$

Proof. Since Binet's formula (2.5), we obtain

$$\begin{aligned} P_{n+r}P_{n-r} - P_n^2 &= \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha - \beta} \cdot \frac{\alpha^{n-r} - \beta^{n-r}}{\alpha - \beta} \\ &\quad - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 \\ &= -\frac{\alpha^{n-r}\beta^{n-r}\left(\alpha^r - \beta^r\right)^2}{\left(\alpha - \beta\right)^2} \\ &= \gamma^{n-r+1}P_r^2. \end{aligned}$$

Similarly, by Binet's formula (2.6), we obtain

$$\begin{aligned} Q_{n+r}Q_{n-r} - Q_n^2 &= \left(\alpha^{n+r} + \beta^{n+r}\right) \cdot \left(\alpha^{n-r} + \beta^{n-r}\right) \\ &\quad - \left(\alpha^n + \beta^n\right)^2 \\ &= \alpha^{n-r}\beta^{n-r}\left(\alpha^r - \beta^r\right)^2 \\ &= \alpha^{n-r}\beta^{n-r}\left(\alpha^{2r} + \beta^{2r}\right) \\ &\quad -2\alpha^{n-r}\beta^{n-r}\alpha^r\beta^r \\ &= \gamma^{n-r}Q_{2r} - 2\gamma^n. \end{aligned}$$

Similarly, by Binet's formula (2.7), we obtain

$$\begin{aligned} q_{n+r}q_{n-r} - q_n^2 &= \frac{\alpha^{n+r} + \beta^{n+r}}{\alpha + \beta} \cdot \frac{\alpha^{n-r} + \beta^{n-r}}{\alpha + \beta} \\ &\quad - \left(\frac{\alpha^n + \beta^n}{\alpha + \beta}\right)^2 \\ &= \frac{\alpha^{n+r}\beta^{n-r}}{\left(\alpha + \beta\right)^2} + \frac{\alpha^{n-r}\beta^{n+r}}{\left(\alpha + \beta\right)^2} \\ &\quad - \frac{2\alpha^n\beta^n}{\left(\alpha + \beta\right)^2} \\ &= \frac{\alpha^{n-r}\beta^{n-r}}{\alpha + \beta} \cdot \frac{\alpha^{2r} + \beta^{2r}}{\alpha + \beta} \\ &\quad - \frac{\alpha^n\beta^n}{\alpha + \beta} \\ &= \frac{\gamma^{n-r}q_{2r} - \gamma^n}{2}. \end{aligned}$$

The proof completed. □

Theorem 2.4: (Catalan's identity or Simpson's identity) Let n is not a negative integer. Then

$$P_{n+1}P_{n-1} - P_n^2 = \gamma^n, \quad (2.11)$$

and

$$Q_{n+1}Q_{n-1} - Q_n^2 = 8\gamma^{n-1}, \quad (2.12)$$

and

$$q_{n+1}q_{n-1} - q_n^2 = 2\gamma^{n-1}. \quad (2.13)$$

Proof. Taking $r = 1$ in Catalan's identity (2.8), (2.9) and (2.10), the proof completed. □

Theorem 2.5: (d'Ocagne's identity)

Let m, n are not a negative integer and $m > n$. Then

$$P_mP_{n+1} - P_{m+1}P_n = \gamma^nP_{m-n}, \quad (2.14)$$

and

$$Q_mQ_{n+1} - Q_{m+1}Q_n = 2\sqrt{2}\gamma^m \left(Q_{n-m} + 2\gamma\beta^{n-m}\right), \quad (2.15)$$

and

$$q_mq_{n+1} - q_{m+1}q_n = \sqrt{2}\gamma^m\left(q_{n-m} - \beta^{n-m}\right). \quad (2.16)$$

Proof. By Binet's formula (2.5), we have

$$\begin{aligned} P_mP_{n+1} - P_{m+1}P_n &= \frac{\alpha^m - \beta^m}{\alpha - \beta} \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &\quad - \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} \cdot \frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &= \frac{\alpha^m\beta^n - \alpha^n\beta^m}{\alpha - \beta} \end{aligned}$$

$$=\alpha^n\beta^n\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}$$
$$=\gamma^n P_{m-n}.$$

Similarly, by Binet's formula (2.6), we obtain

$$\begin{aligned}Q_mQ_{n+1}-Q_{m+1}Q_n &= \left(\alpha^m+\beta^m\right)\cdot\left(\alpha^{n+1}+\beta^{n+1}\right)\\ &\quad -\left(\alpha^{m+1}+\beta^{m+1}\right)\\ &\quad \cdot\left(\alpha^n+\beta^n\right)\\ &=\left(\alpha-\beta\right)\\ &\quad \left(\alpha^n\beta^m-\alpha^m\beta^n\right)\\ &=\left(\alpha-\beta\right)\alpha^m\beta^m\\ &\quad \left(\alpha^{n-m}-\beta^{n-m}\right)\\ &=2\sqrt{2}\gamma^m\\ &\quad \left(Q_{n-m}+2\gamma\beta^{n-m}\right).\end{aligned}$$

Similarly, by Binet's formula (2.7), we obtain

$$\begin{aligned}q_mq_{n+1}-q_{m+1}q_n &= \frac{\alpha^m+\beta^m}{\alpha+\beta}\cdot\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha+\beta}\\ &\quad -\frac{\alpha^{m+1}+\beta^{m+1}}{\alpha+\beta}\cdot\frac{\alpha^n+\beta^n}{\alpha+\beta}\\ &=\frac{\left(\alpha-\beta\right)\left(\alpha^n\beta^m-\alpha^m\beta^n\right)}{\left(\alpha+\beta\right)^2}\\ &=\left(\alpha-\beta\right)\alpha^m\beta^m\\ &\quad \frac{\left(\alpha^{n-m}-\beta^{n-m}\right)}{\left(\alpha+\beta\right)^2}\\ &=\sqrt{2}\gamma^m\left(q_{n-m}-\beta^{n-m}\right).\end{aligned}$$

The proof completed. □

Lemma 2.6 Let m, n are not a negative integer and $m>n$. Then

$$P_mP_{n+1}-P_{m+1}P_n=\frac{\gamma^n}{2\sqrt{2}}, \left(Q_{m-n}+2\gamma\beta^{m-n}\right) \quad (2.17)$$

and

$$P_mP_{n+1}-P_{m+1}P_n=\frac{\gamma^n}{\sqrt{2}}\left(q_{m-n}+\gamma\beta^{m-n}\right). \quad (2.18)$$

Proof. The Proof same as Theorem 2.5.

Lemma 2.7 Let m, n are not a negative integer and $m>n$. Then

$$Q_mQ_{n+1}-Q_{m+1}Q_n=8\gamma^m P_{n-m}, \quad (2.19)$$

and

$$Q_mQ_{n+1}-Q_{m+1}Q_n=4\sqrt{2}\gamma^m \left(q_{n-m}+\gamma\beta^{n-m}\right). \quad (2.20)$$

Proof. The Proof same as Theorem 2.5.

Lemma 2.8 Let m, n are not a negative integer and $m>n$. Then

$$q_mq_{n+1}-q_{m+1}q_n=2\gamma^m P_{n-m}, \quad (2.21)$$

and

$$q_mq_{n+1}-q_{m+1}q_n=\sqrt{2}\gamma^m. \left(q_{n-m}+\gamma\beta^{n-m}\right) \quad (2.22)$$

Proof. The Proof same as Theorem 2.5.

Theorem 2.9: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences, m and n are not a negative integer and $m>n$. Then

$$\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\alpha, \quad (2.23)$$

and

$$\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\alpha, \quad (2.24)$$

and

$$\lim_{n\to\infty}\frac{q_n}{q_{n-1}}=\alpha. \quad (2.25)$$

Proof. By Binet's formula (2.5), we have

$$\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n-\beta^n}{\alpha^{n-1}-\beta^{n-1}}$$

$$=\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}-\frac{1}{\beta}\cdot\left(\frac{\beta}{\alpha}\right)^n}.$$

But $\alpha>\beta$, then $\frac{\beta}{\alpha}<1$ and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0.$

Therefor $\lim_{n\to\infty}\frac{P_n}{P_{n-1}}=\alpha$.

Similarly, $\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}}$

$$=\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}=\alpha.$$

Similarly, $\lim_{n\to\infty}\frac{Q_n}{Q_{n-1}}=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n-1}+\beta^{n-1}}$

$$=\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{\frac{1}{\alpha}+\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^n}=\alpha.$$

This completes the proof. □

Lemma 2.10: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences and n is not a negative integer. Then

$$\lim_{n\to\infty}\frac{P_n}{Q_{n-1}}=\frac{\alpha}{\alpha-\beta}, \tag{2.26}$$

and

$$\lim_{n\to\infty}\frac{P_n}{q_{n-1}}=\frac{\alpha+\beta}{\alpha-\beta}, \tag{2.27}$$

and

$$\lim_{n\to\infty}\frac{Q_n}{q_{n-1}}=\frac{\alpha-\beta}{\alpha}. \tag{2.28}$$

Proof. The Proof same as Theorem 2.9.

In this paper, the generating function for Pell, Pell-Lucas and modified Pell sequences are given as a result, these sequence are seen and the coefficients of the power series of the corresponding generating function.

The generating function for Pell, Pell-Lucas and modified Pell sequences. We can also find the generating function for all three sequences by suppose that the Pell, Pell-Lucas and modified Pell sequences are the coefficients of a potential series center at the origin, and let us consider the corresponding analytic $\{P_n\},\{Q_n\}$ and $\{q_n\}$ of the function, which the function is as follows Theorem.

Theorem 2.11: Let $\{P_n\},\{Q_n\}$ and $\{q_n\}$ be Pell, Pell–Lucas and Modified Pell sequences and n is not a negative integer. Then the generating function defined by

$$P_n(x)=\frac{x}{1-2x-x^2}, \tag{2.29}$$

and

$$Q_n(x)=\frac{2-2x}{1-2x-x^2}, \tag{2.30}$$

and

$$q_n(x)=\frac{1-x}{1-2x-x^2}. \tag{2.31}$$

Proof. Let n is a not negative integer and

$$P_n(x)=P_0+P_1x+P_2x^2+P_3x^3+P_4x^4+\ldots+P_nx^n+\ldots.$$

Then

$$2xP_n(x) = 2P_0x + 2P_1x^2 + 2P_2x^3 + \ldots + 2P_nx^{n+1} + \ldots$$

$$x^2P_n(x) = P_0x^2 + P_1x^3 + P_2x^4 + \ldots + P_nx^{n+2} + \ldots$$

$$P_n(x) - 2xP_n(x) - x^2P_n(x) = x$$

$$(1 - 2x - x^2)P_n(x) = x.$$

Thus $P_n(x) = \sum_{n=0}^{\infty} P_nx^n = \dfrac{x}{1-2x-x^2}$.

Similarly, we have

$$Q_n(x) = Q_0 + Q_1x + Q_2x^2 + Q_3x^3 + Q_4x^4 + \ldots + Q_nx^n + \ldots.$$

Then, we obtain

$$2xQ_n(x) = 2Q_0x + 2Q_1x^2 + 2Q_2x^3 + \ldots + 2Q_nx^{n+1} + \ldots$$

$$x^2Q_n(x) = Q_0x^2 + Q_1x^3 + Q_2x^4 + \ldots + Q_nx^{n+2} + \ldots$$

$$Q_n(x) - 2xQ_n(x) - x^2Q_n(x) = 2 - 2x$$

$$(1 - 2x1 - x^2)Q_n(x) = 2 - 2x.$$

Thus $Q_n(x) = \sum_{n=0}^{\infty} Q_nx^n = \dfrac{2-2x}{1-2x-x^2}$.

Similarly, we have

$$q_n(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4 + \ldots + q_nx^n + \ldots.$$

Then, we write

$$2xq_n(x) = 2q_0x + 2q_1x^2 + 2q_2x^3 + \ldots + 2q_nx^{n+1} + \ldots$$

$$x^2q_n(x) = q_0x^2 + q_1x^3 + q_2x^4 + \ldots + q_nx^{n+2} + \ldots$$

$$q_n(x) - 2xq_n(x) - x^2q_n(x) = 1 - x$$

$$(1 - 2x - x^2)q_n(x) = 1 - x.$$

Thus $q_n(x) = \sum_{n=0}^{\infty} q_nx^n = \dfrac{1-x}{1-2x-x^2}$.

This completes the proof. □

From the Theorem 2.11 used to find the generating function. Next will be the polynomial of Pell, Pell–Lucas and Modified Pell sequences from the generating function, which using Maclaurin series helps to find the following theorem**.**

Theorem 2.12: The equality

$$\frac{x}{1-2x-x^2} = P_0 + P_1x + P_2x^2 + P_3x^3 + P_4x^4 + \ldots, \tag{2.32}$$

and

$$\frac{2-2x}{1-2x-x^2} = Q_0 + Q_1x + Q_2x^2 + Q_3x^3, + Q_4x^4 + \ldots, \tag{2.33}$$

and

$$\frac{1-x}{1-2x-x^2} = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4 + \ldots. \tag{2.34}$$

Proof. Since Maclaurin series, $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$. Let $f(x) = \dfrac{x}{1-2x-x^2}$ for all x, we obtain

$$f'(x) = -\frac{(-2-2x)x}{(1-2x-x^2)^2} + \frac{1}{1-2x-x^2}$$

$$f''(x) = -\frac{2(-2-2x)}{(1-2x-x^2)^2} + \frac{2x(-2-2x)^2}{(1-2x-x^2)^3} + \frac{2x}{(1-2x-x^2)^2}$$

$$f'''(x) = -\frac{6x(-2-2x)^3}{(1-2x-x^2)^4} - \frac{12x(-2-2x)}{(1-2x-x^2)^3}$$

$$+\frac{6\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^3}$$

$$+\frac{6}{\left(1-2x-x^2\right)^2}$$

$$f^{(4)}=\frac{24x\left(-2-2x\right)^4}{\left(1-2x-x^2\right)^5}+\frac{72x\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^4}$$

$$+\frac{24x}{\left(1-2x-x^2\right)^3}-\frac{24\left(-2-2x\right)^3}{\left(1-2x-x^2\right)^4}$$

$$-\frac{48\left(-2-2x\right)}{\left(1-2x-x^2\right)^3}$$

and so on.

Then

$$f\left(0\right)=0,\ f'\left(1\right)=1,\ f''\left(0\right)=4,$$

$f'''\left(0\right)=30,\ f^{(4)}\left(0\right)=288$ and so on,

$$\frac{x}{1-2x-x^2}=0+\frac{1}{1!}x+\frac{4}{2!}x^2+\frac{30}{3!}x^3$$

$$+\frac{288}{4!}x^4+\ldots.$$

Thus $\dfrac{x}{1-2x-x^2}=P_0+P_1x+P_2x^2+P_3x^3$

$$+P_4x^4+\ldots.$$

Similarly, $f\left(x\right)=\dfrac{2-2x}{1-2x-x^2}$ for all x, we obtain

$$f'\left(x\right)=-\frac{\left(-2-2x\right)\left(2-2x\right)}{\left(1-2x-x^2\right)^2}-\frac{2}{1-2x-x^2}$$

$$f''\left(x\right)=\frac{4\left(-2-2x\right)}{\left(1-2x-x^2\right)^2}$$

$$+\frac{2\left(2-2x\right)\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^3}$$

$$+\frac{2\left(2-2x\right)}{\left(1-2x-x^2\right)^2}$$

$$f'''\left(x\right)=-\frac{6\left(2-2x\right)\left(-2-2x\right)^3}{\left(1-2x-x^2\right)^4}$$

$$-\frac{12\left(2-2x\right)\left(-2-2x\right)}{\left(1-2x-x^2\right)^3}$$

$$-\frac{12\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^3}-\frac{12}{\left(1-2x-x^2\right)^2}$$

$$f^{(4)}=\frac{24\left(2-2x\right)\left(-2-2x\right)^4}{\left(1-2x-x^2\right)^5}$$

$$+\frac{72\left(2-2x\right)\left(-2-2x\right)^2}{\left(1-2x-x^2\right)^4}$$

$$+\frac{24\left(2-2x\right)}{\left(1-2x-x^2\right)^3}+\frac{48\left(-2-2x\right)^3}{\left(1-2x-x^2\right)^4}$$

$$+\frac{96\left(-2-2x\right)}{\left(1-2x-x^2\right)^3},$$

and so on.

Then

$$f\left(0\right)=2,\ f'\left(1\right)=2,\ f''\left(0\right)=12,$$

$f'''\left(0\right)=84,\ f^{(4)}\left(0\right)=816$ and so on,

$$\frac{2-2x}{1-2x-x^2}=2+\frac{2}{1!}x+\frac{12}{2!}x^2+\frac{84}{3!}x^3$$

$$+\frac{816}{4!}x^4+\ldots.$$

Thus

$$\frac{2-2x}{1-2x-x^2}=Q_0+Q_1x+Q_2x^2+Q_3x^3$$

$$+Q_4x^4+\ldots.$$

Similarly, $f(x)=\frac{1-x}{1-2x-x^2}$ for all x, we obtain

$$f'(x)=-\frac{(-2-2x)(1-x)}{(1-2x-x^2)^2}-\frac{1}{1-2x-x^2}$$

$$f''(x)=\frac{2(-2-2x)}{(1-2x-x^2)^2}+\frac{2(1-x)(-2-2x)^2}{(1-2x-x^2)^3}+\frac{2(1-x)}{(1-2x-x^2)^2}$$

$$f'''(x)=-\frac{6(1-x)(-2-2x)^3}{(1-2x-x^2)^4}-\frac{12(1-x)(-2-2x)}{(1-2x-x^2)^3}-\frac{6(-2-2x)^2}{(1-2x-x^2)^3}-\frac{6}{(1-2x-x^2)^2}$$

$$f^{(4)}=\frac{24(1-x)(-2-2x)^4}{(1-2x-x^2)^5}+\frac{72(1-x)(-2-2x)^2}{(1-2x-x^2)^4}+\frac{24(1-x)}{(1-2x-x^2)^3}+\frac{24(-2-2x)^3}{(1-2x-x^2)^4}+\frac{48(-2-2x)}{(1-2x-x^2)^3},$$

and so on.

Then

$f(0)=1,\ f'(1)=1,\ f''(0)=6,$

$f'''(0)=42,\ f^{(4)}(0)=408$ and so on,

$$\frac{1-x}{1-2x-x^2}=1+\frac{1}{1!}x+\frac{6}{2!}x^2+\frac{42}{3!}x^3+\frac{408}{4!}x^4+\ldots.$$

Thus $\frac{1-x}{1-2x-x^2}=q_0+q_1x+q_2x^2+q_3x^3+q_4x^4+\ldots.$

This completes the proof. □

Next, we will discuss the power series, including $\sum_{n=0}^{\infty}P_nx^n,\sum_{n=0}^{\infty}Q_nx^n$ and $\sum_{n=0}^{\infty}q_nx^n$ in $x-x_0$ but $x_0=0$ converges is always an interval center at $x=0$. We test convergence of such series by complete convergence and series converges if $|x|<\frac{1}{\alpha}$ and series diverges if $|x|>\frac{1}{\alpha}$. The series convergences absolutely open interval $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$, making the convergence radius equal $\frac{1}{\alpha}$, as the following Theorem.

Theorem 2.13: Let $\sum_{n=0}^{\infty}P_nx^n,\sum_{n=0}^{\infty}Q_nx^n$ and $\sum_{n=0}^{\infty}q_nx^n$ are power series. Then interval of convergence for the given series is $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence is $\frac{1}{\alpha}$.

Proof. By absolute convergence and Theorem 2.9, we have

$$\lim_{n\to\infty}\left|\frac{P_nx^n}{P_{n-1}x^{n-1}}\right|=\lim_{n\to\infty}|\alpha x|=\alpha|x|.$$

So the $\sum_{n=0}^{\infty}P_nx^n$ converges absolute if

$\lim_{n\to\infty}\left|\frac{P_n x^n}{P_{n-1}x^{n-1}}\right|<1$, then $\alpha|x|<1$ and $|x|<\frac{1}{\alpha}$.

The test value $x=\frac{1}{\alpha}$ or $x=-\frac{1}{\alpha}$, when replaced in series, will be

$\sum_{n=0}^{\infty}P_n\left(\frac{1}{\alpha}\right)^n=P_0+\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}+\ldots$ and

$$\sum_{n=0}^{\infty}P_n\left(-\frac{1}{\alpha}\right)^n=\sum_{n=0}^{\infty}(-1)^n P_n\left(\frac{1}{\alpha}\right)^n$$
$$=P_0-\frac{P_1}{\alpha}+\frac{P_2}{\alpha^2}-\ldots.$$

So both of diverge, therefore the interval of convergence for the given power series is $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence is $\frac{1}{\alpha}$. The $\sum_{n=0}^{\infty}Q_n x^n$ and $\sum_{n=0}^{\infty}q_n x^n$ will prove similarly, then the interval of convergence for the given power series are $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$ and the radius of convergence are $\frac{1}{\alpha}$, this completes the proof. □

From the theorem 2.13, found that the polynomial of Pell, Pell–Lucas and Modified Pell sequences are convergence to $\frac{1}{\alpha}$ and there is scope for convergence during the opening period $\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$.

Lemma 2.14: The equality

$$P_n(x)=P_0+P_1x+P_2x^2+P_3x^3+P_4x^4+\ldots, \tag{2.35}$$

and

$$Q_n(x)=Q_0+Q_1x+Q_2x^2+Q_3x^3+Q_4x^4+\ldots, \tag{2.36}$$

and

$$q_n(x)=q_0+q_1x+q_2x^2+q_3x^3+q_4x^4+\ldots. \tag{2.37}$$

for all $x\in\left(-\frac{1}{\alpha},\frac{1}{\alpha}\right)$.

Then $P_n(0)=P_0$, $Q_n(0)=Q_0$ and $q_n(0)=q_0$.

Proof. Taking $x=0$ in (2.35), (2.36) and (2.37). The proof completed. □

Furthermore, how to find $P_n(0)$, $Q_n(0)$ and $q_n(0)$. We also use the equation (2.29), (2.30) and (2.31) in the Theorem 2.11 by giving x = 0, then $P_n(0)=P_0$, $Q_n(0)=Q_0$ and $q_n(0)=q_0$, respectively.

3. Conclusion

In this article, first of all, we consider the generality of Pell, Pell-Lucas and modified Pell sequence by the result of the previous three terms. Then we introduced the Pell, Pell-Lucas and modified Pell number. Until finally, we got the Binet's formula and the generating function of

Pell, Pell-Lucas and the modified Pell sequence. In addition, we also received the information some important identities involving the terms of these sequences.

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Option Pricing for Jump in Volatility and Stochastic Intensity

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Abstract—**An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion model with stochastic volatility and stochastic intensity. The stochastic volatility follows the jump-diffusion. We find a formulation for the European-style option in terms of characteristic functions. The closed-form formulae of pricing for option are derived.**

Keywords— Jump-diffusion model; Stochastic volatility; Intensity; Characteristic functions

I. INTRODUCTION

All processes in this section will be defined in a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F} = (\mathcal{F}_t)_{0\le t\le T}$. An asset price model with stochastic volatility has been defined by Heston [1] which has the following dynamics

$$dS_t = S_t(\mu dt + \sqrt{v_t}\, dW_t^S),$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\, dW_t^v,$$

where S_t is the asset price, $\mu \in \Re$ is the rate of return of the asset, v_t is the volatility of asset returns, $\kappa > 0$ is a mean-reverting rate, $\theta \in \Re$ is the long term variance, $\sigma > 0$ is the volatility of volatility, W_t^S and W_t^v are standard Brownian motions corresponding to the processes S_t and v_t, respectively, with constant correlation ρ . In 1996, Bate [2] introduced the jump-diffusion by adding log normal jump Y_t to the Heston model. The model has the following form

$$dS_t = S_t(\mu dt + \sqrt{v_t}\, dW_t^S) + S_{t-}Y_t dN_t^S,$$

where N_t^S is the Poisson process which corresponds to the underlying asset S_t, Y_t is the jump size of asset price return with log normal distribution and S_{t-} means that there is a jump the value of the process before the jump is used on the left-hand side of the formula. M. Nonthiya et al. [3] extended Bate's work by adding jump in volatility. The model has the following equation

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\, dW_t^v + Z_t dN_t^v .$$

Fang [4] studies the Bates model with a stochastic jump intensity rate. J. Huang et al. [5] considered the double exponential jump model with stochastic volatility and stochastic intensity. In this paper, we will find a formulation for a European call option where the above models satisfy stochastic intensity which has the following equation.

$$d\lambda_t = \kappa_\lambda(\theta_\lambda - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}\, dW_t^\lambda .$$

where $\kappa_\lambda > 0$ is a mean-reverting rate of intensity, $\theta_\lambda \in \Re$ is the long term intensity, $\sigma_\lambda > 0$ is the volatility of intensity, a standard Brownian motions W_t^λ corresponding to the processes λ_t. The rest of the paper is organized as follows. In section 2, we briefly discuss the model descriptions for the option pricing. The relationship between stochastic differential equations and partial differential equations for the jump-diffusion process with jump stochastic volatility and stochastic intensity is presented in section 3. In section 4, a formula for a European call option in terms of characteristic functions is presented. The paper is concluded in Section 5.

II. MODEL DESCRIPTIONS

Assume a risk-neutral probability measure $\mathcal{M}$ exists. The asset price S_t under this measure follows a jump-diffusion process, v_t and λ_t have the following dynamics

$$dS_t = S_t\left((r - \lambda_t m)dt + \sqrt{v_t}\, dW_t^S\right) + S_{t-}Y_t dN_t^S \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\, dW_t^v + Z_t dN_t^v \tag{2}$$

$$d\lambda_t = \kappa_\lambda(\theta_\lambda - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}\, dW_t^\lambda \tag{3}$$

where r is the risk-free interest rate, N_t^S and N_t^v are independent Poisson processes with intensities λ_t and λ^v respectively. Y_t is the jump size of the asset price return with density $\phi_Y(y)$ and $E[Y_t] := m < \infty$ and Z_t is the jump size of the volatility with density $\phi_Z(z)$. Assume that the jump processes N_t^S and N_t^v are independent of standard Brownian motions W_t^S, W_t^v and W_t^λ. Brownian motions W_t^S, W_t^v independent of W_t^λ.

III. PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

Consider the process $\vec{X}_t = \left(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}\right)$ where $X_t^{(1)}, X_t^{(2)}$ and $X_t^{(3)}$ are processes in $\Re$ and satisfy the following equations:

$$dX_t^{(1)} = f_1(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, t)dt + g_1(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, t)dW_t^{(1)}$$

$$+X_{t-}^{(1)}Y_t dN_t^{(1)}$$
$$dX_t^{(2)} = f_2(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, t)dt + g_2(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, t)dW_t^{(2)}$$
$$+Z_t dN_t^{(2)}$$
$$dX_t^{(3)} = f_3(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, t)dt + g_3(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, t)dW_t^{(3)}$$

where f_1, f_2, f_3, g_1, g_2 and g_3 are all continuously differentiable, $W_t^{(1)}, W_t^{(2)}$ and $W_t^{(3)}$ are standard Brownian motions with $Corr[dW_t^{(1)}, dW_t^{(2)}] = \rho$, $N_t^{(1)}$ and $N_t^{(2)}$ are independent Poisson processes with constant intensities $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively.

Since every compound Poisson process can be represented as an integral form of a Poisson random measure [4] then the last term on the right hand side of equations can be written as follows:

$$\int_0^t X_{s-}^{(1)}Y_s dN_s^{(1)} = \sum_{n=1}^{N_t^{(1)}} X_{n-}^{(1)}Y_n = \int_0^t \int_{\Re} X_{s-}^{(1)} q J_Q(ds\,dq),$$
$$\int_0^t Z_s dN_s^{(2)} = \sum_{n=1}^{N_t^{(2)}} Z_n = \int_0^t \int_{\Re} r J_R(ds\,dr)$$

where Y_n are i.i.d. random variables with density $\phi_Y(y)$ and J_Q is a Poisson random measure of the process $Q_t = \sum_{n=1}^{N_t^{(1)}} Y_n$ with intensity measure $\lambda^{(1)}\phi_Y(dq)dt$, Z_n are i.i.d. random variables with density $\phi_Z(z)$, and J_R is a Poisson random measure of the process $R_t = \sum_{n=1}^{N_t^{(2)}} Z_n$ with intensity measure $\lambda^{(2)}\phi_Z(dr)dt$.

Let $U(x_1, x_2, x_3)$ be a bounded real-valued function and twice continuously differentiable with respect to x_1, x_2 and x_3 and

$$u(x_1, x_2, x_3, t) = E[U(X_T^{(1)}, X_T^{(2)}, X_T^{(3)}) \mid X_t^{(1)} = x_1, X_t^{(2)} = x_2, X_t^{(3)} = x_3]$$

By Dynkin formula [5], u is a solution of the partial integro-differential equation (PIDE)

$$0 = \frac{\partial u(x_1, x_2, x_3, t)}{\partial t} + \bar{\mathcal{A}}u(x_1, x_2, x_3, t)$$
$$+\lambda^{(1)}\int_{\Re}[u(x_1 + y, x_2, x_3, t) - u(x_1, x_2, x_3, t)]\phi_Y(y)dy$$
$$+\lambda^{(2)}\int_{\Re}[u(x_1, x_2 + z, x_3, t) - u(x_1, x_2, x_3, t)]\phi_Z(z)dz$$

subject to the final condition $u(x_1, x_2, x_3, T) = U(x_1, x_2, x_3)$. The notation $\bar{\mathcal{A}}$ is defined by

$$\bar{\mathcal{A}}u(x_1, x_2, x_3, t) = f_1\frac{\partial u(x_1, x_2, x_3, t)}{\partial x_1} + f_2\frac{\partial u(x_1, x_2, x_3, t)}{\partial x_2}$$
$$+f_3\frac{\partial u(x_1, x_2, x_3, t)}{\partial x_3} + \rho g_1 g_2\frac{\partial^2 u(x_1, x_2, t)}{\partial x_1 \partial x_2}$$
$$+\frac{1}{2}g_1^2\frac{\partial^2 u(x_1, x_2, t)}{\partial x_1^2} + \frac{1}{2}g_2^2\frac{\partial^2 u(x_1, x_2, t)}{\partial x_2^2}$$
$$+\frac{1}{2}g_3^2\frac{\partial^2 u(x_1, x_2, x_3, t)}{\partial x_3^2}$$

IV. A FORMULA FOR THE PRICE OF EUROPEAN CALL OPTION

Let C denote the price at time t of a European style call option on the current price of the underlying asset S_t with strike price K and expiration time T. The terminal payoff of a European call option on the underlying stock S_t with strike price K is $\max(S_T - K, 0)$. This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \le K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assume the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff

$$C(S_t, v_t, \lambda_t, t; K, T)$$
$$= e^{-r(T-t)}E_{\mathcal{M}}[\max(S_T - K, 0) \mid S_t, v_t, \lambda_t]$$
$$= e^{-r(T-t)}\left(\int_K^\infty S_T P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T - K\int_K^\infty P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T\right)$$
$$= S_t\left(\frac{1}{e^{r(T-t)}S_t}\int_K^\infty S_T P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T\right)$$
$$-Ke^{-r(T-t)}\int_K^\infty P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T$$
$$= S_t\left(\frac{1}{E_{\mathcal{M}}[S_T \mid S_t, v_t, \lambda_t]}\int_K^\infty S_T P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T\right)$$
$$-Ke^{-r(T-t)}\int_K^\infty P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T$$
$$= S_t P_1(S_t, v_t, \lambda_t, t; K, T) - Ke^{-r(T-t)}P_2(S_t, v_t, \lambda_t, t; K, T) \qquad (4)$$

where $E_{\mathcal{M}}$ is the expectation with respect to the risk-neutral probability measure, $P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)$ is the corresponding conditional density given (S_t, v_t, λ_t) and

$$P_1(S_t, v_t, \lambda_t, t; K, T) = \frac{1}{E_{\mathcal{M}}[S_T \mid S_t, v_t, \lambda_t]}\int_K^\infty S_T P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T$$

and

$$P_2(S_t, v_t, \lambda_t, t; K, T) = \int_K^\infty P_{\mathcal{M}}(S_T \mid S_t, v_t, \lambda_t)dS_T.$$

Assume that the asset price S_t, the volatility v_t and intensity λ_t satisfy (1), (2) and (3), respectively. We will compute the price of a European call option with strike price K and maturity T. We set $L_t = \ln S_t$, i.e., $S_t = e^{L_t}$. Denote $k = \ln K$ the logarithm of the strike price. By the jump-diffusion chain rule, $\ln S_t$ satisfies the SDE

$$d\ln S_t = (r - \lambda_t m - \frac{v_t}{2})dt + \sqrt{v_t}dW_t^S + \ln(1 + Y_t)dN_t^S. \qquad (5)$$

Applying Dynkin formula [5] for the price dynamics, we obtain the value of a European-style option, as a function of the stock log-return L_t denoted by

$$\begin{aligned}\tilde{C}(L_t, v_t, \lambda_t, t; k, T) &\equiv C(e^{L_t}, v_t, \lambda_t, t; e^k, T)\\ &= C(e^{\ln S_t}, v_t, \lambda_t, t; e^{\ln K}, T)\\ &= C(S_t, v_t, \lambda_t, t; K, T),\end{aligned}$$

i.e.,

$$\tilde{C}(l,v,\lambda,t;k,T) = e^{-r(T-t)} E_{\mathcal{M}}[\max(e^{L_T} - K, 0) \mid L_t = l, v_t = v, \lambda_t = \lambda]$$

and satisfies the following PIDE:

$$\begin{aligned}0 &= \frac{\partial \tilde{C}}{\partial t} + \bar{\mathcal{A}}[\tilde{C}](l,v,\lambda,t;k,T)\\ &+\lambda \int_{\Re} [\tilde{C}(l+y, v, \lambda, t; k, T) - \tilde{C}(l, v, \lambda, t; k, T)]\phi_Y(y)dy\\ &+\lambda^v \int_{\Re} [\tilde{C}(l, v+z, \lambda, t; k, T) - \tilde{C}(l, v, \lambda, t; k, T)]\phi_Z(z)dz \qquad (6)\end{aligned}$$

Here the operator $\bar{\mathcal{A}}$ is defined by

$$\begin{aligned}\bar{\mathcal{A}}[\tilde{C}](l,v,\lambda,t;k,T) &= (r - \lambda m - \frac{1}{2}v)\frac{\partial \tilde{C}}{\partial l} + \kappa(\theta - v)\frac{\partial \tilde{C}}{\partial v}\\ &+\kappa_\lambda(\theta_\lambda - \lambda)\frac{\partial \tilde{C}}{\partial \lambda} + \rho\sigma v \frac{\partial^2 \tilde{C}}{\partial l \partial v}\\ &+\frac{1}{2}v\frac{\partial^2 \tilde{C}}{\partial l^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \tilde{C}}{\partial v^2}\\ &+\frac{1}{2}\sigma_\lambda^2 \lambda \frac{\partial^2 \tilde{C}}{\partial \lambda^2} - r\tilde{C}.\end{aligned}$$

The last line of (4) becomes

$$\tilde{C}(l,v,\lambda,t;k,T) = e^l \tilde{P}_1(l,v,\lambda,t;k,T) - e^{k - r(T-t)}\tilde{P}_2(l,v,\lambda,t;k,T) \qquad (7)$$

where $\tilde{P}_j(l,v,\lambda,t;k,T) := P_j(e^l, v, \lambda, t; e^k, T)$, $j = 1, 2$.

The following lemma shows the relationship between $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of (7).

Lemma 1. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of (7) satisfy the following PIDEs*

$$\begin{aligned}0 &= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}[\tilde{P}_1](l,v,\lambda,t;k,T) + v\frac{\partial \tilde{P}_1}{\partial l} + \rho\sigma v \frac{\partial \tilde{P}_1}{\partial v} + (r - \lambda^S m)\tilde{P}_1\\ &+\lambda \int_{\Re} [(e^y - 1)\tilde{P}_1(l+y, v, \lambda, t : k, T)]\phi_Y(y)dy\end{aligned}$$

and subject to the boundary condition at expiration time $t = T$ ***;***

$$\tilde{P}_1(l,v,\lambda,T;k,T) = 1_{l>k}. \qquad (8)$$

$\tilde{P}_2$ *satisfies the equation*

$$0 = \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}[\tilde{P}_2](l,v,\lambda,t;k,T) + r\tilde{P}_2,$$

and subject to the boundary condition at expiration time $t = T$;

$$\tilde{P}_2(l,v,\lambda,T;k,T) = 1_{l>k}, \qquad (9)$$

The operator $\mathcal{A}$ is defined by

$$\begin{aligned}&\mathcal{A}[f](l,v,\lambda,t;k,T)\\ &:= (r - \lambda m - \frac{1}{2}v)\frac{\partial f}{\partial l} + \kappa(\theta - v)\frac{\partial f}{\partial v} + \kappa_\lambda(\theta_\lambda - \lambda)\frac{\partial f}{\partial \lambda} + \frac{1}{2}v\frac{\partial^2 f}{\partial l^2}\\ &+\rho\sigma v\frac{\partial^2 f}{\partial l \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f}{\partial v^2} + \frac{1}{2}\sigma_\lambda{}^2 \lambda \frac{\partial^2 f}{\partial \lambda^2} - rf\\ &+\lambda \int_{\Re} [f(l+y, v, \lambda, t; k, T) - f(l, v, \lambda, t; k, T)]\phi_Y(y)dy\\ &+\lambda^v \int_{\Re} [f(l, v+z, \lambda, t; k, T) - f(l, v, \lambda, t; k, T)]\phi_Z(z)dz. \qquad (10)\end{aligned}$$

Proof. We substitute (7) into (6) and separate it by assumed independent terms of $\tilde{P}_1$ and $\tilde{P}_2$. This gives two PIDEs for the risk-neutralized probability for $\tilde{P}_j(l,v,\lambda,t;k,T), j = 1,2$:

$$\begin{aligned}0 &= \frac{\partial \tilde{P}_1}{\partial t} + \left(r - \lambda^S m - \frac{1}{2}v\right)\left(\frac{\partial \tilde{P}_1}{\partial l} + \tilde{P}_1\right)\\ &+\kappa(\theta - v)\frac{\partial \tilde{P}_1}{\partial v} + \kappa_\lambda(\theta_\lambda - \lambda)\frac{\partial \tilde{P}_1}{\partial v} + \frac{1}{2}v\left(\frac{\partial^2 \tilde{P}_1}{\partial l^2} + 2\frac{\partial \tilde{P}_1}{\partial l} + \tilde{P}_1\right)\\ &+\rho\sigma v\left(\frac{\partial^2 \tilde{P}_1}{\partial l \partial v} + \frac{\partial \tilde{P}_1}{\partial v}\right) + \frac{1}{2}\sigma^2 v \frac{\partial^2 \tilde{P}_1}{\partial v^2} + \frac{1}{2}\sigma_\lambda{}^2 \lambda \frac{\partial^2 \tilde{P}_1}{\partial \lambda^2} - r\tilde{P}_1\\ &+\lambda \int_{\Re} [(e^y - 1)\tilde{P}_1(l+y, v, \lambda, t; k, T) + \tilde{P}_1(l+y, v, \lambda, t; k, T)\\ &-\tilde{P}_1(l, v, \lambda, t; k, T)]\phi_Y(y)dy\\ &+\lambda^v \int_{\Re} [\tilde{P}_1(l, v+z, \lambda, t; k, T) - \tilde{P}_1(l, v, \lambda, t; k, T)]\phi_Z(z)dz \qquad (11)\end{aligned}$$

subject to the boundary condition at the expiration time $t = T$ according to (8). By using the notation in (10), PIDE (11) becomes

$$\begin{aligned}0 &= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}[\tilde{P}_1](l,v,\lambda,t;k,T) + v\frac{\partial \tilde{P}_1}{\partial l} + \rho\sigma v\frac{\partial \tilde{P}_1}{\partial v} + (r - \lambda^S m)\tilde{P}_1\\ &+\lambda \int_{\Re} [(e^y - 1)\tilde{P}_1(l+y, v, \lambda, t; k, T)]\phi_Y(y)dy\\ &:= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}_1[\tilde{P}_1](l,v,t;k,T).\end{aligned}$$

For $\tilde{P}_2(l,v,t;k,T)$:

$$\begin{aligned}0 &= \frac{\partial \tilde{P}_2}{\partial t} + r\tilde{P}_2 + \left(r - \lambda m - \frac{1}{2}v\right)\frac{\partial \tilde{P}_2}{\partial l} + \kappa(\theta - v)\frac{\partial \tilde{P}_2}{\partial v} + \kappa_\lambda(\theta_\lambda - \lambda)\frac{\partial \tilde{P}_2}{\partial \lambda}\\ &+\frac{1}{2}v\frac{\partial^2 \tilde{P}_2}{\partial l^2} + \rho\sigma v \frac{\partial^2 \tilde{P}_2}{\partial l \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \tilde{P}_2}{\partial v^2} + \frac{1}{2}\sigma_\lambda^2 \lambda \frac{\partial^2 \tilde{P}_2}{\partial \lambda^2} - r\tilde{P}_2\\ &+\lambda \int_{\Re} [\tilde{P}_2(l+y, v, \lambda, t; k, T) - \tilde{P}_2(l, v, \lambda, t; k, T)]\phi_Y(y)dy\\ &+\lambda^v \int_{\Re} [\tilde{P}_2(l, v+z, \lambda, t; k, T) - \tilde{P}_2(l, v, \lambda, t; k, T)]\phi_Z(z)dz \qquad (12)\end{aligned}$$

subject to the boundary condition at the expiration time $t = T$ according to (9). Again, by using the notation in (10), PIDE (12) becomes

$$0 = \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}[\tilde{P}_2](l,v,\lambda,t;k,T) + r\tilde{P}_2$$

$$:= \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}_2[\tilde{P}_2](l,v,\lambda,t;k,T).$$

The proof is now completed.

For $j=1,2$ the characteristic functions for $\tilde{P}_j(l,v,\lambda,t;k,T)$, with respect to the variable k are defined by

$$f_j(l,v,\lambda,t;x,T) := -\int_{-\infty}^{\infty} e^{ixk} d\tilde{P}_j(l,v,\lambda,t;k,T),$$

with a minus sign to account for the negativity of the measure $d\tilde{P}_j$. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t} + \mathcal{A}_j[f_j](l,v,\lambda,t;k,T) = 0, \tag{13}$$

with the respective boundary conditions

$$f_j(l,v,\lambda,T;x,T) = -\int_{-\infty}^{\infty} e^{ixk} d\tilde{P}_j(l,v,\lambda,T;k,T) = e^{ixl}$$

The following lemma shows how to calculate the functions $\tilde{P}_1$ and $\tilde{P}_2$ as they appeared in Lemma 1.

Lemma 2. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ can be calculated by the inverse Fourier transforms of the characteristic function, i.e.,*

$$\tilde{P}_j(l,v,\lambda,t;k,T) = \frac{1}{2} + \frac{1}{\pi}\int_{0^+}^{+\infty} \mathrm{Re}\left[\frac{e^{-ixk} f_j(l,v,\lambda,t;x,T)}{ix}\right] dx,$$

for $j=1,2$ with $\mathrm{Re}[\cdot]$ denoting the real part of a complex number.

(i) The characteristic function f_1 is given by

$$f_1(l,v,\lambda,t;x,t+\tau) = \exp(g_1(\tau) + vh_1(\tau) + \lambda w_1(\tau) + ixl),$$

where

$$h_1(\tau) = \frac{(\eta_1^2 - \Delta_1^2)(e^{\Delta_1\tau} - 1)}{\sigma^2(\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau})}$$

$$g_1(\tau) = rix\tau - \left(\frac{\kappa\theta}{\sigma^2} + \frac{\kappa_\lambda\theta_\lambda}{\sigma_\lambda^2}\right) \times \left(2\ln\left(1 - \frac{(\Delta_1+\eta_1)(1-e^{-\Delta_1\tau})}{2\Delta_1}\right) + (\Delta_1+\eta_1)\tau\right) + \lambda^v \tau \int_{-\infty}^{\infty} (e^{zh_1(\tau)} - 1)\phi_Z(z)dz$$

$$w_1(\tau) = \frac{(\kappa_\lambda^2 - \xi_1^2)(e^{\xi_1\tau} - 1)}{\sigma_\lambda^2(-\kappa_\lambda + \xi_1 - (\kappa_\lambda - \xi_1)e^{\xi_1\tau})} + \tau\int_{-\infty}^{\infty} (e^{(ix+1)y} - 1)\phi_Y(y)dy$$

$$\eta_1 = \rho\sigma(ix+1) - \kappa, \quad \Delta_1 = \sqrt{\eta_1^2 - \sigma^2 ix(ix+1)}$$

$$\xi_1 = \sqrt{\kappa_\lambda^2 - 2\sigma_\lambda^2 m(ix-1)} \quad \text{and} \quad \tau = T - t.$$

(ii) The characteristic function f_2 is given by

$$f_2(l,v,\lambda,t;x,t+\tau) = \exp(g_2(\tau) + vh_2(\tau) + \lambda w_2(\tau) + ixl + r\tau),$$

where

$$h_2(\tau) = \frac{(\eta_2^2 - \Delta_2^2)(e^{\Delta_2\tau} - 1)}{\sigma^2(\eta_2 + \Delta_2 - (\eta_2 - \Delta_2)e^{\Delta_2\tau})},$$

$$g_2(\tau) = rix\tau + \lambda^v\tau\int_{-\infty}^{\infty} (e^{zh_2(\tau)} - 1)\phi_Z(z)dz$$

$$w_2(\tau) = \frac{(\kappa_\lambda^2 - \xi_2^2)(e^{\xi_2\tau} - 1)}{\sigma_\lambda^2(-\kappa_\lambda + \xi_2 - (\kappa_\lambda - \xi_2)e^{\xi_2\tau})} + \tau\int_{-\infty}^{\infty} (e^{(ix+1)y} - 1)\phi_Y(y)dy$$

$$\eta_2 = i\rho\sigma x - \kappa, \quad \Delta_2 = \sqrt{\eta_2^2 - \sigma^2 ix(ix-1)}$$

$$\xi_2 = \sqrt{\kappa_\lambda^2 - 2\sigma_\lambda^2 m(ix+1)} \quad \text{and} \quad \tau = T - t.$$

Proof. (i) To solve for the characteristic function explicitly, letting $\tau = T - t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1(l,v,\lambda,t;x,t+\tau) = \exp(g_1(\tau) + vh_1(\tau) + \lambda w_1(\tau) + ixl) \tag{14}$$

and the boundary condition

$$g_1(0) = w_1(0) = h_1(0) = 0.$$

This conjecture exploits the linearity of the coefficient in PIDE (13). Substituting (14) into (13) and after canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned}0 &= -g_1'(\tau) - vh_1'(\tau) - \lambda w_1'(\tau) + (r - \lambda m + \frac{1}{2}v)ix \\ &+ (\kappa(\theta - v) + \rho\sigma v)h_1(\tau) + \kappa_\lambda(\theta_\lambda - \lambda)w_1(\tau) - \frac{1}{2}vx^2 \\ &+ \rho\sigma vixh_1(\tau) + \frac{1}{2}\sigma^2 vh_1^2(\tau) + \frac{1}{2}\sigma_\lambda^2\lambda w_1^2(\tau) - \lambda m \\ &+ \lambda\int_{\Re}(e^{(ix+1)y} - 1)\phi_Y(y)dy + \lambda^v\int_{\Re}(e^{zh_1(\tau)} - 1)\phi_Z(z)dz.\end{aligned}$$

By separating the order v,λ and ordering the remaining terms, we can reduce it to ordinary differential equations

$$h_1'(\tau) = \frac{1}{2}\sigma^2 h_1^2(\tau) + (\rho\sigma(1+ix) - \kappa)h_1(\tau) + \frac{1}{2}ix - \frac{1}{2}x^2 \tag{15}$$

$$g_1'(\tau) = \kappa\theta h_1(\tau) + rix + \kappa_\lambda\theta_\lambda w_1(\tau) + \lambda^v\int_{-\infty}^{\infty}(e^{zh_1(\tau)} - 1)\phi_Z(z)dz \tag{16}$$

$$w_1'(\tau) = m(xi-1) - \kappa_\lambda w_1(\tau) + \frac{1}{2}\sigma_\lambda^2 w_1^2(\tau) + \int_{-\infty}^{\infty}(e^{(ix+1)y} - 1)\phi_Y(y)dy \tag{17}$$

Let $\eta_1 = \rho\sigma(ix+1) - \kappa$ and substitute it into (15). We get

$$\begin{aligned}h_1'(\tau) &= \frac{1}{2}\sigma^2\left(h_1^2 + \frac{2\eta_1}{\sigma^2}h_1 + \frac{1}{\sigma^2}ix(ix+1)\right) \\ &= \frac{1}{2}\sigma^2\left(h_1 + \frac{2\eta_1 + \sqrt{4\eta_1^2 - 4\sigma^2 ix(ix+1)}}{2\sigma^2}\right) \times \left(h_1 + \frac{2\eta_1 - \sqrt{4\eta_1^2 - 4\sigma^2 ix(ix+1)}}{2\sigma^2}\right) \\ &= \frac{1}{2}\sigma^2\left(h_1 + \frac{\eta_1 + \Delta_1}{\sigma^2}\right)\left(h_1 + \frac{\eta_1 - \Delta_1}{\sigma^2}\right),\end{aligned}$$

where $\Delta_1 = \sqrt{\eta_1^2 - \sigma^2 ix(ix+1)}$.

By the method of variable separation, we have

$$\frac{2dh_1}{\left(h_1+\frac{\eta_1+\Delta_1}{\sigma^2}\right)\left(h_1+\frac{\eta_1-\Delta_1}{\sigma^2}\right)}=\sigma^2 d\tau .$$

Using partial fractions, we get

$$\frac{1}{\Delta_1}\left(\frac{1}{h_1+\frac{\eta_1-\Delta_1}{\sigma^2}}-\frac{1}{h_1+\frac{\eta_1+\Delta_1}{\sigma^2}}\right)dh_1=d\tau .$$

Integrating both sides, we obtain

$$\ln\left(\frac{h_1+\frac{\eta_1-\Delta_1}{\sigma^2}}{h_1+\frac{\eta_1+\Delta_1}{\sigma^2}}\right)=\Delta_1\tau+C .$$

Using boundary condition $h_1(\tau=0)=0$, we get

$$C=\ln\left(\frac{\eta_1-\Delta_1}{\eta_1+\Delta_1}\right).$$

Solving for h_1, we obtain

$$h_1(\tau)=\frac{(\eta_1^2-\Delta_1^2)(e^{\Delta_1\tau}-1)}{\sigma^2(\eta_1+\Delta_1-(\eta_1-\Delta_1)e^{\Delta_1\tau})} .$$

Similarly to $w_1(\tau)$, we have $w_1(\tau)$ as in lemma. In order to solve $g_1(\tau)$ explicitly, we substitute $h_1(\tau)$ and $w_1(\tau)$ into (16) and integrate with respect to τ on both sides. Then we get

$$\begin{aligned} g_1(\tau) &= rxi\tau \\ &-\left(\frac{\kappa\theta}{\sigma^2}+\frac{\kappa_\lambda\theta_\lambda}{\sigma_\lambda^2}\right)\left(2\ln\left(1-\frac{(\Delta_1+\eta_1)(1-e^{-\Delta_1\tau})}{2\Delta_1}\right)+(\Delta_1+\eta_1)\tau\right) \\ &+\lambda^v\tau\int_{-\infty}^{\infty}(e^{zh_1(\tau)}-1)\phi_Z(z)dz . \end{aligned}$$

(ii). The details of the proof are similar to case (i). Hence, we have

$$f_2(l,v,\lambda,t;x,t+\tau)=\exp(g_2(\tau)+vh_2(\tau)+\lambda w_2(\tau)+ixl+r\tau) .$$

where $g_2(\tau)$, $h_2(\tau)$, $w_2(\tau)$, η_2, Δ_2 and ξ_2 are as given in the Lemma.

The following theorem gives a valuation formula for a European call option under the asset prices follow the jump-diffusion model with jump in stochastic volatility and stochastic intensity.

Theorem 3. *The value of a European call option of (4) is*

$$\tilde{C}(l,v,\lambda,t;k,T)=e^l\tilde{P}_1(l,v,\lambda,t;k,T)-e^{k-r(T-t)}\tilde{P}_2(l,v,\lambda,t;k,T)$$

where $\tilde{P}_1$ *and* $\tilde{P}_2$ *are given in Lemma 2.*

V. CONCLUSION

In this paper we investigate the asset prices follow the jump-diffusion model with jump in stochastic volatility and stochastic intensity. We obtain the closed-form of the pricing formulae for option.

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The generalized viscosity explicit rules for solving variational inclusion problems in Banach spaces

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The generalized viscosity explicit rules for solving variational inclusion problems in Banach spaces

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ABSTRACT

In this paper, we propose a generalized viscosity explicit method for finding zeros of the sum of two accretive operators in the framework of Banach spaces. The strong convergence theorem of such method is proved under some suitable assumption on the parameters. As applications, we apply our main result to the variational inequality problem, the convex minimization problem and the split feasibility problem. The numerical experiments to illustrate the behaviour of the proposed method including compare it with other methods are also presented.

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1. Introduction

The starting point of this paper, we consider initial value problem (IVP) in the following form:

$$x'(t) = f\big(x(t)\big), \quad x(t_0) = x_0. \tag{1}$$

In real life, many mathematical model have been formulated as this problem. It is well known that most of ordinary differential equations (ODEs) are not analytically solvable. Numerical methods have become a powerful method for numerically solving time-dependent ordinary and partial differential equations, as is required in computer simulations of physical processes such as groundwater flow and the wave equation. One of famous method is known as *implicit midpoint method* (or modified Euler's method) (see [1–3] for more detail). Given a time interval $[t_0, T]$, the method firstly computes the step size $h = (T - t_0)/N$, where N is the number of steps of h and select the mesh $\{t_n\}_{n=0}^{N}$ of time steps $t_n \in [t_0, T]$, through the formula $t_n = t_0 + nh$ for $n = 0, 1, \dots, N-1$. It provides to generate a sequence $\{y_n\}_{n=0}^{N}$ of approximation of solution at each time

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step t_n, i.e. $y_n \approx x(t_n)$. The implicit midpoint method (IMM) is given by the following procedure:

$$\begin{aligned} &y_0 = x_0, \\ &y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right), \quad n = 0, 1, \ldots, N-1. \end{aligned} \tag{2}$$

It is known that if $f : \mathbb{R}^M \to \mathbb{R}^M$ is a Lipschitz continuous and sufficiently smooth function, then the sequence $\{y_n\}$ converges to the exact solution of (1) as $h \to 0$ uniformly on $t \in [t_0, T]$. If the function f is written as $f(x) = x - g(x)$, then (2) becomes

$$\begin{aligned} &y_0 = x_0, \\ &y_{n+1} = y_n + h\left[\frac{y_n + y_{n+1}}{2} - g\left(\frac{y_n + y_{n+1}}{2}\right)\right], \quad n = 0, 1, \ldots, N-1 \end{aligned} \tag{3}$$

and the critical points of (1) is the fixed point problems $x = g(x)$.

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. We denote by I the identity operator on H. Let $T : C \to C$ be a non-linear mapping. The fixed points set of T is denoted by $F(T) := \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is called *non-expansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall\, x, y \in C.$$

A mapping $f : C \to C$ is called a *contraction*, if there exists constant $\theta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \theta\|x - y\| \quad \forall\, x, y \in C.$$

In recent years, several types of iterative method have been constructed for fixed point problems in various settings. One classical method, due to Mann' iteration [4] which is defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \quad \forall\, n \ge 0, \tag{4}$$

where T is a self-mapping on C and $\{\alpha_n\}$ is a sequence in $[0, 1]$. It is know that Mann's iteration process has only weak convergence.

Motivated by IMM (2) and Mann's iteration (4), Alghamdi et al. [5] introduced the following two algorithms for a non-expansive mapping T: for given $x_0 \in H$ and

$$x_{n+1} = x_n - t_n\left[\frac{x_n + x_{n+1}}{2} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right] \quad \forall\, n \ge 0, \tag{5}$$

$$x_{n+1} = (1 - t_n)x_n + t_nT\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall\, n \ge 0, \tag{6}$$

where $\{t_n\} \subset (0, 1)$. They proved that the above two algorithms converge weakly to a point in $F(T)$.

In 2015, Xu et al. [6] applied the viscosity approximation method introduced by Moudafi [7] to the IMM for a non-expansive mapping T. They proposed the following *viscosity implicit midpoint method*: for given $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right) \quad \forall\, n \geq 0, \tag{7}$$

where f is a contractive mapping on C and $\{\alpha_n\}$ is a sequence in $(0,1)$. It was proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a fixed point of T.

Later, Ke and Ma [8] improved the viscosity implicit midpoint method (7) by replacing the midpoint by any point of interval $[x_n, x_{n+1}]$. They introduced the following *generalized viscosity implicit method* to approximating the fixed point of a non-expansive mapping T: for given $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n x_n + (1-t_n)x_{n+1}) \quad \forall\, n \geq 0. \tag{8}$$

They also proved that the sequence $\{x_n\}$ generated by (8) converges strongly to a point in $F(T)$.

However, it is noted that the computation by implicit method is not a simple task in general because this method need to compute at every time steps and it can be much harder to implement. To overcome this difficulty, we consider the method so-called an *explicit midpoint method* (EMM) which given by the following finite difference scheme [9, 10]:

$$\begin{aligned} y_0 &= x_0, \\ \bar{y}_{n+1} &= y_n + hf(y_n), \\ y_{n+1} &= y_n + hf\left(\frac{y_n + \bar{y}_{n+1}}{2}\right) \quad \forall\, n \geq 0. \end{aligned} \tag{9}$$

It is generally remarked that the EMM (9) calculates the system status at a future time from the currently known system status while IMM (2) calculates the system status involving both the current state of the system and the later one (see [9, 11]).

In 2017, Marino et al. [12] combined the generalized viscosity implicit midpoint method (8) with the EMM (9) for solving the fixed point problem of a quasi-non-expansive mapping T. They introduced the following *generalized viscosity explicit midpoint method*: for any $x_1 \in C$ and

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)T(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1. \end{aligned} \tag{10}$$

They also showed that the sequence $\{x_n\}$ generated by (10) converges strongly to a fixed point of T under certain assumptions imposed on the parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$.

On the other hand, let us consider the following *variational inclusion problems*: find $x^* \in H$ such that

$$0 \in (A + B)x^*, \tag{11}$$

where $A : H \to H$ is an operator, $B : H \to 2^H$ is a set-valued operator and 0 is a zero vector in H. The solutions set of (11) is denoted by $(A + B)^{-1}0 := \{x \in H : 0 \in (A + B)x\}$. This problem includes, as special cases, convex programming, variational inequalities, equilibrium problem, split feasibility problem and minimization problem. To be more precise, some concrete problems in signal processing, image recovery, statistical regression and machine learning can be modelled mathematically as this form (see [13–16]).

One of the most successful methods for solving problem (11) is *forward–backward algorithm* (FBA) ([17–20]) which is given by $x_1 \in H$ and

$$x_{n+1} = (I + \lambda B)^{-1}(x_n - \lambda A x_n) \quad \forall\, n \geq 1, \tag{12}$$

where $\lambda > 0$. In the context of this method, the operators $(I + \lambda B)^{-1}$ and $I - \lambda A$ are often referred to as the backward and forward operators, respectively. However, this method has only weak convergence.

In order to obtain strong convergence result, Takahashi et al. [21] (see also [22]) proposed the following modified FBA based on Halpern's iteration: for any $u, x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n) \quad \forall\, n \geq 1, \tag{13}$$

where $A : H \to H$ is a monotone operator, $B : H \to 2^H$ is a maximal monotone operator and $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$ is a resolvent operator of B. They proved that the sequence $\{x_n\}$ generated by (13) converges strongly to a point in $(A + B)^{-1}0$.

López et al. [23] proposed the following modified FBA with error sequences $\{a_n\}, \{b_n\}$ in q-uniformly smooth and uniformly convex Banach spaces E: for given $u, x_1 \in E$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J^B_{\lambda_n}(x_n - \lambda_n(A x_n + a_n)) + b_n) \quad \forall\, n \geq 1. \tag{14}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}, \{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$. They proved that the sequence $\{x_n\}$ generated by (14) converges strongly to a point in $(A + B)^{-1}0$.

In [24], Cholamjiak proposed the following new general type of FBA for accretive operators with error $\{e_n\}$ in Banach spaces E: for given $u, x_1 \in E$ and

$$x_{n+1} = \alpha_n u + \eta_n x_n + \delta_n J^B_{\lambda_n}(x_n - \lambda_n A x_n) + e_n \quad \forall\, n \geq 1, \tag{15}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\eta_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \eta_n + \delta_n = 1$. He proved that the sequence $\{x_n\}$ generated by (15) converges strongly to a point in $(A + B)^{-1}0$ under some appropriate conditions.

Shehu and Cai [25] extended iterative method (13) by combining the viscosity approximation method and FBA in a uniformly smooth and uniformly convex Banach space E: for given $x_1 \in E$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n) \quad \forall\, n \geq 1, \tag{16}$$

where $f : E \to E$ is a contraction with a constant $\theta \in (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset (0,1)$. It was proved that the sequence $\{x_n\}$ generated by (16) converges strongly to a point in $(A+B)^{-1}0$ under some appropriate conditions.

In 2018, Chang et al. [26] proposed the following strong convergence theorem of a generalized viscosity implicit rules for solving the variational inclusion problem (11) in a q-uniformly smooth and uniformly convex Banach space.

Theorem 1.1: *Let E be a q-uniformly smooth and uniformly convex Banach space. Let $A : E \to E$ be an α-isa of order q and $B : E \to 2^E$ be an m-accretive operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $f : E \to E$ be a θ-contractive mapping with $\theta q \in (0,1)$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda}(I - \lambda A)(t_n x_n + (1-t_n)x_{n+1}) \quad \forall\, n \geq 1, \tag{17}$$

where $J^B_\lambda := (I+\lambda B)^{-1}$, κ_q is the q-uniform smoothness coefficient of E, $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ and λ is a positive real number satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C3) $0 < \epsilon \leq t_n \leq t_{n+1} < 1$;
(C4) $0 < \lambda \leq (\alpha q/\kappa_q)^{1/(q-1)}$.

Then $\{x_n\}$ converges strongly to $x^* = Q_{(A+B)^{-1}0} f(x^*)$, where $Q_{(A+B)^{-1}0}$ is a sunny non-expansive retraction of E onto $(A+B)^{-1}0$.

In this paper, motivated and inspired by the works of Chang et al. [26] and Marino et al. [12], we propose a generalized viscosity explicit method for solving the variational inclusion problem (11) in the framework of Banach spaces. We prove its strong convergence of the proposed algorithm under some suitable assumption on the parameters. As applications, we apply our main result to the variational inequality problem, the convex minimization problem and the split feasibility problem. Finally, we provide several numerical experiments to illustrate the behaviour of the proposed method and compare it with other methods. The result obtained in this paper improves and extends many known results in the literature.

2. Basic definitions and preliminaries

In this section, we collect some preliminary results which will be used throughout the paper.

Let E and E^* be a real Banach space and the dual space of E, respectively. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}.$$

The *modulus of smoothness* of E is the function $\theta_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by

$$\theta_E(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

Definition 2.1: Suppose that $p, q > 1$. A Banach space E is said to be

(1) *Uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$.
(2) *p-Uniformly convex* if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0,2]$.
(3) *Uniformly smooth* if $\lim_{\tau\to 0}\theta_E(\tau)/\tau = 0$.
(4) *q-Uniformly smooth* if there exists a $c_q > 0$ such that $\theta_E(\tau) \leq c_q\tau^q$ for all $\tau > 0$.

If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth. Further, E is p-uniformly convex (q-uniformly smooth) if and only if E^* is q-uniformly smooth (p-uniformly convex), where $p \geq 2$ and $1 < q \leq 2$ satisfy $1/p + 1/q = 1$. It is well known that a Hilbert space H is 2-uniformly smooth. Typical examples of both uniformly convex and uniformly smooth Banach spaces are ℓ_p and L_p spaces, where $p > 1$. More precisely, ℓ_p and L_p spaces are $\min\{p,2\}$-uniformly smooth for every $p > 1$.

The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^q, \|\bar{x}\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between elements of E and E^*.

In particular, $J_2 := J$ is called the *normalized duality mapping*. If E is smooth, then J_q is single-valued, which is denoted by j_q. If $E: = H$ is a real Hilbert space, then $J = I$.

Using the concept of sub-differentials, we know the following inequality:

Lemma 2.2 ([27]): *Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in E$, we have*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle, \tag{18}$$

where $j_q(x+y) \in J_q(x+y)$.

Definition 2.3: Let C a be non-empty, closed and convex subsets of E and Q be a mapping of E onto C. Then Q is said to be:

(1) *Sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$.
(2) *Retraction* if $Qx = x$ for all $x \in C$.
(3) A sunny non-expansive retraction if Q is sunny, non-expansive and a retraction from E onto C.

It is known that if $E := H$ is a real Hilbert space, then a sunny non-expansive retraction Q is coincident with the metric projection from E onto C. Moreover, if E is uniformly smooth and T is a non-expansive mapping of C into itself with $F(T) \neq \emptyset$, then $F(T)$ is a sunny non-expansive retract from E onto C (see [28]). We know that in a uniformly smooth Banach space E, a retraction $Q : E \to C$ is sunny and non-expansive, if and only if $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$ for all $x \in E$ and $y \in C$ (see [29]).

Let $A : E \to 2^E$ be a set-valued operator. We denote the domain of an operator A by $\mathcal{D}(A) = \{x \in E : Ax \neq \emptyset\}$. Let $q > 1$. An operator A is said to be *accretive* of order q if for each $x, y \in \mathcal{D}(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq 0, \quad u \in Ax \text{ and } v \in Ay.$$

An accretive operator A is said to be α*-inverse strongly accretive* (α-isa) of order q if for each $x, y \in \mathcal{D}(A)$, there exists $\alpha > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha\|Ax - Ay\|^q, \quad u \in Ax \text{ and } v \in Ay.$$

In a real Hilbert space H, $A : C \to H$ is called α-inverse strongly monotone (α-ism).

An accretive operator A is said to be *m-accretive* if and only if A is accretive and $\mathcal{R}(I + \lambda A) = E$ for all $\lambda > 0$, where $\mathcal{R}(I + \lambda A)$ is the range of $I + \lambda A$ (see [30]). For an accretive operator A, we can define a mapping $J_\lambda^A : \mathcal{R}(I + \lambda A) \to \mathcal{D}(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$ for each $\lambda > 0$. Such J_λ^A are called the *resolvents* of A for $\lambda > 0$.

Lemma 2.4 ([31]): *The following statements hold*:

(1) *If J_λ^A is a resolvent of A for $\lambda > 0$, then J_λ^A is a single valued non-expansive mapping with $F(J_\lambda^A) = A^{-1}0$, where $A^{-1}0 = \{x \in \mathcal{D}(A) : 0 \in Ax\}$.*
(2) *In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.*

Let $A : E \to E$ be an α-isa of order q and $B : E \to 2^E$ an m-accretive operator. In what follows, we shall use the following notation:

$$T_\lambda = J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \quad \lambda > 0.$$

Lemma 2.5 ([23]): *The following statements hold:*

(i) *For* $\lambda > 0$, $F(T_\lambda) = (A+B)^{-1}0$.
(ii) *For* $0 < \lambda \leq s$ *and* $x \in E$, $\|x - T_\lambda x\| \leq 2\|x - T_s x\|$.

Lemma 2.6 ([23]): *Let E be a uniformly convex and q-uniformly smooth Banach spaces. Assume that A is a single-valued* α*-isa of order q in E. Let* $r > 0$, *there exists a continuous, strictly increasing and convex function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(0) = 0$ *such that*

$$\|T_\lambda x - T_\lambda y\|^q \leq \|x-y\|^q - \lambda(\alpha q - \lambda^{q-1}\kappa_q)\|Ax - Ay\|^q \\ - \phi(\|(I - J_\lambda^B)(I - \lambda A)x - (I - J_\lambda^B)(I - \lambda A)y\|)$$

for all $x, y \in B_r := \{x \in E : \|x\| \leq r\}$, *where* κ_q *is the q-uniform smoothness coefficient of E. In particular, if* $0 < \lambda < (\alpha q/\kappa_q)^{1/(q-1)}$, *then* T_λ *is non-expansive.*

Lemma 2.7 ([32]): *Let C be a non-empty, closed and convex subset of a uniformly smooth Banach space E. Let* $T : C \to C$ *be a non-expansive self-mapping such that* $F(T) \neq \emptyset$ *and* $f : C \to C$ *be a contraction with coefficient* $\theta \in (0,1)$. *Then a net sequence defined by* $z_t = tf(z_t) + (1-t)Tz_t$, $\forall\ t \in (0,1)$ *converges strongly as* $t \to 0$ *to a point* $x^* \in F(T)$.

Lemma 2.8 ([33]): *Assume* $\{s_n\}$ *is a sequence of non-negative real numbers such that*

$$s_{n+1} \leq (1 - \delta_n)s_n + \delta_n \tau_n \quad \forall\, n \geq 1$$

and

$$s_{n+1} \leq s_n - \eta_n + \theta_n \quad \forall\, n \geq 1,$$

where $\{\delta_n\}$ *is a sequence in* $(0,1)$, $\{\eta_n\}$ *is a sequence of non-negative real numbers and* $\{\tau_n\}$, *and* $\{\theta_n\}$ *are real sequences such that*

(i) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(ii) $\lim_{n\to\infty} \theta_n = 0$;
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence of real numbers* $\{n_k\}$ *of* $\{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main result

In this section, we propose a generalized viscosity implicit rule for solving the variational inclusion problem (11) and prove its strong convergence theorem of the generated sequence by the proposed method.

Theorem 3.1: *Let E be a real uniformly convex and q-uniformly smooth Banach space E. Let $A : E \to E$ be an α-isa of order q and let $B : E \to 2^E$ is an m-accretive operator. Let $f : E \to E$ be a contraction with a constant $\theta \in (0,1)$. Assume that $(A+B)^{-1}0 \neq \emptyset$. For any $x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)J^B_{\lambda_n}(x_n - \lambda_n A x_n),$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J^B_{\lambda_n}(I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \tag{19}$$

where $\{\lambda_n\} \subset (0,\infty)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0,1)$ which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < (\alpha q/\kappa_q)^{1/(q-1)}$.

Then $\{x_n\}$ converges strongly to an element $x^* = Q_{(A+B)^{-1}}f(x^*)$, where $Q_{(A+B)^{-1}0}$ is a sunny non-expansive retraction of E onto $(A+B)^{-1}0$.

Proof: For each $n \geq 1$, put $T_n := J^B_{\lambda_n}(I - \lambda_n A)$. Let $z \in (A+B)^{-1}0$ and by the non-expansivity of T_n, we have

$$\begin{aligned}
\|\bar{x}_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(T_n x_n - T_n z)\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)\|T_n x_n - T_n z\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z)\| \\
&\leq \alpha_n \|f(x_n) - z\| + (1-\alpha_n)\|T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z\| \\
&\leq \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| \\
&\quad + (1-\alpha_n)\|t_n(z_n - z) + (1-t_n)(\bar{x}_{n+1} - z)\| \\
&\leq \alpha_n \theta \|x_n - z\| + (1-\alpha_n)(t_n\|x_n - z\| + (1-t_n)\|\bar{x}_{n+1} - z\|) \\
&\quad + \alpha_n \|f(z) - z\| \\
&= \alpha_n \theta \|x_n - z\| + (1-\alpha_n)t_n\|x_n - z\| + (1-\alpha_n)(1-t_n)\|x_n - z\| \\
&\quad + \alpha_n \|f(z) - z\| \\
&= (1-(1-\theta)\alpha_n)\|x_n - z\| + (1-\theta)\alpha_n \frac{\|f(z) - z\|}{1-\theta} \\
&\leq \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1-\theta}\right\}.
\end{aligned}$$

By induction, we obtain

$$\|x_n - z\| \leq \max\left\{\|x_1 - z\|, \frac{\|f(z) - z\|}{1-\theta}\right\} \quad \forall\, n \geq 1.$$

Hence $\{x_n\}$ is bounded. For each $n \geq 1$, put $z_n := t_n x_n + (1-t_n)\bar{x}_{n+1}$. Let $x^* = Q_{(A+B)^{-1}0} f(x^*)$. By Lemma 2.6, we have

$$\begin{aligned}\|T_n z_n - x^*\|^q &= \|J^B_{\lambda_n}(I - \lambda_n A)z_n - J^B_{\lambda_n}(I-\lambda_n A)x^*\|^q \\ &\leq \|z_n - x^*\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\ &\quad - \phi(\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n A x^*\|). \end{aligned} \tag{20}$$

In a similar way, we also have

$$\begin{aligned}\|T_n x_n - x^*\|^q &\leq \|x_n - x^*\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\ &\quad - \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|).\end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - x^*\|^q &\leq t_n\|x_n - x^*\|^q + (1-t_n)\|\bar{x}_{n+1} - x^*\|^q \\ &\leq t_n\|x_n - x^*\|^q + (1-t_n) \\ &\quad \times \left[\beta_n\|x_n - x^*\|^q + (1-\beta_n)\|T_n x_n - x^*\|^q\right] \\ &\leq t_n\|x_n - x^*\|^q + (1-t_n) \\ &\quad \times \Big[\beta_n\|x_n - x^*\|^q + (1-\beta_n)\Big(\|x_n - x^*\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q) \\ &\quad \times\|Ax_n - Ax^*\|^q - \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|)\Big)\Big] \\ &\leq \|x_n - x^*\|^q - (1-t_n)(1-\beta_n)\Big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\ &\quad + \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|)\Big). \end{aligned} \tag{21}$$

Substituting (21) into (20), we get

$$\begin{aligned}\|T_n z_n - x^*\|^q &\leq \|x_n - x^*\|^q - (1-t_n)(1-\beta_n)\Big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q) \\ &\quad \times \|Ax_n - Ax^*\|^q + \phi(\|x_n - \lambda_n A x_n - T_n x_n + \lambda_n A x^*\|)\Big) \\ &\quad - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\ &\quad - \phi(\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n A x^*\|). \end{aligned} \tag{22}$$

From Lemma 2.2 and (22), we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad = \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(T_n z_n - x^*)\|^q \\
&\quad = \|\alpha_n(f(x_n) - f(x^*)) + \alpha_n(f(x^*) - x^*) + (1-\alpha_n)(T_n z_n - x^*)\|^q \\
&\quad \le \|\alpha_n(f(x_n) - f(x^*)) + (1-\alpha_n)(T_n z_n - x^*)\|^q \\
&\qquad + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le \alpha_n\|f(x_n) - f(x^*)\|^q + (1-\alpha_n)\|T_n z_n - x^*\|^q \\
&\qquad + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le \alpha_n\|f(x_n) - f(x^*)\|^q + (1-\alpha_n)\big[\|x_n - x^*\|^q - (1-t_n)(1-\beta_n) \\
&\qquad \times \big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\
&\qquad + \phi(\|x_n - \lambda_n Ax_n - T_n x_n + \lambda_n Ax^*\|)\big) - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\
&\qquad - \phi(\|z_n - \lambda_n Az_n - T_n z_n + \lambda_n Ax^*\|)\big] + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le \big(1 - (1-\theta)\alpha_n\big)\|x_n - x^*\|^q - K_n\big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\
&\qquad + \phi(\|x_n - \lambda_n Ax_n - T_n x_n + \lambda_n Ax^*\|)\big) \\
&\qquad - (1-\alpha_n)\big(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\
&\qquad + \phi(\|z_n - \lambda_n Az_n - T_n z_n + \lambda_n Ax^q\|)\big) \\
&\qquad + q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle,
\end{aligned} \tag{23}
$$

where $K_n := (1-\alpha_n)(1-t_n)(1-\beta_n)$. We note that $\liminf_{n\to\infty} K_n > 0$ and $\liminf_{n\to\infty} \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q) > 0$. For each $n \ge 1$, we set

$$
\begin{aligned}
s_n &:= \|x_n - x^*\|^q, \\
\delta_n &:= (1-\theta)\alpha_n, \\
\tau_n &:= \frac{q}{1-\theta}\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle, \\
\eta_n &:= K_n(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ax_n - Ax^*\|^q \\
&\qquad + \phi(\|x_n - \lambda_n Ax_n - T_n x_n + \lambda_n Ax^*\|)) \\
&\qquad + (1-\alpha_n)(\lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Az_n - Ax^*\|^q \\
&\qquad + \phi(\|z_n - \lambda_n Az_n - T_n z_n + \lambda_n Ax^*\|)), \\
\theta_n &:= q\alpha_n\langle f(x^*) - x^*, j_q(x_{n+1} - x^*)\rangle.
\end{aligned}
$$

Then (23) reduces to the following formulae:

$$
s_{n+1} \le (1-\delta_n)s_n + \delta_n\tau_n \quad \forall\, n \ge 1 \tag{24}
$$

and

$$s_{n+1} \le s_n - \eta_n + \theta_n \quad \forall\, n \ge 1. \tag{25}$$

By $(C1)$, we see that $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\lim_{n\to\infty} \theta_n = 0$. In order to complete the proof, using Lemma 2.8, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies that $\limsup_{k\to\infty} \tau_{n_k} \le 0$ for any subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. So by our assumptions and the properties of ϕ, we obtain

$$\lim_{k\to\infty} \|Az_{n_k} - Ax^*\| = \lim_{k\to\infty} \|z_{n_k} - \lambda_{n_k} Az_{n_k} - T_{n_k} z_{n_k} + \lambda_{n_k} Ax^*\| = 0$$

and

$$\lim_{k\to\infty} \|Ax_{n_k} - Ax^*\| = \lim_{k\to\infty} \|x_{n_k} - \lambda_{n_k} Ax_{n_k} - T_{n_k} x_{n_k} + \lambda_{n_k} Ax^*\| = 0.$$

Consequently,

$$\lim_{k\to\infty} \|T_{n_k} z_{n_k} - z_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T_{n_k} x_{n_k} - x_{n_k}\| = 0. \tag{26}$$

Since $\liminf_{n\to\infty} \lambda_n > 0$, there is $\lambda > 0$ such that $\lambda_n \ge \lambda$ for all $n \ge 1$. In particular, $\lambda_{n_k} \ge \lambda$ for all $k \ge 1$. Then, by Lemma 2.5 (ii), we have

$$\|T_\lambda x_{n_k} - x_{n_k}\| \le 2\|T_{n_k} x_{n_k} - x_{n_k}\|.$$

From (26), we obtain

$$\lim_{k\to\infty} \|T_\lambda x_{n_k} - x_{n_k}\| = 0. \tag{27}$$

Let $z_t = tf(z_t) + (1-t)T_\lambda z_t$, $\forall t \in (0,1)$. Then it follows from Lemma 2.7 that $\{z_t\}$ converges strongly to a fixed point $x^* \in F(T_\lambda) = (A+B)^{-1}0$. From Lemma 2.2, we have

$$\begin{aligned}
\|z_t - x_{n_k}\|^q &= \|t(f(z_t) - x_{n_k}) + (1-t)(T_\lambda z_t - x_{n_k})\|^q \\
&\le (1-t)^q \|T_\lambda z_t - x_{n_k}\|^q + qt\langle f(z_t) - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\
&= (1-t)^q \|T_\lambda z_t - x_{n_k}\|^q + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle \\
&\quad + qt\langle z_t - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\
&\le (1-t)^q \big(\|T_\lambda z_t - T_\lambda x_{n_k}\| + \|T_\lambda x_{n_k} - x_{n_k}\|\big)^q \\
&\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt\|z_t - x_{n_k}\|^q \\
&\le (1-t)^q \big(\|z_t - x_{n_k}\| + \|T_\lambda x_{n_k} - x_{n_k}\|\big)^q \\
&\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt\|z_t - x_{n_k}\|^q,
\end{aligned}$$

which implies that

$$\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle \leq \frac{(1-t)^q}{qt}\left(\|z_t - x_{n_k}\| + \|T_\lambda x_{n_k} - x_{n_k}\|\right)^q + \frac{qt-1}{qt}\|z_t - x_{n_k}\|^q.$$

From (27), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle &\leq \frac{(1-t)^q}{qt}M + \frac{qt-1}{qt}M \\ &= \left(\frac{(1-t)^q + qt - 1}{qt}\right)M, \end{aligned} \tag{28}$$

where $M = \limsup_{k\to\infty}\|z_t - x_{n_k}\|^q$, $t \in (0,1)$. We see that $((1-t)^q + qt - 1)/qt \to 0$ as $t \to 0$. Since j_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_t \to x^*$, we have

$$\|j_q(x_{n_k} - z_t) - j_q(x_{n_k} - x^*)\| \to 0 \quad \text{as } t \to 0.$$

So we have

$$\begin{aligned}&|\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle - \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle| \\ &\quad = |\langle f(z_t) - f(x^*) + f(x^*) - x^* + x^* - z_t, j_q(x_{n_k} - z_t)\rangle \\ &\qquad - \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle| \\ &\quad = |\langle f(z_t) - f(x^*), j_q(x_{n_k} - z_t)\rangle + \langle f(x^*) - x^*, j_q(x_{n_k} - z_t)\rangle \\ &\qquad + \langle x^* - z_t, j_q(x_{n_k} - z_t)\rangle - \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle| \\ &\quad \leq |\langle f(x^*) - x^*, j_q(x_{n_k} - z_t) - j_q(x_{n_k} - x^*)\rangle| + |\langle f(z_t) - f(x^*), j_q(x_{n_k} - z_t)\rangle| \\ &\qquad + |\langle x^* - z_t, j_q(x_{n_k} - z_t)\rangle| \\ &\quad \leq \|f(x^*) - x^*\|\|j_q(x_{n_k} - z_t) - j_q(x_{n_k} - x^*)\| \\ &\qquad + (1+\theta)\|z_t - x^*\|\|x_{n_k} - z_t\|^{q-1}.\end{aligned}$$

Hence as $t \to 0$, we have

$$\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle \to \langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle.$$

From (28), as $t \to 0$, it follows that

$$\limsup_{k\to\infty}\langle f(x^*) - x^*, j_q(x_{n_k} - x^*)\rangle \leq 0. \tag{29}$$

On the other hand, we have

$$\begin{aligned}\|T_{n_k}z_n - x_{n_k}\| &\leq \|T_{n_k}z_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \\ &\leq \|T_{n_k}z_{n_k} - z_{n_k}\| + (1-t_{n_k})(1-\beta_{n_k})\|T_{n_k}x_{n_k} - x_{n_k}\| \\ &\leq \|T_{n_k}z_{n_k} - z_{n_k}\| + \|T_{n_k}x_{n_k} - x_{n_k}\|.\end{aligned}$$

This together with (26) implies that

$$\lim_{k\to\infty} \|T_{n_k} z_{n_k} - x_{n_k}\| = 0. \tag{30}$$

Further, we have

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\le \|x_{n_k+1} - T_{n_k} z_{n_k}\| + \|T_{n_k} z_{n_k} - x_{n_k}\| \\ &\le \alpha_{n_k}\|f(x_{n_k}) - T_{n_k} z_{n_k}\| + \|T_{n_k} z_{n_k} - x_{n_k}\|.\end{aligned}$$

This together with (30) implies

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{31}$$

Combining (29) and (31), we get

$$\limsup_{k\to\infty} \langle f(x^*) - x^*, j_q(x_{n_k+1} - x^*)\rangle \le 0. \tag{32}$$

This implies that $\lim_{k\to\infty} \tau_{n_k} \le 0$. Then, by Lemma 2.8, we conclude that $\lim_{n\to\infty} s_n = 0$. Hence $x_n \to x^*$ as $n \to \infty$. This completes the proof. ■

Remark 3.2: We point out main issue on our work as follows:

(1) The method of proof of our result is very different from ones of [12, 21, 26, 34–37]. In particular, we remove the assumptions '$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$' and '$0 < \epsilon \le t_n \le t_{n+1} < 1$' in Theorem 3.1 of [8, 26, 34]. Moreover, we remove the assumption $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ in Theorem 3.1 of [37].
(2) The method of proof of our result is more simple with respect to the result of Chang et al. [26].

From [38], we obtain the following results.

Corollary 3.3: *Let $E := \ell_q$ (or L_q) with $1 < q \le 2$. Let $A : E \to E$ be an α-isa of order q and let $B : E \to 2^E$ is an m-accretive operator. Let $f : E \to E$ be a contraction with a constant $\theta \in (0, 1)$. Assume that $(A + B)^{-1}0 \ne \emptyset$. For any $x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}\bar{x}_{n+1} &= \beta_n x_n + (1 - \beta_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) J^B_{\lambda_n}(I - \lambda_n A)(t_n x_n + (1 - t_n)\bar{x}_{n+1}) \ \forall n \ge 1,\end{aligned} \tag{33}$$

where $\{\lambda_n\} \subset (0, \infty)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$ which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1 - t_n)(1 - \beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < (\alpha q/\kappa_q)^{1/(q-1)}$, *where* $\kappa_q = (1+t_q^{q-1})/((1+t_q)^{q-1})$ *and* t_q *is the unique solution of the equation* $(q-2)t^{q-1} + (q-1)t^{q-2} - 1 = 0,\ 0 < t < 1$.

Then $\{x_n\}$ *converges strongly to an element* $x^* = Q_{(A+B)^{-1}}f(x^*)$, *where* $Q_{(A+B)^{-1}0}$ *is a sunny non-expansive retraction of* E *onto* $(A+B)^{-1}0$.

Corollary 3.4: *Let* $E := \ell_p$ *(or* L_p*) with* $2 \le p < \infty$*. Let* $A : E \to E$ *be an* α*-isa of order 2 and let* $B : E \to 2^E$ *is an m-accretive operator. Let* $f : E \to E$ *be a contraction with a constant* $\theta \in (0,1)$*. Assume that* $(A+B)^{-1}0 \ne \emptyset$*. For any* $x_1 \in E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda_n}(I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \ge 1, \end{aligned} \tag{34}$$

where $\{\lambda_n\} \subset (0,\infty)$*, and* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha/(p-1)$.

Then $\{x_n\}$ *converges strongly to an element* $x^* = Q_{(A+B)^{-1}}f(x^*)$, *where* $Q_{(A+B)^{-1}0}$ *is a sunny non-expansive retraction of* E *onto* $(A+B)^{-1}0$.

Corollary 3.5: *Let* H *be a Hilbert space* H*. Let* $A : H \to H$ *be an* α*-ism and let* $B : H \to 2^H$ *be a maximal monotone operator. Let* $f : H \to H$ *be a contraction mapping with a constant* $\theta \in (0,1)$*. Suppose that* $(A+B)^{-1}0 \ne \emptyset$*. For any* $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1-\beta_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) J^B_{\lambda_n}(I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \ge 1, \end{aligned} \tag{35}$$

where $\{\lambda_n\} \subset (0,2\alpha)$*, and* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha$.

Then $\{x_n\}$ *converges strongly to an element* $x^* = P_{(A+B)^{-1}0}f(x^*)$, *where* $P_{(A+B)^{-1}0}$ *is a metric projection of* H *onto* $(A+B)^{-1}0$.

4. Some applications

4.1. Application to variational inequality problem

Let C be a non-empty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear monotone operator. The *variational inequality problem* (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, z - x^* \rangle \geq 0 \quad \forall\, z \in C. \tag{36}$$

The set of solutions of VIP is denoted by $VI(C, A)$. Let i_C be an indicator function of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C. \end{cases} \tag{37}$$

Denote N_C the normal cone of C, i.e.

$$N_C(u) = \{z \in H : \langle z, v - u \rangle \leq 0,\ \forall\, v \in C\}.$$

It is also known that i_C is proper convex and lower semi continuous function and sub-differential ∂i_C is maximal monotone operator (see [39]). We define the resolvent operator $J_\lambda^{\partial i_C}$ of i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) := (I + \lambda \partial i_C)^{-1}(x) \quad \forall\, x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{z \in H : i_C(v) + \langle z,\ v - u \rangle \leq i_C(u),\ \forall\, u \in H\} \\ &= \{z \in H : \langle z, v - u \rangle \leq 0,\ \forall v \in C\} = N_C(u), \quad u \in C. \end{aligned}$$

So we have

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) &\Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, v - u \rangle \leq 0 \quad \forall\, v \in C \\ &\Leftrightarrow u = P_C(x), \end{aligned}$$

where P_C is the metric projection from H onto C. Further, we also have $(A + \partial i_C)^{-1}0 = VI(C, A)$ (see [37]).

If we set $B = \partial i_C$ in Theorem 3.1, then we obtain the following result.

Theorem 4.1: *Let $A : C \to H$ be an α-ism such that $VI(C, A) \neq \emptyset$. Let $f : C \to C$ be a contraction with a constant $\theta \in (0, 1)$. For any $x_1 \in C$, let $\{x_n\}$ be a sequence* *generated by*

$$\bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)P_C(x_n - \lambda_n A x_n),$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_C(I - \lambda_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \tag{38}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$.

Then $\{x_n\}$ *converges strongly to a point in* $VI(C, A)$.

4.2. Application to convex minimization problem

Let $g : H \to \mathbb{R}$ be a convex smooth function and $h : H \to \mathbb{R}$ be a proper convex and lower semicontinuous function. The *convex minimization problem* is to find $x^* \in H$ such that

$$g(x^*) + h(x^*) = \min_{x\in H}\{g(x) + h(x)\}. \tag{39}$$

By Fermat's rule, it is known that the problem (39) is equivalent to the problem of finding $x^* \in H$ such that

$$0 \in \nabla g(x^*) + \partial h(x^*),$$

where ∇g is a gradient of g and ∂h is a subdifferential of h. It is also known if ∇g is $(1/\alpha)$-Lipschitz continuous, then it is also α-ism (see [40]). In fact, we can set $A = \nabla g$ and $B = \partial h$ in Theorem 3.1. So we obtain the following result.

Theorem 4.2: *Let* $g : H \to \mathbb{R}$ *be a convex and differentiable function with* $(1/\alpha)$-*Lipschitz continuous gradient* ∇g *and let* $h : H \to \mathbb{R}$ *be a convex and lower semicontinuous function such that* $g + h$ *attains a minimizer. Let* $f : H \to H$ *be a contraction with a constant* $\theta \in (0,1)$. *For any* $x_1 \in H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)J_{\lambda_n}^{\partial h}(x_n - \lambda_n \nabla g(x_n)),$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{\lambda_n}^{\partial h}(I - \lambda_n \nabla g)(t_n x_n + (1-t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \tag{40}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\alpha$.

Then $\{x_n\}$ *converges strongly to minimizer of* $g + h$.

4.3. Application to split feasibility problem

Let C and Q be non-empty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a linear bounded operator with its adjoint T^*. The *split feasibility problem* (SFP) is to find

$$x^* \in C \quad \text{such that } Tx^* \in Q. \tag{41}$$

The SFP can be used to model the intensity-modulated radiation therapy (see [41–43]). To solve the SFP (41), we can rewrite it as the following convexly

Table 1. Comparison of Algorithm (19), Algorithm (17), Algorithm (15) and Algorithm (16) for Example 5.1.

		Algorithm (19)	Algorithm (17)	Algorithm (16)	Algorithm (15)
Case 1	No. of Iter.	34	43	44	1806
Case 2	No. of Iter.	34	43	44	803

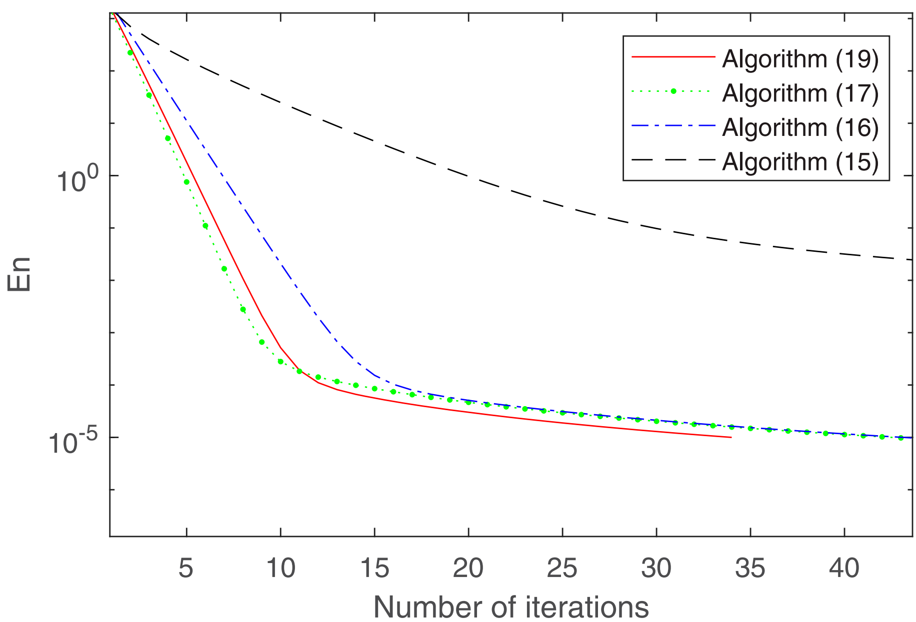

Figure 1. The error plotting of iterations in Case 1.

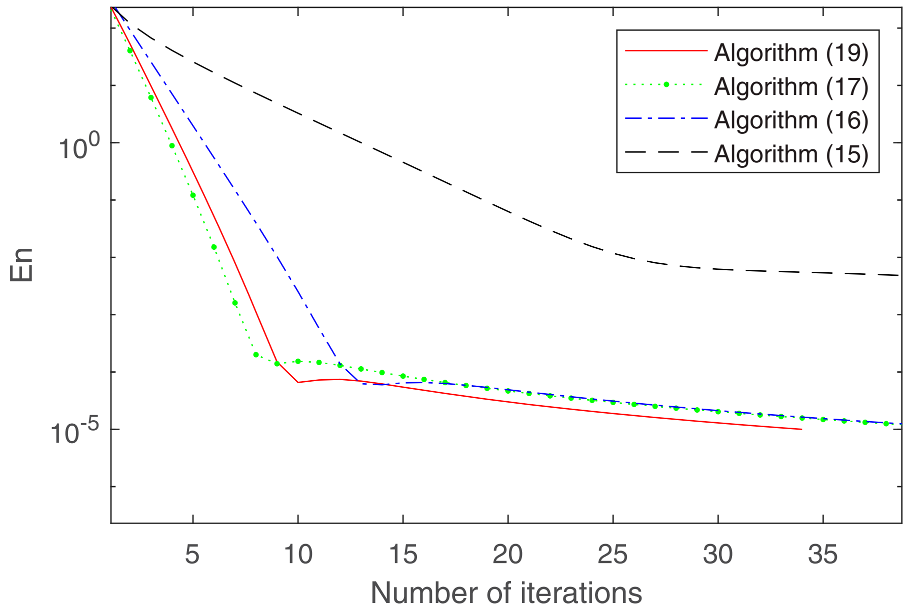

Figure 2. The error plotting of iterations in Case 2.

Table 2. Comparison of Algorithm (19), Algorithm (16) and Algorithm (15) for Example 5.3.

		Algorithm (19)	Algorithm (16)	Algorithm (15)
Case 1	No. of Iter.	2028	4052	12,204
	CPU	0.4311	0.4707	1.4134
Case 2	No. of Iter.	3776	7547	22,718
	CPU	13.8184	13.9072	41.6642
Case 3	No. of Iter.	7347	14,688	44,201
	CPU	100.2134	100.3403	302.2716

constrained minimization problem:

$$\min_{x\in C} g(x),$$

where $g(x) := \frac{1}{2}\|(I-P_Q)Tx\|^2$. Note that the function g is differentiable convex and has a Lipschitz gradient given by $\nabla g = T^*(I-P_Q)T$. Further, ∇g is $1/\|T\|^2$-ism, where $\|T\|^2$ is the spectral radius of T^*T (see [13]). Thus, we have the SFP equivalent to the variational inclusion problem (11) with $A = \nabla g$ and $B = \partial i_C$. It follows that

$$\begin{aligned} 0 \in \nabla g(x^*) + \partial i_C(x^*) &\Leftrightarrow 0 \in x^* + \lambda\partial i_C(x^*) - (x^* - \lambda\nabla g(x^*)) \\ &\Leftrightarrow x^* - \lambda\nabla g(x^*) \in x^* + \lambda\partial i_C(x^*) \\ &\Leftrightarrow x^* = (I + \lambda\partial i_C)^{-1}(x^* - \lambda\nabla g(x^*)) \\ &\Leftrightarrow x^* = P_C(x^* - \lambda\nabla g(x^*)). \end{aligned}$$

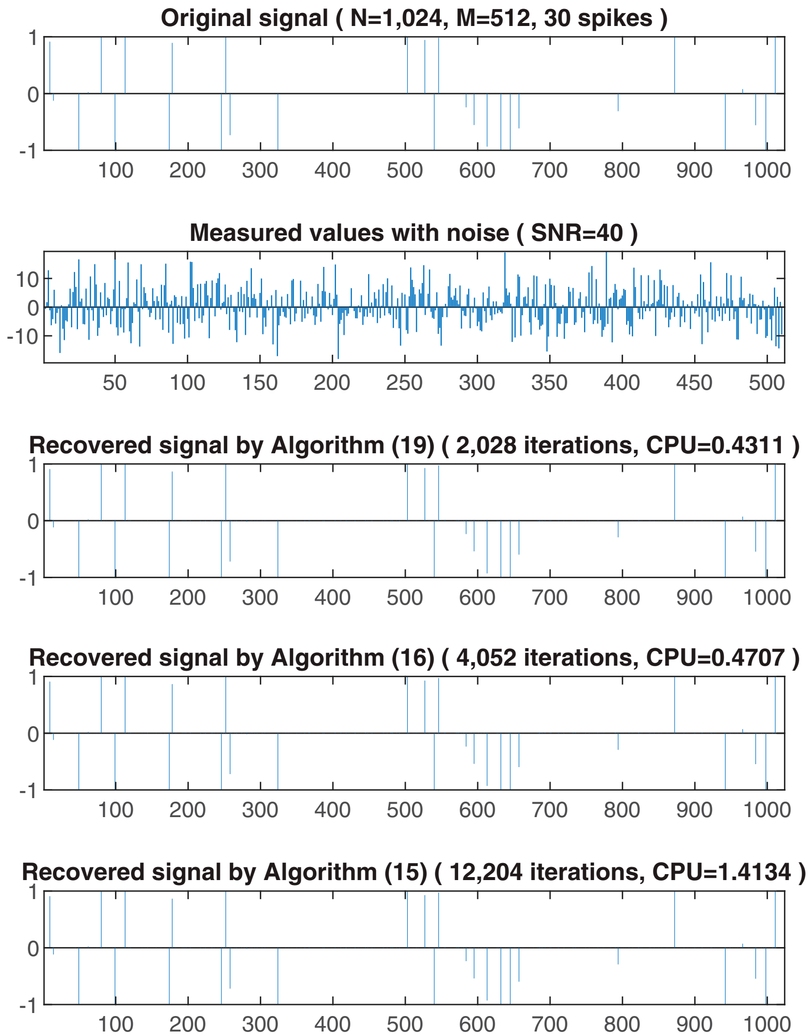

Figure 3. Comparison of recovered signal by using different algorithms in Case 1.

Theorem 4.3: *Let C and Q be non-empty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator with its adjoint T^* and $T \neq 0$. Let $f : C \to C$ be a contraction with a constant $\theta \in (0, 1)$. Suppose that the solution sets of SFP is non-empty. For any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} \bar{x}_{n+1} &= \beta_n x_n + (1 - \beta_n) P_C(x_n - \lambda_n T^*(I - P_Q)Tx_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) P_C(I - \lambda_n T^*(I - P_Q)T) \\ &\quad \times (t_n x_n + (1 - t_n)\bar{x}_{n+1}) \quad \forall\, n \geq 1, \end{aligned} \tag{42}$$

where $\{\lambda_n\} \subset (0, 2/\|T\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0, 1)$ which satisfy the following conditions:

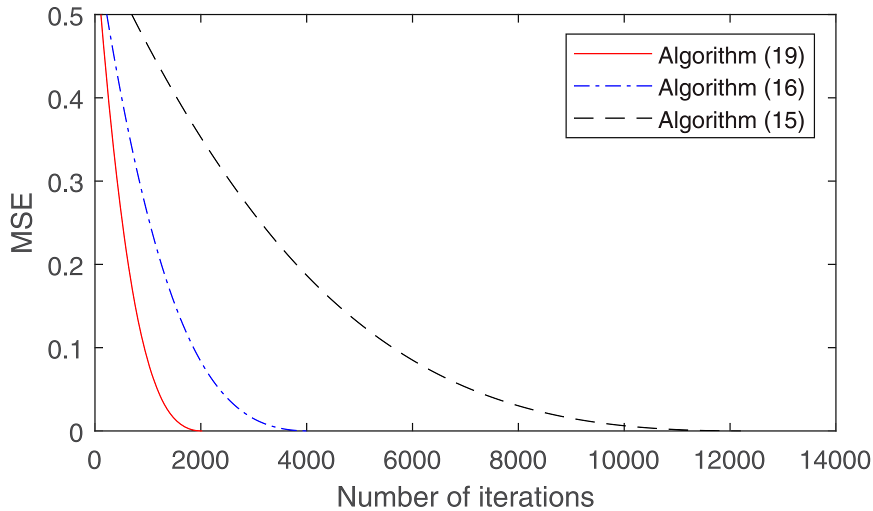

Figure 4. MSE versus number of iterations in Case 1.

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;
(C3) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 2/\|T\|^2$.

Then $\{x_n\}$ *converges strongly to solution of SFP.*

5. Numerical experiments

In this section, we provide numerical experiments to illustrate the behaviour of the our Algorithm (19) and also compare it with Algorithm (17) in [26], Algorithm (15) in [24] and Algorithm (16) in [25].

Example 5.1: We consider the example in an infinite dimensional Banach spaces outside Hilbert spaces which is taken from [24] (see also [44]). Let $E = \ell_3$ and $x = (x_1, x_2, x_3, \ldots) \in \ell_3$. Let $A, B : \ell_3 \to \ell_3$ be defined by

$$Ax = 2x + (1, 1, 1, 0, 0, 0, 0, \ldots) \quad \text{and} \quad Bx = 5x \quad \text{for } x \in \ell_3.$$

It is to see that A is 1/2-isa of order 2 and B is an m-accretive operator with $\mathcal{R}(I + \lambda B) = \ell_3$ for all $\lambda > 0$. Moreover,

$$J_\lambda^B(x - \lambda Ax) = \frac{1-2\lambda}{1+5\lambda}x - \frac{\lambda}{1+5\lambda}(1, 1, 1, 0, 0, 0, 0, \ldots),$$

for all $x \in \ell_3$. It is not difficult to check that $(A+B)^{-1}0 = \{(-\frac{1}{7}, -\frac{1}{7}, -\frac{1}{7}, 0, 0, 0, 0, \ldots)\}$.

Since, in ℓ_3, we have $q = 2$ and $\kappa_2 = 2$. Due to $\alpha = \frac{1}{2}$, then we can choose $\lambda_n = \frac{1}{10}$ for all $n \in \mathbb{N}$. We take $\alpha_n = (1/2n)$, $\beta_n = 1/(3(n+1))$, $\delta_n = n/(3(n+3))$,

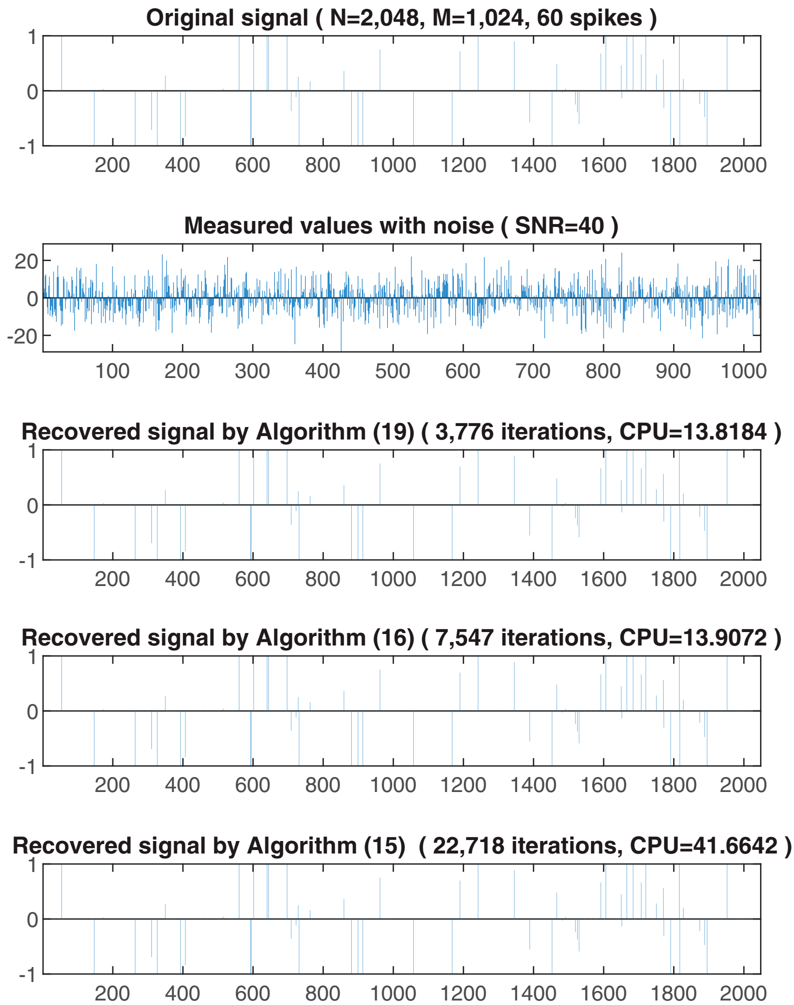

Figure 5. Comparison of recovered signal by using different algorithms in Case 2.

$\eta_n = 1 - (1/2n) - (n/(3(n+3)))$, $t_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ and $f(x) = x/2$ in those algorithms. In our numerical experiments, we consider the following two cases of starting point x_1:

Case 1 : $x_1 = (71.23, -42.51, -1.42, 0, 0, 0, \ldots)$;

Case 2 : $x_1 = (-27.53, -22.47, 4.64, 0, 0, 0, \ldots)$.

Let u be randomly generated in ℓ_3. We choose the stopping criterion is $E_n = \|x_{n+1} - x_n\| < 10^{-5}$. The numerical results are reported in Table 1 and Figures 1 and 2.

Remark 5.2: From Table 1 and Figures 1 and 2, we see that our Algorithm (19) has a number of iterations less than Algorithm (17) of Chang et al. [26], Algorithm (16) of Shehu and Cai [25] and Algorithm (15) of Cholamjiak [24].

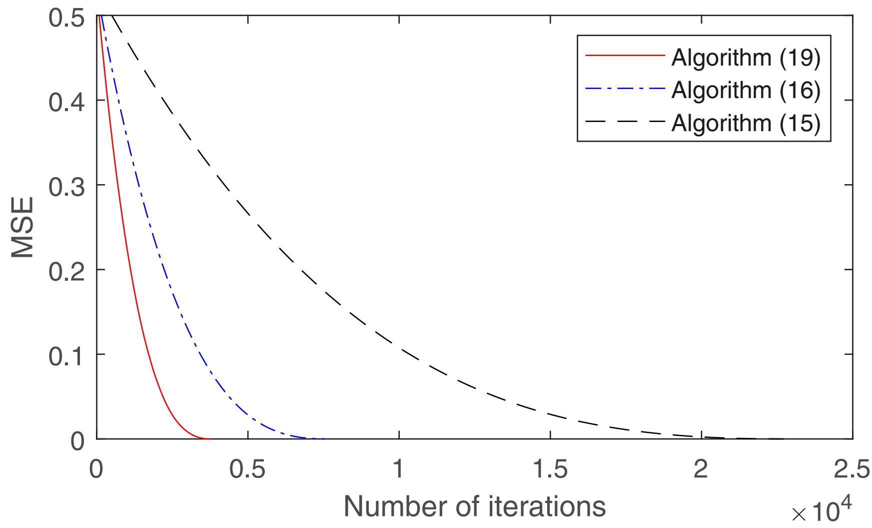

Figure 6. MSE versus number of iterations in Case 2.

It is shown that our proposed algorithm has good convergence behaviour.

Example 5.3: In this example, we consider the signal recovery by compressed sensing which refers to a signal acquisition and reconstruction technique. In signal processing, compressed sensing can be modelled as the following under determinated linear equation system:

$$y = Tx + \varepsilon, \tag{43}$$

where $x \in \mathbb{R}^N$ is a vector with m non-zero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $T : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear observation operator. It is know that problem (43) can be seen as solving the following LASSO problem:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Tx\|_2^2 \quad \text{subject to } \|x\|_1 \leq t, \tag{44}$$

where $t > 0$ is a given constant. In particular, if $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$, then the LASSO problem can be considered as the SFP (41).

The sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m non-zero elements. The matrix $T \in \mathbb{R}^{M \times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio SNR $= 40$. The process is started with $t = m$ and starting point x_1 is randomly generated in $\mathbb{R}^N$. The

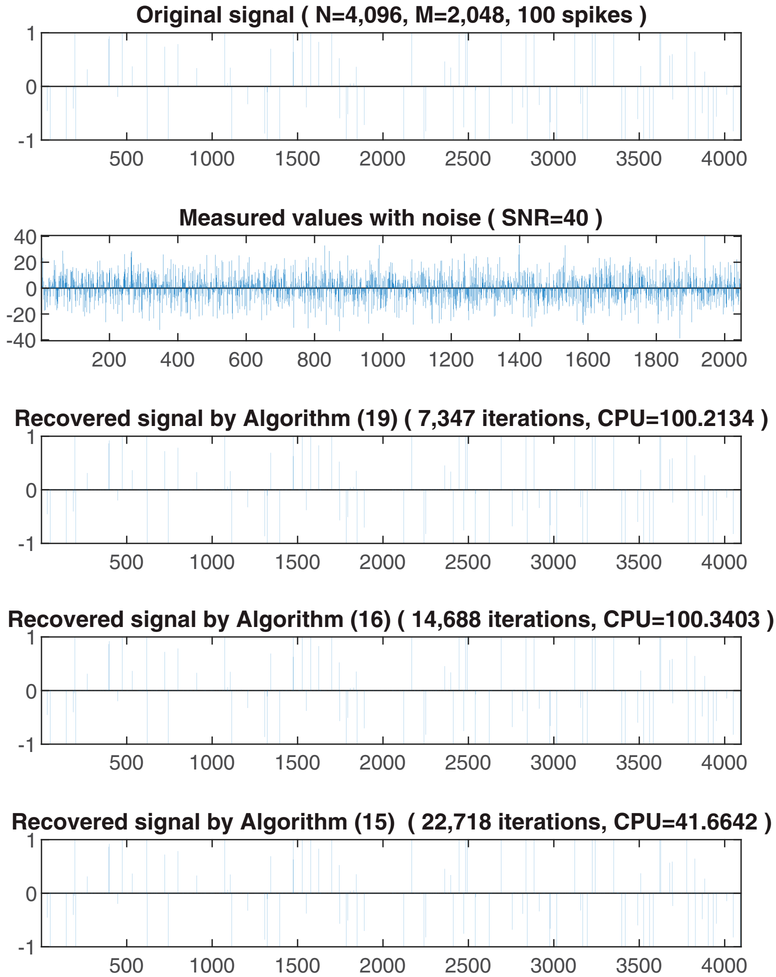

Figure 7. Comparison of recovered signal by using different algorithms in Case 3.

restoration accuracy is measured by the mean squared error as follows:

$$E_n = \frac{1}{N}\|x_n - x^*\|_2^2 < 10^{-5}, \tag{45}$$

where x^* is an estimated signal of x.

We perform numerical computations for Algorithm (19) and also compare with Algorithm (15) and Algorithm (16). Take $\alpha_n = (1/(1500(n+5)))$, $\beta_n = (1/(3(n+10)))$, $t_n = (n/(5700(n+1)))$, $\delta_n = (n/(3(n+3)))$, $\lambda_n = 1/\|T\|^2$ for all $n \in \mathbb{N}$, $f(x) = x/2$ and $u = (1, 1, \ldots, 1) \in \mathbb{R}^N$.

In our numerical experiments, we consider the following three cases of N, M and m:

Case 1 : $N = 1024$, $M = 512$ and $m = 30$;

Case 2 : $N = 2048$, $M = 1024$ and $m = 60$;

Case 3 : $N = 4096$, $M = 2048$ and $m = 100$.

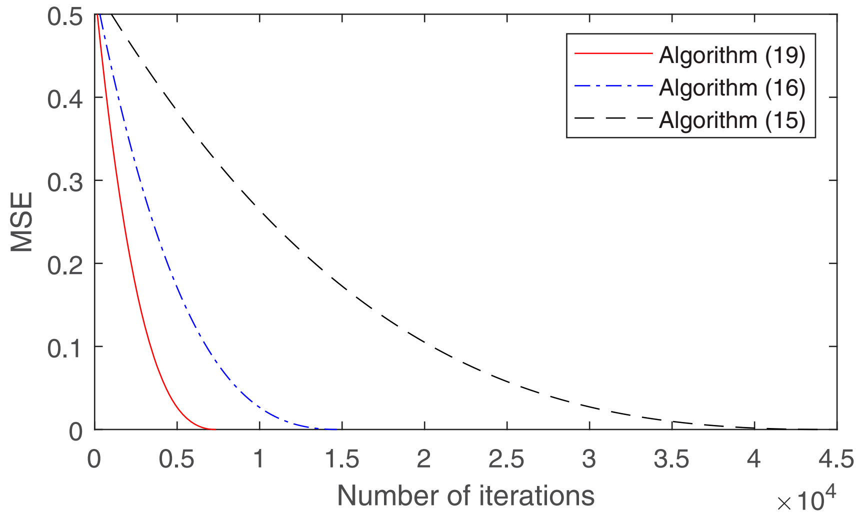

Figure 8. MSE versus number of iterations in Case 3.

Then the numerical results are reported in Table 2 and Figures 3–8.

Remark 5.4: From Table 2 and Figures 3–8, we see that our Algorithm (19) has a number of iterations and cpu time less than Algorithm (16) of Shehu and Cai [25] and Algorithm (15) of Cholamjiak [24]. It is shown that our algorithm highly improves those algorithms. This is the primary advantage of our algorithm in comparison with other algorithms.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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Some new (s,t)-Pell and (s,t)-Pell-Lucas identites by matrix methods

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Abstract— **In this paper, we characterize the 2×2 matrices X which satisfy the relation $X^2 = 2sX + tI$, where s and t are any real numbers with $s^2 + t > 0$, $s > 0$ and $t \neq 0$ and we obtain some new identities concerning with (s,t)-Pell and (s,t)-Pell-Lucas numbers.**

Keywords—Pell-numbers; Pell-Lucas number; (s,t)-Pell number; (s,t)-Pell-Lucas numbers; Matrix methods.

I. Introduction

Let s and t be any real number with $s^2 + t > 0$, $s > 0$ and $t \neq 0$ The (s, t)-Pell sequences $\{P_n(s,t)\}$ [1] is defined by the recurrence relation

$$P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t), \quad \text{for all } n \geq 2, \tag{I.1}$$

with initial conditions $P_0(s,t) = 0$ and $P_1(s,t) = 1$. In particular, if $s = \frac{1}{2}$ and $t = 1$, then the classical Fibonacci sequence is obtained, and if $s = t = 1$, then the classical Pell sequence is obtained. The first few terms of $\{P_n(s,t)\}$ are $0, 1, 2s, 4s^2 + t, 8s^3 + 4t$ and so on. The terms of this sequence are called (s,t)-Pell numbers and we denoted the n^{th} (s,t)-Pell numbers by $P_n(s,t)$. The Binet's formula for the n^{th} (s,t)-Pell numbers [2] is given by

$$P_n(s,t) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ for all } n \in \mathbb{Z}, \tag{I.2}$$

where $\alpha = s + \sqrt{s^2 + t}$ and $\beta = s - \sqrt{s^2 + t}$ are the roots of the characteristic equation $x^2 = 2sx + t$. We note that $\alpha + \beta = 2s$, $\alpha - \beta = 2\sqrt{s^2 + t}$ and $\alpha\beta = -t$.

Also, (s,t)–Pell-Lucas sequences $\{Q_n(s,t)\}$ [1] is defined by

$$Q_n(s,t) = 2sQ_{n-1}(s,t) + tQ_{n-2}(s,t) \text{ for all } n \geq 2, \tag{I.3}$$

with initial conditions $Q_0(s,t) = 2$ and $Q_1(s,t) = 2s$. In particular, if $s = \frac{1}{2}$ and $t = 1$, then the classical Lucas sequence is obtained, and if $s = t = 1$, then the classical Pell-Lucas sequence is obtained. The first few terms of $\{Q_n(s,t)\}$ are $2, 2s, 4s^2 + 2t, 8s^3 + 6st$ and so on. The terms of this sequence are called (s,t)-Pell-Lucas numbers and we denoted the n^{th} (s,t)-Pell-Lucas numbers by $Q_n(s,t)$. It can be seen that $Q_n(s,t) = 2sP_n(s,t) + 2tP_{n-1}(s,t)$ and $Q_n(s,t) = P_{n+1}(s,t) + 2tP_{n-1}(s,t)$ for all $n \in \mathbb{Z}$. The Binet's formula for the n^{th} (s,t) Pell-Lucas numbers [2] is given by

$$Q_n(s,t) = \alpha^n + \beta^n \quad \text{for all } n \in \mathbb{Z}, \tag{I.4}$$

where α and β are the roots of the characteristic equation which are given in (I.2). Recently, many identities for (s,t)-Pell and (s,t)-Pell-Lucas numbers have been studied and proved in the different ways such as induction, using Binet's formula and using matrix methods (see [1–4]).

In this paper, we will establish some identities for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using matrix method. Firstly, we characterize all the 2×2 matrices X which satisfying the relation $X^2 = 2sX + tI$. After that we establish some new identities for (s, t)-Pell and (s, t)-Pell-Lucas numbers by using this property. Throughout this paper, for convenience we will use the symbol P_n and Q_n instead of $P_n(s,t)$ and $Q_n(s,t)$ respectively.

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II. MAIN RESULTS

In this section, we first characterize all the 2×2 matrices X which satisfying the relation $X^2 = 2sX + tI$ and then by using this property, we obtain some new identities for (s, t)-Pell and (s, t)-Pell-Lucas numbers. We begin this section with the following Lemma which is given in [2].

Lemma 2.1 [2] *If X is a square matrix with $X^2 = 2sX + tI$, then $X^n = P_n X + tP_{n-1} I$ for all $n \in \mathbb{Z}$.*

In the following theorem, we characterize the 2×2 matrices X which satisfying the relation $X^2 = 2sX + tI$.

Theorem 2.2 *Let X be arbitrary 2×2 matrix. Then $X^2 = 2sX + tI$, if and only if X is of the form*

$$X = \begin{bmatrix} a & b \\ c & 2s-a \end{bmatrix}, \text{ with } \det(X) = -t$$

or $X = \lambda I$, where $\lambda \in \{\alpha, \beta\}$.

Proof. Assume that $X^2 = 2sX + tI$. Then the minimum polynomial of X must divides $x^2 - 2sx - t$. Therefore, it must be $x-\alpha$ or $x-\beta$ or $x^2-2sx-t$. In the first case $X=\alpha I$, in the second case $X = \beta I$, and in the third case, since X is 2×2 matrix, its characteristic polynomial must be $x^2-2sx-t$, so its trace is $2s$ and its determinant is $-t$. The argument reverses. □

From Theorem 2.2 and Lemma 2.1, we get the following three corollaries.

Corollary 2.3 *If $X = \begin{bmatrix} a & b \\ c & 2s-a \end{bmatrix}$, is a matrix with $\det(X) = -t$, then*

$$X^n = \begin{bmatrix} aP_n + tP_{n-1} & bP_n \\ cP_n & P_{n+1} - aP_n \end{bmatrix} \text{ for all } n \in \mathbb{Z}.$$

Proof. By Theorem 2.2, we have $X^2 = 2sX + tI$. Then the result follows from Lemma 2.1. □

Corollary 2.4 *Let $A = \begin{bmatrix} 2s & t \\ 1 & 0 \end{bmatrix}$, then*

$$A^n = \begin{bmatrix} P_{n+1} & tP_n \\ P_n & tP_{n-1} \end{bmatrix} \text{ for all } n \in \mathbb{Z}.$$

Proof. The result follows from Theorem 2.2 and Lemma 2.1. □

Corollary 2.5 [2] Let $W = \begin{bmatrix} s & 2(s^2+t) \\ \frac{1}{2} & s \end{bmatrix}$. *Then*

$$W^n = \begin{bmatrix} \frac{1}{2}Q_n & 2(s^2+t)P_n \\ \frac{1}{2}P_n & \frac{1}{2}Q_n \end{bmatrix} \text{ for all } n \in \mathbb{Z}.$$

Proof. By Theorem 2.2, we have $W^2 = 2sW + tI$. Then the result follows from Lemma 2.1 and the identities $Q_n = 2sP_n + 2tP_{n-1}$. □

Let us consider the following lemma, which will be needed for the results in this section.

Lemma 2.6 *Let $A = \begin{bmatrix} 2s & t \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2s & 2t \\ 2 & -2s \end{bmatrix}$, then*

$$A^n B = BA^n = \begin{bmatrix} Q_{n+1} & tQ_n \\ Q_n & tQ_{n-1} \end{bmatrix} \text{ for all } n \in \mathbb{Z} \text{ and } B^2 = 4(s^2+t)I.$$

Proof. From Corollary 2.4, $A^n = \begin{bmatrix} P_{n+1} & tP_n \\ P_n & tP_{n-1} \end{bmatrix}$. It is clear that

$$A^n B = BA^n = \begin{bmatrix} Q_{n+1} & tQ_n \\ Q_n & tQ_{n-1} \end{bmatrix} \text{ and } B^2 = 4(s^2+t)I. \quad □$$

Theorem 2.7 *Let $m,k \in \mathbb{Z}$* with $m \neq 0$ and $m \neq 1$. Then for all $n \in \mathbb{N}$, we have

$$(i)\ P_{mn+k} = \sum_{r=0}^{n} \binom{n}{r} t^{n-r} P_m^r P_{m-1}^{n-r} P_{r+k},$$

$$(ii)\ Q_{mn+k} = \sum_{r=0}^{n} \binom{n}{r} t^{n-r} P_m^r P_{m-1}^{n-r} Q_{r+k}.$$

Proof. Note that

$$\begin{aligned} W^{mn+k} &= (W^m)^n W^k = (P_m W + tP_{m-1} I)^n W^k \\ &= \left(\sum_{r=0}^{n} \binom{n}{r} t^{n-r} P_m^r P_{m-1}^{n-r} W^r \right) W^k \\ &= \sum_{r=0}^{n} \binom{n}{r} t^{n-r} P_m^r P_{m-1}^{n-r} W^{r+k}. \end{aligned}$$

Then the results follow from Corollary 2.5. □

Theorem 2.8 *Let $m,k \in \mathbb{Z}$* with $m \neq 0$ and $m \neq 1$. Then for all $n \in \mathbb{N}$, we have

(i) $(4s^2+4t)^n P_{2mn+k} = \sum_{r=0}^{2n} \binom{2n}{r} t^{2n-r} Q_m^r Q_{m-1}^{2n-r} P_{r+k}$,

(ii) $(4s^2+4t)^n Q_{2mn+k} = \sum_{r=0}^{2n} \binom{2n}{r} t^{2n-r} Q_m^r Q_{m-1}^{2n-r} Q_{r+k}$,

(iii) $(4s^2+4t)^{n+1} P_{(2n+1)m+k} = \sum_{r=0}^{2n+1} \binom{2n+1}{r} t^{2n+1-r} Q_m^r Q_{m-1}^{2n+1-r} Q_{r+k}$,

(iv) $(4s^2+4t)^n Q_{(2n+1)m+k} = \sum_{r=0}^{2n+1} \binom{2n+1}{r} t^{2n+1-r} Q_m^r Q_{m-1}^{2n+1-r} P_{r+k}$.

Proof. From Lemma 2.6, we have

$$BA^m = \begin{bmatrix} Q_{m+1} & tQ_m \\ Q_m & tQ_{m-1} \end{bmatrix} \text{ and } B^2 = 4(s^2+t)I.$$

Then we get that

$$\begin{aligned}(BA^m)^{2n} A^k &= \begin{bmatrix} Q_{m+1} & tQ_m \\ Q_m & tQ_{m-1} \end{bmatrix}^{2n} A^k \\ &= (Q_m A + tQ_{m-1})^{2n} A^k \\ &= \sum_{r=0}^{2n} \binom{2n}{r} t^{2n-r} Q_m^r Q_{m-1}^{2n-r} A^{r+k}.\end{aligned}$$

Since $(BA^m)^{2n} A^k = (B^2)^n A^{2mn} A^k = (4s^2+4t)^n A^{2mn+k}$, the identities (i) and (ii) are obtained. The identities (iii) and (iv) are proved in a similar way. □

Theorem 2.9 *Let $n \in \mathbb{N}$ and $m,k \in \mathbb{Z}$ with $m \neq 0$. Then*

(i) $$2^n Q_{mn+k} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} (4s^2+4t)^r P_m^{2r} Q_m^{n-2r} Q_k + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} (4s^2+4t)^{r+1} P_m^{2r+1} Q_m^{n-2r-1} P_k,$$

(ii) $$2^n P_{mn+k} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} (4s^2+4t)^r P_m^{2r} Q_m^{n-2r} P_k + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} (4s^2+4t)^r P_m^{2r+1} Q_m^{n-2r-1} Q_k.$$

Proof. Let $H = W + tW^{-1}$, we have

$$H = \begin{bmatrix} 0 & 4(s^2+t) \\ 1 & 0 \end{bmatrix}. \text{ Then } H^2 = (4s^2+4t)I,$$

$$H^{2r} = (4s^2+4t)^r I \text{ and } H^{2r+1} = (4s^2+4t)^r H.$$

Thus $$W^m = \begin{bmatrix} \frac{1}{2}Q_m & 2(s^2+t)P_m \\ \frac{1}{2}P_m & \frac{1}{2}Q_m \end{bmatrix} = \frac{1}{2}(P_m H + Q_m I),$$

and therefore

$$\begin{aligned} W^{mn} = (W^m)^n &= \frac{1}{2^n}(P_m H + Q_m I)^n \\ &= \frac{1}{2^n} \sum_{r=0}^{n} \binom{n}{r} P_m^r Q_m^{n-r} H^r \\ &= \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} P_m^{2r} Q_m^{n-2r} H^{2r} \\ &\quad + \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} P_m^{2r+1} Q_m^{n-2r-1} H^{2r+1}, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} P_m^{2r} Q_m^{n-2r} H^{2r} + \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} P_m^{2r+1} Q_m^{n-2r-1} H^{2r+1} \\ &= \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} P_m^{2r} Q_m^{n-2r} (4s^2+4t)^r I \\ &\quad + \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} P_m^{2r+1} Q_m^{n-2r-1} (4s^2+4t)^r H. \end{aligned}$$

Thus,

$$\begin{aligned} W^{mn} &= \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} P_m^{2r} Q_m^{n-2r} (4s^2+4t)^r I \\ &\quad + \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} P_m^{2r+1} Q_m^{n-2r-1} (4s^2+4t)^r H. \quad \text{(II.1)} \end{aligned}$$

Multiplying each side of (II.1) by W^k, we get

$$\begin{aligned} W^{mn+k} &= \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} (4s^2+4t)^r P_m^{2r} Q_m^{n-2r} W^k \\ &\quad + \frac{1}{2^n} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2r+1} (4s^2+4t)^r P_m^{2r+1} Q_m^{n-2r-1} H W^k. \end{aligned}$$

Since

$$W^{mn+k} = \begin{bmatrix} \frac{1}{2}Q_{mn+k} & 2(s^2+t)P_{mn+k} \\ \frac{1}{2}P_{mn+k} & \frac{1}{2}Q_{mn+k} \end{bmatrix}$$

and

$$HW^k = \begin{bmatrix} 2(s^2+t)P_k & 2(s^2+t)Q_k \\ \frac{1}{2}Q_k & 2(s^2+t)P_k \end{bmatrix},$$

we obtain the result. □

Theorem 2.10 *Let $m,n \in \mathbb{Z}$. Then*

$(i)\ P_{n+1}P_{n-1} - P_n^2 = -(-t)^{n-1}$,

$(ii)\ Q_{n+1}Q_{n-1} - Q_n^2 = 4(s^2+t)(-t)^{n-1}$,

$(iii)\ Q_{m+n} = Q_{m+1}P_n + tQ_mP_{n-1}$,

$(iv)\ (-t)^n Q_{m-n} = P_{m+1}Q_n - P_mQ_{n+1}$,

$(v)\ P_{m+n} = P_mP_{n+1} + tP_{m-1}P_n$,

$(vi)\ (-t)^n P_{m-n} = P_mP_{n+1} - P_{m+1}P_n$.

Proof. Since $\det(A^n) = (\det(A))^n$ and $\det(BA^n) = \det(B)(\det(A))^n$, the identities (i) and (ii) are obtained. Next, Since $BA^{m+n} = (BA^m)A^n$ and $BA^{m-n} = B(A^mA^{-n}) = A^m(BA^{-n})$, the identities (iii) and (iv) follow from Lemma 2.6. Also, the identities (v) and (vi) are proved in the same fashion as above by using Corollary 2.4 and the properties $A^{m+n} = A^m A^n$ and $A^{m-n} = A^m A^{-n}$, and therefore the proof is completed. □

Theorem 2.11 *Let* $n \in \mathbb{N}$ *and* $m,k \in \mathbb{Z}$ *with* $(-t)^m - Q_m \neq -1$. *Then*

$(i)\ Q_kQ_{m+n+k} = Q_{m+k}Q_{n+k} + 4(s^2+t)(-t)^k P_mP_n$,
$(ii)\ Q_kQ_{m+n-k} = 4(s^2+t)P_mP_n + (-t)^k Q_{m-k}Q_{n-k}$,
$(iii)\ Q_kP_{m+n} = P_nQ_{m+k} + (-t)^k P_mQ_{n-k}$.

Proof. Let $a = \dfrac{Q_{k+1}}{Q_k}$ and $X = \begin{bmatrix} a & b \\ c & 2s-a \end{bmatrix}$ with $\det(X) = -t$. Then by Corollary 2.3, we get

$$X^n = \begin{bmatrix} aP_n + tP_{n-1} & bP_n \\ cP_n & P_{n+1} - aP_n \end{bmatrix} = \frac{1}{Q_k}\begin{bmatrix} Q_{k+1}P_n + tQ_kP_{n-1} & bQ_kP_n \\ cQ_kP_n & Q_kP_{n+1} - Q_{k+1}P_n \end{bmatrix}.$$

By Lemma 2.10 (iii) and (iv), we have

$$X^n = \frac{1}{Q_k}\begin{bmatrix} Q_{n+k} & bQ_kP_n \\ cQ_kP_n & (-t)^kQ_{n-k} \end{bmatrix}.$$

Since $\det(X) = -t$ and $a = \dfrac{Q_{k+1}}{Q_k}$, it follows that

$$bc = \frac{Q_{k+2}Q_k - Q_{k+1}^2}{Q_k^2} = \frac{4(s^2+t)(-t)^k}{Q_k^2},$$ by Lemma 2.10 (ii).

If we consider the matrix multiplication $X^{m+n} = X^mX^n$, then we get the result. □

Theorem 2.12 *Let* $n \in \mathbb{N}$ *and* $m,k \in \mathbb{Z}$ *with* $(-t)^m - Q_m \neq -1$. *Then*

$(i)\ P_kP_{m+n+k} = P_{m+k}P_{n+k} - (-t)^k P_mP_n$,
$(ii)\ P_kP_{m+n-k} = P_mP_n - (-t)^k P_{m-k}P_{n-k}$,
$(iii)\ P_kP_{m+n} = P_nP_{m+k} - (-t)^k P_mP_{n-k}$.

Proof. Let $a = \dfrac{P_{k+1}}{P_k}$ and $X = \begin{bmatrix} a & b \\ c & 2s-a \end{bmatrix}$ with $\det(X) = -t$. Then by Corollary 2.3, we get

$$X^n = \begin{bmatrix} aP_n + tP_{n-1} & bP_n \\ cP_n & P_{n+1} - aP_n \end{bmatrix} = \frac{1}{P_k}\begin{bmatrix} P_{k+1}P_n + tP_kP_{n-1} & bP_kP_n \\ cP_kP_n & P_kP_{n+1} - P_{k+1}P_n \end{bmatrix}.$$

By Lemma 2.10 (v) and (vi), we have

$$X^n = \frac{1}{P_k}\begin{bmatrix} P_{k+1} & bP_kP_n \\ cP_kP_n & -(-t)^kP_{n-k} \end{bmatrix}.$$

Since $\det(X) = -t$ and $a = \dfrac{P_{k+1}}{P_k}$, it follows that

$$bc = \frac{P_{k+2}P_k - P_{k+1}^2}{P_k^2} = \frac{-(-t)^k}{P_k^2},$$ by Lemma 2.10 (i).

If we consider the matrix multiplication $X^{m+n} = X^mX^n$, then we get the result. □

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Self-dual codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ and applications*

Research Article

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Abstract: Self-dual codes over finite fields and over some finite rings have been of interest and extensively studied due to their nice algebraic structures and wide applications. Recently, characterization and enumeration of Euclidean self-dual linear codes over the ring $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with $u^3 = 0$ have been established. In this paper, Hermitian self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ are studied for all square prime powers q. Complete characterization and enumeration of such codes are given. Subsequently, algebraic characterization of H-quasi-abelian codes in $\mathbb{F}_q[G]$ is studied, where $H \leq G$ are finite abelian groups and $\mathbb{F}_q[H]$ is a principal ideal group algebra. General characterization and enumeration of H-quasi-abelian codes and self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ are given. For the special case where the field characteristic is 3, an explicit formula for the number of self-dual $A \times \mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ is determined for all finite abelian groups A and B such that $3 \nmid |A|$ as well as their construction. Precisely, such codes can be represented in terms of linear codes and self-dual linear codes over $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m} + u^2\mathbb{F}_{3^m}$. Some illustrative examples are provided as well.

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1. Introduction

Self-dual linear codes over finite fields form an interesting class of linear codes that have been extensively studied due to their nice algebraic structures and wide applications (see [8], [11], [12], [22] and references therein). Codes over finite rings have been of interest after it was shown that some binary

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non-linear codes such as the Kerdock, Preparata and Goethal codes are the Gray images of linear codes over $\mathbb{Z}_4$ in [7]. In general, families of linear codes and self-dual linear codes over finite chain rings are now become of interest. In [16], the mass formula for Euclidean self-dual linear codes over $\mathbb{Z}_{p^3}$ has been studied. Characterization and enumeration of Euclidean self-dual linear codes over the ring $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with $u^3 = 0$ have been given in [3].

Algebraically structured codes over finite fields such as cyclic codes, abelian codes and quasi-abelian codes are another important family of linear codes that have been extensively studied for both theoretical and practical reasons (see [2], [8], [10], [11], [12] and references therein). In [10], H-quasi-abelian codes and self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ have been studied in the case where $\mathbb{F}_q[H]$ is semisimple

To the best of our knowledge, Hermitian self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ and non-semisimple H-quasi-abelian codes in $\mathbb{F}_q[G]$ have not been well studied. The goals of this paper are to investigate the following families of linear codes and their links. 1) Hermitian self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ where q is a square prime power. 2) H-quasi-abelian codes and self-dual H-quasi-abelian codes in group algebras $\mathbb{F}_q[G]$, where $H \leq G$ are finite abelian groups and $\mathbb{F}_q[H]$ is a principal ideal group algebra.

The paper is organized as follows. In Section 2, some results on linear codes and Euclidean self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ are recalled. In Section 3, characterization and enumeration Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ are established for all square prime powers q together with an algorithm to determine all Hermitian self-dual codes and illustrative examples. In Section 4, the study of H-quasi-abelian codes in $\mathbb{F}_q[G]$ is given, where $\mathbb{F}_q[H]$ is a principal ideal group algebra. In the special case where the field characteristic is 3, the characterization and enumeration of $A \times \mathbb{Z}_3$-quasi-abelian codes and self-dual $A \times \mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ are completely determined in terms of linear and self-dual linear codes over $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m} + u^2\mathbb{F}_{3^m}$ obtained in Section 3 for all finite abelian groups A and B such that $3 \nmid |A|$. Summary and remarks are given in Section 5.

2. Preliminaries

In this section, basic results on linear codes and Euclidean self-dual linear codes over rings are recalled.

2.1. Linear codes over $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$

For a prime power q, denote by $\mathbb{F}_q$ the finite field of order q. Let $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q := \{a_0 + ua_1 + \cdots + u^{e-1}a_{e-1} \mid a_i \in \mathbb{F}_q \text{ for all } 0 \leq i < e\}$ be a ring, where the addition and multiplication are defined as in the usual polynomial ring over $\mathbb{F}_q$ with indeterminate u together with the condition $u^e = 0$. It is easily seen that $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ is isomorphic to $\mathbb{F}_q[u]/\langle u^e\rangle$ as rings. The *Galois extension* of $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ of degree m is defined to be the quotient ring $(\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q)[x]/\langle f(x)\rangle$, where $f(x)$ is an irreducible polynomial of degree m over $\mathbb{F}_q$. It is not difficult to see that the Galois extension of $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ of degree m is isomorphic to $\mathbb{F}_{q^m} + u\mathbb{F}_{q^m} + \cdots + u^{e-1}\mathbb{F}_{q^m}$. The ring $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ is a finite chain ring with maximal ideal $\langle u\rangle$, nilpotency index e and residue field $\mathbb{F}_q$. In addition, if q is a square, the mapping $^- : \mathbb{F}_q \to \mathbb{F}_q$ defined by $a \mapsto a^{\sqrt{q}}$ is a field automorphism on $\mathbb{F}_q$ of order 2. Extend $^-$ to be a ring automorphism of order 2 on $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ of the form $\overline{a_0 + ua_1 + \cdots + u^{e-1}a_{e-1}} = \overline{a_0} + u\overline{a_1} + \cdots + u^{e-1}\overline{a_{e-1}}$.

Let n be a positive integer and let R be a finite ring. The *Euclidean inner product* of $\boldsymbol{u} = (u_0, u_1, \ldots, u_{n-1})$ and $\boldsymbol{v} = (v_0, v_1, \ldots, v_{n-1})$ in R^n is defined to be

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathrm{E}} := \sum_{i=0}^{n-1} u_i v_i.$$

In the case where q is a square and $R \in \{\mathbb{F}_q, \mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q\}$, the *Hermitian inner product* of $\boldsymbol{u}$

and $\boldsymbol{v}$ in R^n is defined to be

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathrm{H}} := \sum_{i=0}^{n-1} u_i \overline{v_i}.$$

A *linear code* $\mathcal{C}$ of length n over the ring R is defined to be an R-submodule of the R-module R^n. A linear code over R is said to be *free* if it is a free R-module. Denote by $\mathrm{wt}(\boldsymbol{v})$ the Hamming weight of an element $\boldsymbol{v} \in R^n$. Precisely, $\mathrm{wt}(\boldsymbol{v})$ is the number of non-zero components in $\boldsymbol{v}$. For a linear code $\mathcal{C}$ over R, let $\mathrm{wt}(\mathcal{C}) = \min\{\mathrm{wt}(\boldsymbol{c}) \mid \boldsymbol{c} \in \mathcal{C}\}$ be the minimum Hamming weight of $\mathcal{C}$. If $R = \mathbb{F}_q$, a linear code $\mathcal{C}$ of length n and dimension k over R with $\mathrm{wt}(\mathcal{C}) = d$ is referred as an $[n, k, d]_q$ code. The parameters of a linear code $\mathcal{C}$ of length n over R satisfies the Singleton bond [14], i.e., $\mathrm{wt}(\mathcal{C}) \leq n - \log_{|R|}(|\mathcal{C}|) + 1$. A linear code $\mathcal{C}$ is called a *Maximum Distance Separable (MDS) code* if the equality in the Singleton bound holds. A matrix G over R is called a *generator matrix* for $\mathcal{C}$ if the rows of G generate all the elements of $\mathcal{C}$ and none of the rows can be written as a linear combination of the others. Linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ over R are said to be *equivalent* if there exists a monomial matrix P such that $\mathcal{C}_2 = \mathcal{C}_1 P := \{\boldsymbol{c}P \mid \boldsymbol{c} \in \mathcal{C}_1\}$. Denote by $\mathcal{C}^{\perp_{\mathrm{E}}} = \{\boldsymbol{v} \in R^n \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathrm{E}} = 0\}$ and $\mathcal{C}^{\perp_{\mathrm{H}}} = \{\boldsymbol{v} \in R^n \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathrm{H}} = 0\}$ the *Euclidean* and *Hermitian duals* of $\mathcal{C}$, respectively. A linear code $\mathcal{C}$ is said to be *Euclidean* (resp., *Hermitian*) *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^{\perp_{\mathrm{E}}}$ (resp., $\mathcal{C} \subseteq \mathcal{C}^{\perp_{\mathrm{H}}}$). It is called *Euclidean* (resp., *Hermitian*) *self-dual* if $\mathcal{C} = \mathcal{C}^{\perp_{\mathrm{E}}}$ (resp., $\mathcal{C} = \mathcal{C}^{\perp_{\mathrm{H}}}$).

In Section 3 and the remaining parts of this section, we focus on linear and self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$. In [19], it has been shown that every linear code of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ is permutation equivalent to a code $\mathcal{C}$ with generator matrix

$$G = \begin{bmatrix} I_k & A_2 & A_3 & A_4 \\ 0 & uI_l & uB_3 & uB_4 \\ 0 & 0 & u^2 I_m & u^2 C_4 \end{bmatrix} = \begin{bmatrix} A' \\ uB' \\ u^2 C \end{bmatrix}, \tag{1}$$

where I_r is the identity matrix of order r, $A_3 = A_{30} + uA_{31}, B_4 = B_{40} + uB_{41}, A_4 = A_{40} + uA_{41} + u^2 A_{42}$, and A_2, B_3, C_4, A_{ij} and B_{ij} are matrices of appropriate sizes over $\mathbb{F}_q$. In this case, the code $\mathcal{C}$ is said to be of type $\{k, l, m\}$ and it contains $q^{3k+2l+m}$ codewords.

For each linear code $\mathcal{C}$ of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ and $i \in \{0, 1, 2\}$, the ith *torsion code* of $\mathcal{C}$ is a linear code of length n over $\mathbb{F}_q$ defined to be

$$\mathrm{Tor}_i(\mathcal{C}) = \left\{ \boldsymbol{v} (\mathrm{mod}\, u) \mid \boldsymbol{v} \in \left(\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q\right)^n \text{ and } u^i \boldsymbol{v} \in \mathcal{C} \right\}.$$

The code $\mathrm{Tor}_0(\mathcal{C})$ is sometime called the *residue code* of $\mathcal{C}$ and denoted it by $\mathrm{Res}(\mathcal{C})$. From the definitions, it is obvious that $\mathrm{Res}(\mathcal{C}) = \mathrm{Tor}_0(\mathcal{C}) \subseteq \mathrm{Tor}_1(\mathcal{C}) \subseteq \mathrm{Tor}_2(\mathcal{C})$.

For a linear code $\mathcal{C}$ of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with generator matrix G given in (1), the residue code $\mathrm{Res}(\mathcal{C})$ has dimension k and generator matrix

$$G = \begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \end{bmatrix}, \tag{2}$$

the first torsion code $\mathrm{Tor}_1(\mathcal{C})$ has dimension $k + l$ and generator matrix

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \\ 0 & I_l & B_3 & B_{40} \end{bmatrix}, \tag{3}$$

and the second torsion code $\mathrm{Tor}_2(\mathcal{C})$ has dimension $k + l + m$ and generator matrix

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \\ 0 & I_l & B_3 & B_{40} \\ 0 & 0 & I_m & C_4 \end{bmatrix}. \tag{4}$$

For $0 \leq k \leq n$, the *Gaussian coefficient* is defined to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$

Let $N_e(q,n)$ denote the number of distinct linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$. The number $N_e(q,n)$ has been studied and summarized in [4]. For $e = 3$, we have the following result.

Proposition 2.1 ([4, Lemma 2.2])**.** *Let q be a prime power and let n be a positive integer. Then*

$$N_3(q,n) = 1 + \sum_{t=1}^{3} \sum_{n \geq h_1 \geq h_2 \geq \cdots \geq h_t > h_{t+1}=0} \prod_{j=1}^{t} \begin{bmatrix} n - h_{j+1} \\ h_j - h_{j+1} \end{bmatrix}_q q^{h_{j+1}(n-h_j)}.$$

2.2. Euclidean self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$

Let $\mathcal{C}$ be a linear code of length n and type $\{k,l,m\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ and let $h = n-(k+l+m)$. In [3], it has been shown that the Euclidean dual $\mathcal{C}^{\perp_{\mathrm{E}}}$ of $\mathcal{C}$ is of type $\{h,m,l\}$ and it contains $q^{3h+2m+l}$ codewords. Therefore, $k = h$ and $l = m$ whenever $\mathcal{C}$ is Euclidean self-dual. Consequently, every Euclidean self-dual code of type $\{k,l,m\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ has even length $n = 2(k+l)$.

Characterization of Euclidean self-dual linear codes of even length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ has been established in [3].

Proposition 2.2 ([3, Proposition 1])**.** *Let q be a prime power and let $\mathcal{C}$ be a linear code of length n and type $\{k,l,m\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with generator matrix G in the form of* (1)*. Then $\mathcal{C}$ is Euclidean self-dual if and only if $k = h, l = m$ and the following conditions hold:*

$$A'A'^T \equiv 0 \pmod{u^3}, \tag{5}$$
$$A'B'^T \equiv 0 \pmod{u^2}, \tag{6}$$
$$B'B'^T \equiv 0 \pmod{u}, \tag{7}$$
$$A'C^T \equiv 0 \pmod{u}. \tag{8}$$

For a positive integer n and a prime power q, let $\sigma_{\mathrm{E}}(q,n)$ denote the number of Euclidean self-dual linear codes of length n over $\mathbb{F}_q$. Further, if q is a square prime power, let $\sigma_{\mathrm{H}}(q,n)$ denote the number of Hermitian self-dual linear codes of length n over $\mathbb{F}_q$. The following results in [21] and [22] are useful in the enumeration of self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$.

Lemma 2.3 ([21] and [22])**.** *Let q be a prime power and let n be a positive integer. Then*

$$\sigma_{\mathrm{E}}(q,l) = \begin{cases} \prod_{i=1}^{\frac{n}{2}-1}(q^i+1) & \text{if } q \text{ and } n \text{ are even,} \\ 2\prod_{i=1}^{\frac{n}{2}-1}(q^i+1) & \text{if } q \equiv 1 \,(\mathrm{mod}\, 4) \text{ and } 2 \mid n, \\ 2\prod_{i=1}^{\frac{n}{2}-1}(q^i+1) & \text{if } q \equiv 3 \,(\mathrm{mod}\, 4) \text{ and } 4 \mid n, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

If q is square, then

$$\sigma_{\mathrm{H}}(q,n) = \begin{cases} \prod_{i=0}^{\frac{n}{2}-1}(q^{i+\frac{1}{2}}+1) & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The empty product is regarded as 1.

Let $NE_e(q,n)$ denote the number of distinct Euclidean self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$. The value of $NE_3(q,n)$ has been completely determined in [3].

Theorem 2.4 ([3, Theorem 1])**.** *Let q be a prime power and let n be a positive integer. Then*

$$NE_3(q,n) = \begin{cases} \sigma_{\mathrm{E}}(q,n) \displaystyle\sum_{k=0}^{n/2} \begin{bmatrix} \frac{n}{2} \\ k \end{bmatrix}_q q^{kn/2} & \text{if } q \text{ is even and } n \text{ is even,} \\ \sigma_{\mathrm{E}}(q,n) \displaystyle\sum_{k=0}^{n/2} \begin{bmatrix} \frac{n}{2} \\ k \end{bmatrix}_q q^{k(n/2-1)} & \text{if } q \text{ is odd and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

3. Hermitian self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$

In this section, we focus on characterization and enumeration of Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$.

Throughout this section, we assume that q is a square prime power. For each positive integer n, let $NH_e(q,n)$ denote the number of distinct Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$. By extending techniques introduced in [3], the characterization and the number $NH_3(q,n)$ of Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ are established.

Let $\mathcal{C}$ be a linear code of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ of type $\{k,l,m\}$ and let $h = n-(k+l+m)$. Using argument similar to those in Section 2 of [3], it can be deduced that the Hermitian dual $\mathcal{C}^{\perp_{\mathrm{H}}}$ of $\mathcal{C}$ is of type $\{h,m,l\}$ and it contains $q^{3h+2m+l}$ codewords. It follows that $k=h$ and $l=m$ if $\mathcal{C}$ is Hermitian self-dual. Hence, every Hermitian self-dual code of type $\{k,l,m\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ has even length $n = 2(k+l)$.

For a matrix $A = [a_{ij}]_{s\times t}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$, let $\overline{A} := [\overline{a_{ij}}]_{s\times t}$ and $A^\dagger := \overline{A}^T$. Characterization of Hermitian self-dual linear codes of even length n over $\mathbb{F}_q+u\mathbb{F}_q+u^2\mathbb{F}_q$ is given in the following proposition.

Proposition 3.1. *Let q be a square prime power and let $\mathcal{C}$ be a linear code of even length n and type $\{k,l,m\}$ over $\mathbb{F}_q+u\mathbb{F}_q+u^2\mathbb{F}_q$ with generator matrix G in the form of* (1)*. Then $\mathcal{C}$ is Hermitian self-dual if and only if $k=h$, $l=m$ and the following hold:*

$$A'A'^\dagger \equiv 0 \pmod{u^3}, \tag{11}$$
$$A'B'^\dagger \equiv 0 \pmod{u^2}, \tag{12}$$
$$B'B'^\dagger \equiv 0 \pmod{u}, \tag{13}$$
$$A'C^\dagger \equiv 0 \pmod{u}. \tag{14}$$

Proof. Assume that $\mathcal{C}$ is Hermitian self-dual. By the above discussion, we have $k=h$, $l=m$ and

$$\begin{bmatrix} A' \\ uB' \\ u^2C \end{bmatrix} \begin{bmatrix} A' \\ uB' \\ u^2C \end{bmatrix}^\dagger \equiv [0] \pmod{u^3}$$

which are equivalent to the conditions (11)–(14).

Conversely, assume that $\mathcal{C}$ is a linear code such that $k = h, l = m$ and the conditions (11)–(14) hold. From (11)–(14), it is not difficult to see that $\mathcal{C}$ is Hermitian self-orthogonal. Equivalently, $\mathcal{C} \subseteq \mathcal{C}^{\perp_{\rm H}}$. Since $k = h$ and $l = m$, we have $|\mathcal{C}| = |\mathcal{C}^{\perp_{\rm H}}|$ which implies that $\mathcal{C} = \mathcal{C}^{\perp_{\rm H}}$. Therefore, $\mathcal{C}$ is Hermitian self-dual as desired. $\square$

Corollary 3.2. *Let $\mathcal{C}$ be a Hermitian self-dual linear code of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$. Then the following statements holds.*

1) $\mathrm{Tor}_1(\mathcal{C})$ *is a Hermitian self-dual code of length n over $\mathbb{F}_q$.*

2) $\mathrm{Tor}_2(\mathcal{C}) = \mathrm{Res}(\mathcal{C})^{\perp_{\rm H}}$.

Proof. Assume that $\mathcal{C}$ is of type $\{k, l, m\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$. Then From (11)–(13), it follows that $\mathrm{Tor}_1(\mathcal{C})$ is Hermitian self-orthogonal. Since $\dim(\mathrm{Tor}_1(\mathcal{C})) = k + l = \frac{n}{2} = \dim((\mathrm{Tor}_1(\mathcal{C}))^{\perp_{\rm H}})$, $\mathrm{Tor}_1(\mathcal{C})$ is Hermitian self-dual. From (11)–(14), we have $\mathrm{Tor}_2(\mathcal{C}) \subseteq \mathrm{Res}(\mathcal{C})^{\perp_{\rm H}}$. Since $\dim(\mathrm{Tor}_2(\mathcal{C})) = k + 2l = n - k = \dim((\mathrm{Res}(\mathcal{C}))^{\perp_{\rm H}})$, we have $\mathrm{Tor}_2(\mathcal{C}) = \mathrm{Res}(\mathcal{C})^{\perp_{\rm H}}$. $\square$

Since $\mathrm{Res}(\mathcal{C}) = \mathrm{Tor}_0(\mathcal{C}) \subseteq \mathrm{Tor}_1(\mathcal{C}) \subseteq \mathrm{Tor}_2(\mathcal{C})$, it can be concluded further that $\mathrm{Res}(\mathcal{C})$ is Hermitian self-orthogonal for all Hermitian self-dual linear codes $\mathcal{C}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$.

From Corollary 3.2, a Hermitian self-dual code $\mathcal{C}$ of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ can be induced by Hermitian self-dual linear codes of length n over $\mathbb{F}_q$. For a given Hermitian self-dual code $\mathcal{C}_1$ of length n over $\mathbb{F}_q$, we first aim to determine the number of Hermitian self-dual linear codes $\mathcal{C}$ of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ such that $\mathrm{Tor}_1(\mathcal{C}) = \mathcal{C}_1$.

Proposition 3.3. *Let q be a square prime power and let n be an even positive integer. Let $\mathcal{C}_1$ be a Hermitian self-dual linear code of length n over $\mathbb{F}_q$. Then, for each $0 \leq k \leq \frac{n}{2}$, there are $q^{\frac{kn}{2}}$ Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ corresponding to each subspace of $\mathcal{C}_1$ of dimension k.*

Proof. Let $\mathcal{C}_1$ be a Hermitian self-dual linear code of length n over $\mathbb{F}_q$ with dimension $\frac{n}{2} = k + l$ and generator matrix

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \\ 0 & I_l & B_3 & B_{40} \end{bmatrix}, \tag{15}$$

where the columns are grouped into blocks of sizes k, l, l and k.

Since $\mathcal{C}_1$ is Hermitian self-dual, we have

$$I_k + A_2A_2^\dagger + A_{30}A_{30}^\dagger + A_{40}A_{40}^\dagger = 0, \tag{16}$$

$$A_2 + A_{30}B_3^\dagger + A_{40}B_{40}^\dagger = 0, \tag{17}$$

$$I_l + B_3B_3^\dagger + B_{40}B_{40}^\dagger = 0. \tag{18}$$

Let $H = \begin{bmatrix} \overline{A_{30}} & \overline{A_{40}} \\ \overline{B_3} & \overline{B_{40}} \end{bmatrix}$. From (16)–(18), it can be deduced that

$$\begin{aligned} H(-H^\dagger) &= -HH^\dagger \\ &= -H\overline{H}^T \\ &= \begin{bmatrix} -\overline{A_{30}}A_{30}^T - \overline{A_{40}}A_{40}^T & -\overline{A_{30}}B_3^T - \overline{A_{40}}B_{40}^T \\ \left(-\overline{A_{30}}B_3^T - \overline{A_{40}}B_{40}^T\right)^T & -\overline{B_3}B_3^T - \overline{B_{40}}B_{40}^T \end{bmatrix} \\ &= \begin{bmatrix} I_k + \overline{A_2}A_2^T & \overline{A_2} \\ A_2^T & I_l \end{bmatrix}. \end{aligned}$$

Let $J = \begin{bmatrix} I_k & -\overline{A_2} \\ -A_2^T & I_l + A_2^T\overline{A_2} \end{bmatrix}$. Then $H(-H^\dagger)J = \begin{bmatrix} I_k & 0 \\ 0 & I_l \end{bmatrix}$ which implies that H is invertible.

Let $\mathcal{C}_0$ be a k-dimensional $\mathbb{F}_q$-subspace of $\mathcal{C}_1$ with generator matrix A. Since $\mathcal{C}_1$ is Hermitian self-dual, it follows that $\mathcal{C}_0$ is Hermitian self-orthogonal. Up to permutation of the last $(k+l)$ columns (if necessary), its follows that $\mathcal{C}_0^{\perp_\mathrm{H}}$ has a generator matrix of the form

$$\begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \\ 0 & I_l & B_3 & B_{40} \\ 0 & 0 & I_l & C_4 \end{bmatrix}. \tag{19}$$

Then $A_{30} = -A_{40}C_4^\dagger$ which implies that $H = \begin{bmatrix} -\overline{A_{40}}C_4^T & \overline{A_{40}} \\ \overline{B_3} & \overline{B_{40}} \end{bmatrix}$. Since H is invertible, it follows that A_{40} is invertible.

Next, we determined the matrices over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ satisfying conditions (11)–(14) which are equivalent to

$$I_k + A_2A_2^\dagger + A_3A_3^\dagger + A_4A_4^\dagger \equiv 0 \pmod{u^3} \tag{20}$$

$$A_2 + A_3B_3^\dagger + A_4B_4^\dagger \equiv 0 \pmod{u^2} \tag{21}$$

$$I_l + B_3B_3^\dagger + B_4B_4^\dagger \equiv 0 \pmod{u} \tag{22}$$

$$A_3 + A_4C_4^\dagger \equiv 0 \pmod{u}. \tag{23}$$

The matrices A_2, B_3 and C_4 are considered modulo u, i.e. all the entries in A_2, B_3 and C_4 are in $\mathbb{F}_q$. The matrices A_3 and B_4 are considered modulo u^2 while A_4 is considered modulo u^3. From these fact, let $A_3 = A_{30} + uA_{31}, B_4 = B_{40} + uB_{41}$ and $A_4 = A_{40} + uA_{41} + u^2A_{42}$, where A_{31}, B_{41}, A_{41} and A_{42} are matrices of appropriate sizes over $\mathbb{F}_q$. Therefore, we can write (20) as

$$\left(I_k + A_2A_2^\dagger + A_{30}A_{30}^\dagger + A_{40}A_{40}^\dagger\right) + u\left(\widetilde{A_{30}A_{31}^\dagger} + \widetilde{A_{40}A_{41}^\dagger}\right)$$
$$+u^2\left(A_{31}A_{31}^\dagger + A_{41}A_{41}^\dagger + \widetilde{A_{40}A_{42}^\dagger}\right) \equiv 0 \pmod{u^3},$$

where $\widetilde{X} := X + X^\dagger$. We can also rewrite (21) as

$$\left(A_2 + A_{30}B_3^\dagger + A_{40}B_{40}^\dagger\right) + u\left(A_{31}B_3^\dagger + A_{40}B_{41}^\dagger + A_{41}B_{40}^\dagger\right) \equiv 0 \pmod{u^2}$$

By substituting (18) into (21), we obtain that

$$B_{41}^\dagger = -A_{40}^{-1}\left(A_{31}B_3^\dagger + A_{41}B_{40}^\dagger\right),$$

From (23), C_4 is uniquely determined as

$$C_4 = \left(-A_{40}^{-1}A_{30}\right)^\dagger.$$

It is sufficient to focus on (20) because (18) is the same as (22). From (16), we have to determined the matrices satisfying the following:

$$\widetilde{A_{30}A_{31}^\dagger} + \widetilde{A_{40}A_{41}^\dagger} = 0, \tag{24}$$

$$A_{31}A_{31}^\dagger + A_{41}A_{41}^\dagger + \widetilde{A_{40}A_{42}^\dagger} = 0. \tag{25}$$

Hence, the code $\mathcal{C}$ with generator matrix of the form (1) is a Hermitian self-dual linear code if and only if conditions (24) and (25) are satisfied.

Therefore, the number of Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ whose the 1st torsion is $\mathcal{C}_1$ is equal to the number of solutions of the system of matrix equations (24) and (25).

We take an arbitrary matrix $A_{31} \in M_{k\times l}(\mathbb{F}_q)$ and put $[g_{ij}] = \widetilde{A_{30}A_{31}^{\dagger}}$ and $[x_{ij}] = A_{40}A_{41}^{\dagger}$. Then condition (24) is equivalent to

$$g_{ij} + x_{ij} + \overline{x_{ji}} = 0.$$

Then $-g_{ii} = x_{ii} + \overline{x_{ii}} = \mathrm{Tr}(x_{ii}) \in \mathbb{F}_{\sqrt{q}}$ for each $1 \le i \le k$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_{\sqrt{q}}$ is the *trace map* defined by $\alpha \mapsto \overline{\alpha} + \alpha$ for all $\alpha \in \mathbb{F}_q$. Note that $|\,\mathrm{Tr}^{-1}(a)| = \sqrt{q}$ for all $a \in \mathbb{F}_{\sqrt{q}}$. Then we have $x_{ii} \in \mathrm{Tr}^{-1}(-g_{ii})$ for all $1 \le i \le k$, $x_{ji} \in \mathbb{F}_q$ and $x_{ij} = -g_{ij} - \overline{x_{ji}}$ for each $1 \le i < j \le k$. Therefore,

$$A_{41} = (A_{40}^{-1}[x_{ij}])^{\dagger}.$$

Thus we have q^{kl} possible choices for A_{31} and $q^{\frac{k(k-1)}{2}+\frac{k}{2}} = q^{\frac{k^2}{2}}$ for A_{41}.

For fixed matrices A_{31} and A_{41}, let $[h_{ij}] = A_{31}A_{31}^{\dagger} + A_{41}A_{41}^{\dagger}$ and $[y_{ij}] = A_{40}A_{42}^{\dagger}$. Then (25) is equivalent to

$$h_{ij} + y_{ij} + \overline{y_{ji}} = 0.$$

Using a similar argument as above, we have $q^{\frac{k^2}{2}}$ possible choices for

$$A_{42} = (A_{40}^{-1}[y_{ij}])^{\dagger}.$$

Therefore, we have

$$q^{kl} \times q^{\frac{k^2}{2}} \times q^{\frac{k^2}{2}} = q^{k^2+kl} = q^{k(k+l)} = q^{\frac{kn}{2}}$$

possible choices for the matrices A_{31}, A_{41} and A_{42} over $\mathbb{F}_q$. Therefore, the desired result follows immediately. □

The number of distinct Hermitian self-dual linear codes of even length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ can be summarized in the following theorem.

Theorem 3.4. *Let q be a square prime power and let n be a positive integer. Then the number of distinct Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ is*

$$NH_3(q,n) = \begin{cases} \sigma_{\mathrm{H}}(q,n) \displaystyle\sum_{k=0}^{n/2} \begin{bmatrix} \frac{n}{2} \\ k \end{bmatrix}_q q^{kn/2} & \textit{if } n \textit{ is even}, \\ 0 & \textit{otherwise}. \end{cases}$$

From the proof of Theorem 3.3, we obtain not only the number of Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ but also a construction of such Hermitian self-dual linear codes. The construction of Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ induced by a Hermitian self-dual linear code of length n over $\mathbb{F}_q$ in the proof of Theorem 3.3 is summarized in Algorithm 1.

Based on Algorithm 1, an illustrative example of a Hermitian self-dual linear code of length 6 over $\mathbb{F}_9 + u\mathbb{F}_9 + u^2\mathbb{F}_9$ constructed from a Hermitian self-dual linear code of length 6 over $\mathbb{F}_9$ is given as follows.

Example 3.5. *Let $\mathbb{F}_9 = \mathbb{F}_3[\alpha]$ be the finite field of order 9, where α is a root of $x^2 + x + 2$ over $\mathbb{F}_3$. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be linear codes of length 6 over $\mathbb{F}_9$ with generator matrices*

$$\left[\begin{array}{cc|c|c|cc} 1 & 0 & 1 & \alpha^2 & \alpha^5 & \alpha^2 \\ 0 & 1 & 0 & \alpha & 1 & \alpha \end{array}\right] \textit{ and } \left[\begin{array}{cc|c|c|cc} 1 & 0 & 1 & \alpha^2 & \alpha^5 & \alpha^2 \\ 0 & 1 & 0 & \alpha & 1 & \alpha \\ \hline 0 & 0 & 1 & \alpha^5 & \alpha & 1 \end{array}\right],$$

Algorithm 1. **Construction of Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$**

For a given Hermitian self-dual linear code $\mathcal{C}_1$ of length n over $\mathbb{F}_q$ and its linear subcode $\mathcal{C}_0$ of dimension $0 \le k \le \frac{n}{2}$, do the following steps.

1. Define $l = \frac{n}{2} - k$.
2. Construct a generator matrix $A = \begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \end{bmatrix}$ for $\mathcal{C}_0$, where the columns are grouped into blocks of sizes k, l, l and k.
3. Extend A to be a generator matrix $\begin{bmatrix} I_k & A_2 & A_{30} & A_{40} \\ 0 & I_l & B_3 & B_{40} \end{bmatrix}$ for $\mathcal{C}_1$.
4. Set $C_4 = -\left(A_{40}^{-1}A_{30}\right)^\dagger$.
5. Set A_{31} to be a $k \times l$ matrix over $\mathbb{F}_q$.
6. Define $[g_{ij}] = \widetilde{A_{30}A_{31}^\dagger}$ and set $[x_{ij}]$ to be a $k \times k$ matrix over $\mathbb{F}_q$ such that the strictly lower triangular elements are arbitrary in $\mathbb{F}_q$, $x_{ii} \in \mathrm{Tr}^{-1}(-g_{ii})$, and $x_{ij} = -g_{ij} - \overline{x_{ji}}$ for all $i < j$. (If $k = \frac{n}{2}$, set $[g_{ij}]$ to be the $k \times k$ zero matrix over $\mathbb{F}_q$.)
7. Set $A_{41} = (A_{40}^{-1}[x_{ij}])^\dagger$.
8. Set $B_{41} = -\left(A_{40}^{-1}\left(A_{31}B_3^\dagger + A_{41}B_{40}^\dagger\right)\right)^\dagger$.
9. Define $[h_{ij}] = A_{31}A_{31}^\dagger + A_{41}A_{41}^\dagger$ and set $[y_{ij}]$ to be a $k \times k$ matrix over $\mathbb{F}_q$ such that the strictly lower triangular elements are arbitrary in $\mathbb{F}_q$, $y_{ii} \in \mathrm{Tr}^{-1}(-h_{ii})$, and $y_{ij} = -g_{ij} - \overline{x_{ji}}$ for all $i < j$. (If $k = \frac{n}{2}$, set $[h_{ij}] = A_{41}A_{41}^\dagger$.)
10. Set $A_{42} = (A_{40}^{-1}[y_{ij}])^\dagger$.
11. Define $\mathcal{C}$ to be a linear code of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with generator matrix
$$\begin{bmatrix} I_k & A_2 & A_{30} + uA_{31} & A_{40} + uA_{41} + u^2A_{42} \\ 0 & uI_l & uB_3 & uB_{40} + u^2B_{41} \\ 0 & 0 & u^2I_l & u^2C_4 \end{bmatrix}.$$
The $\mathcal{C}$ is Hermitian self-dual by Theorem 3.3.
12. Repeat 5. – 11. with different choices of A_{31}, A_{41}, and A_{42}. The Hermitian self-dual linear codes of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ determined by $\mathcal{C}_0 \subseteq \mathcal{C}_1$ are obtained.

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respectively. Then $\mathcal{C}_1$ is Hermitian self-dual and $\mathcal{C}_0 \subseteq \mathcal{C}_1$. Based on Algorithm 1, we have $k = 2$, $l = 1$, $A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A_{30} = \begin{bmatrix} \alpha^2 \\ \alpha \end{bmatrix}$, $A_{40} = \begin{bmatrix} \alpha^5 & \alpha^2 \\ 1 & \alpha \end{bmatrix}$, $B_3 = \begin{bmatrix} \alpha^5 \end{bmatrix}$, and $B_{40} = \begin{bmatrix} \alpha & 1 \end{bmatrix}$. Then we have

$$C_4 = -\left(A_{40}^{-1}A_{30}\right)^\dagger = -\left(\begin{bmatrix} \alpha^5 & \alpha^2 \\ 1 & \alpha \end{bmatrix}^{-1}\begin{bmatrix} \alpha^2 \\ \alpha \end{bmatrix}\right)^\dagger = \begin{bmatrix} 0 & 2 \end{bmatrix}.$$

By choosing $A_{31} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $[g_{ij}] = \widetilde{A_{30}A_{31}^\dagger} = \widetilde{\begin{bmatrix} \alpha^2 \\ \alpha \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & \alpha \\ \alpha^3 & 2 \end{bmatrix}$. We choose $x_{11} = 0 \in \{0, \alpha^2, \alpha^6\} = \mathrm{Tr}^{-1}(0) = \mathrm{Tr}^{-1}(-g_{11})$, $x_{22} = 2 \in \{2, \alpha^5, \alpha^7\} = \mathrm{Tr}^{-1}(1) = \mathrm{Tr}^{-1}(-g_{22})$, $x_{21} = 1$, and $x_{12} = -g_{12} - \overline{x_{21}} = -\alpha - 1 = \alpha^3$. It follows that $[x_{ij}] = \begin{bmatrix} 0 & \alpha^3 \\ 1 & 2 \end{bmatrix}$ and $A_{41} = (A_{40}^{-1}[x_{ij}])^\dagger = \begin{bmatrix} 2 & \alpha \\ \alpha & 1 \end{bmatrix}$. Consequently, $B_{41} = -\left(A_{40}^{-1}\left(A_{31}B_3^\dagger + A_{41}B_{40}^\dagger\right)\right)^\dagger = \begin{bmatrix} \alpha & 0 \end{bmatrix}$.

Let $[h_{ij}] = A_{31}A_{31}^{\dagger} + A_{41}A_{41}^{\dagger} = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \begin{bmatrix}\alpha^4 & \alpha\\ \alpha & 1\end{bmatrix}\begin{bmatrix}\alpha^4 & \alpha^3\\ \alpha^3 & 1\end{bmatrix} = \begin{bmatrix}1 & \alpha^3\\ \alpha & 1\end{bmatrix}$. *We choose* $y_{11} = 1 \in \{1,\alpha,\alpha^3\} = \mathrm{Tr}^{-1}(2) = \mathrm{Tr}^{-1}(-h_{11})$, $y_{22} = 1 \in \{1,\alpha,\alpha^3\} = \mathrm{Tr}^{-1}(2) = \mathrm{Tr}^{-1}(-h_{22})$, $y_{21} = 1$, *and* $y_{12} = -h_{12} - \overline{y_{21}} = -\alpha^3 - 1 = \alpha$. *Then* $[y_{ij}] = \begin{bmatrix}1 & \alpha\\ 1 & 1\end{bmatrix}$ *and* $A_{42} = (A_{40}^{-1}[y_{ij}])^{\dagger} = \begin{bmatrix}\alpha^3 & \alpha^3\\ 0 & \alpha^5\end{bmatrix}$.

From Algorithm 1, the matrix

$$\begin{bmatrix} I_k & A_2 & A_{30}+uA_{31} & A_{40}+uA_{41}+u^2A_{42}\\ 0 & uI_l & uB_3 & uB_{40}+u^2B_{41}\\ 0 & 0 & u^2I_l & u^2C_4 \end{bmatrix} = \left[\begin{array}{cc|c|c|cc} 1 & 0 & 1 & \alpha^2+u & \alpha^5+2u+\alpha^3u^2 & \alpha^2+\alpha u+\alpha^3u^2\\ 0 & 1 & 0 & \alpha+u & 1+\alpha u & \alpha+u+\alpha^5u^2\\ \hline 0 & 0 & u & \alpha^5u & \alpha u+\alpha u^2 & u\\ \hline 0 & 0 & & u^2 & 0 & 2u^2 \end{array}\right]$$

is a generator matrix for a Hermitian self-dual code of length 6 *over* $\mathbb{F}_9 + u\mathbb{F}_9 + u^2\mathbb{F}_9$ *with type* $\{2, 1, 1\}$.

When $k = \frac{n}{2}$ in Theorem 3.3 (equivalently, in Algorithm 1), we have the following extension on the parameters of Hermitian self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$. Let n be an even positive integer and let $\mathcal{C}_1 = \mathcal{C}_0$ be a Hermitian self-dual code of length n over $\mathbb{F}_q$ with parameters $[n, k = \frac{n}{2}, d]_q$ and generator matrix

$$A = \begin{bmatrix} I_{\frac{n}{2}} & A_{40} \end{bmatrix}, \tag{26}$$

where A_{40} is a $k \times k$ invertible matrix over $\mathbb{F}_q$. Based on Algorithm 1, the linear code $\mathcal{C}$ of length n and type $\{\frac{n}{2}, 0, 0\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with generator matrix

$$G = \begin{bmatrix} I_k & A_{40} + uA_{41} + u^2A_{42} \end{bmatrix} \tag{27}$$

is Hermitian self-dual. Since $\mathcal{C}$ is a free code, $\mathrm{wt}(\mathcal{C}) = \mathrm{wt}(\mathcal{C}_1) = d$ by [18, Corollary 4.3]. Hence, the following two theorems can be derived directly.

Theorem 3.6. *Let q be a prime power and let n be an even positive integer. If there exists an $[n, \frac{n}{2}, \frac{n}{2}+1]_q$ MDS Hermitian self-dual code over $\mathbb{F}_q$, then an MDS Hermitian self-dual code of length n over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ of type $\{\frac{n}{2}, 0, 0\}$ can be constructed with minimum Hamming weight $\frac{n}{2} + 1$.*

Proof. Assume that there exists an $[n, \frac{n}{2}, \frac{n}{2} + 1]_q$ MDS Hermitian self-dual code $\mathcal{C}_1$ over $\mathbb{F}_q$ with generator matrix of the form (26). By Algorithm 1, a linear code $\mathcal{C}$ of length n and type $\{\frac{n}{2}, 0, 0\}$ over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ with generator matrix of the form (27) is Hermitian self-dual. Since $\mathcal{C}$ is free, we have $\mathrm{wt}(\mathcal{C}) = \mathrm{wt}(\mathcal{C}_1) = \frac{n}{2} + 1$ and $\log_{q^3}(|\mathcal{C}|) = \frac{n}{2} = \dim(\mathcal{C}_1)$ by the discussion above. Hence, $\mathrm{wt}(\mathcal{C}) = \frac{n}{2} + 1 = n - \log_{q^3}(|\mathcal{C}|) + 1$ which implies that $\mathcal{C}$ is MDS. □

Using the above theorem, numerous MDS Hermitian self-dual codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ can be construct based on known MDS Hermitian self-dual codes over $\mathbb{F}_q$ (see, for example, [13], [17], [24]).

4. Self-dual quasi-abelian codes over principal ideal group algebras

In this section, the study of quasi-abelian codes over principal ideal group algebras is given. In the special case where the field characteristic is 3 and the Sylow 3-subgroup of the underlying finite abelian group has order 3, complete characterization and enumeration of quasi-abelian codes and self-dual quasi-abelian codes are presented in terms linear codes and self-dual linear codes over $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m} + u^2\mathbb{F}_{3^m}$ obtained in [3], [4] and Section 3.

4.1. Group rings and quasi-abelian codes

Let R be a finite commutative ring with nonzero identity and let G be a finite abelian group. Then

$$R[G] = \left\{ \sum_{g \in G} \alpha_g Y^g \mid \alpha_g \in R, g \in G \right\}$$

is a commutative ring under the addition and multiplication given for the usual polynomial ring over R with indeterminate Y, where the indices are computed additively in G. The ring $R[G]$ is called a *group ring of G over R*. In the case where R is the finite field $\mathbb{F}_{p^m}$, the group ring $\mathbb{F}_{p^m}[G]$ is called a *group algebra of G over $\mathbb{F}_{p^m}$* and it is called a *Principal Ideal Group Algebra (PIGA)* if every ideal in $\mathbb{F}_{p^m}[G]$ is principal. The readers may refer to [15] for more details on group rings. A linear code of length $|G|$ over R can be viewed as an embedded R-submodule of the R-module in $R[G]$ by indexing the $|G|$-tuples by the elements in G. Given a subgroup H of G with index $n = [G : H]$, a linear code $\mathcal{C}$ of length $|G|$ viewed as an R-submodule of $R[G]$ is called an *H-quasi-abelian code* (specifically, an *H-quasi-abelian code of index n*) in $R[G]$ if $\mathcal{C}$ is an $R[H]$-module, i.e., $\mathcal{C}$ is closed under the multiplication by the elements in $R[H]$. Such a code will be called a quasi-abelian code if H is not specified or where it is clear in the context.

Let $\{g_1, g_2, \dots, g_n\}$ be a fixed set of representatives of the cosets of H in G. Let $\mathcal{R} := \mathbb{F}_q[H]$. Define $\Phi : \mathbb{F}_q[G] \to \mathcal{R}^n$ by

$$\Phi\left(\sum_{h \in H} \sum_{i=1}^{n} \alpha_{h+g_i} Y^{h+g_i} \right) = (\alpha_1(Y), \alpha_2(Y), \dots, \alpha_n(Y)),$$

where $\alpha_i(Y) = \sum_{h \in H} \alpha_{h+g_i} Y^h \in \mathcal{R}$ for all $i = 1, 2, \dots, n$. It is well known that Φ is an $\mathcal{R}$-module isomorphism interpreted as follows.

Lemma 4.1 ([10, Lemma 2.1])**.** *The map Φ induces a one-to-one correspondence between H-quasi-abelian codes in $\mathbb{F}_q[G]$ and linear codes of length n over $\mathcal{R}$.*

We note that a group algebra $\mathbb{F}_{p^m}[H]$ is semisimple if and only if the Sylow p-subgroup of H is trivial (see [20, Chapter 2: Theorem 4.2]), and it is a PIGA if and only if he Sylow p-subgroup of H is cyclic (see [6]). In [10], complete characterization and enumeration of H-quasi-abelian codes in $\mathbb{F}_{p^m}[G]$ have been established in the case where $\mathbb{F}_{p^m}[H]$ is semisimple. Here, we focus on a more general case where $\mathbb{F}_{p^m}[H]$ is a PIGA, or equivalently, the Sylow p-subgroup of H is cyclic. Precisely, $H \cong A \times \mathbb{Z}_{p^m i}$ and $G \cong A \times \mathbb{Z}_{p^s} \times B$, where s is a non-negative integer, A and B are finite abelian groups such that $p \nmid |A|$. General characterization is given in Subsection 4.2. In the special case where $p = 3$ and $s = 1$, complete characterization and enumeration of $A \times \mathbb{Z}_3$-quasi-abelian codes and self-dual $A \times \mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ are given in Subsection 4.3.

4.2. $A \times \mathbb{Z}_{p^s}$-Quasi-Abelian Codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$

We focus on H-quasi-abelian codes in $\mathbb{F}_{p^m}[G]$, where $\mathbb{F}_{p^m}[H]$ is a PIGA. Equivalently, $H \cong A \times \mathbb{Z}_{p^s}$ and $G \cong A \times \mathbb{Z}_{p^s} \times B$, where s is a positive integer, A and B are finite abelian groups such that $p \nmid |A|$ (see [6] and [12]).

Note that the group algebra $\mathbb{F}_{p^m}[A]$ is semisimple [2] and it can be decomposed using the Discrete Fourier Transform in [23] (see [12] and [11] for more details). For completeness, the decomposition used in this paper are summarized as follows.

For co-prime positive integers i and j, denote by $\mathrm{ord}_j(i)$ the multiplicative order of i modulo j. For each $a \in A$, denote by $\mathrm{ord}(a)$ the additive order of a in A and the *p^m-cyclotomic class* of A containing

$a \in A$ is defined to be the set

$$S_{p^m}(a) := \{p^{mi} \cdot a \mid i = 0, 1, \dots\} = \{p^{mi} \cdot a \mid 0 \le i < \mathrm{ord}_{\mathrm{ord}(a)}(p^m)\},$$

where $p^{ki} \cdot a := \sum_{j=1}^{p^{mi}} a$ in A. A subset $\{a_1, a_2, \dots, a_t\}$ of A is called a *complete set of representatives* of p^m-cyclotomic classes of A if $S_{p^m}(a_1), S_{p^m}(a_2), \dots, S_{p^m}(a_t)$ are distinct and $\bigcup_{i=1}^{t} S_{p^m}(a_i) = A$.

An *idempotent* in $\mathbb{F}_{p^m}[A]$ is a nonzero element e such that $e^2 = e$. It is called *primitive* if for every other idempotent f, either $ef = e$ or $ef = 0$. The existence of primitive idempotent elements in $\mathbb{F}_{p^m}[A]$ is proved in [5]. They are induced by the p^m-cyclotomic classes of A (see [5, Proposition II.4]). Consequently, $\mathbb{F}_{p^m}[A]$ can be viewed as a direct sum of principal ideals generated by these primitive idempotent elements.

Proposition 4.2 ([5, Corollary III.6])**.** *Let $\{a_1, a_2, \dots, a_t\}$ be a complete set of representatives of p^m-cyclotomic classes of a finite abelian group A where $p \nmid |A|$ and let e_i be the primitive idempotent induced by $S_{p^m}(a_i)$ for all $1 \le i \le t$. Then*

$$\mathbb{F}_{p^m}[A] = \bigoplus_{i=1}^{t} \mathbb{F}_{p^m}[A]e_i \cong \prod_{i=1}^{t} \mathbb{F}_{p^{m_i}},$$

where $m_i = m \cdot \mathrm{ord}_{\mathrm{ord}(a_i)}(p^m)$.

A PIGA $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}]$ can be decomposed in the following theorem.

Theorem 4.3. *Let s be a positive integer. Let $\{a_1, a_2, \dots, a_t\}$ be a complete set of representatives of p^m-cyclotomic classes of a finite abelian group A where $p \nmid |A|$. Then*

$$\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}] \cong \prod_{i=1}^{t} \left(\mathbb{F}_{p^{m_i}} + u\mathbb{F}_{p^{m_i}} + \dots + u^{p^s-1}\mathbb{F}_{p^{m_i}}\right)$$

where $m_i = m \cdot \mathrm{ord}_{\mathrm{ord}(a_i)}(p^m)$ for all $1 \le i \le t$.

Proof. For each $1 \le i \le t$, let e_i be the primitive idempotent induced by $S_{p^m}(a_i)$. From Proposition 4.2, we have

$$\mathbb{F}_{p^m}[A] \cong \mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}]e_i \cong \prod_{i=1}^{t} \mathbb{F}_{p^{m_i}}, \tag{28}$$

and hence,

$$\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}] \cong (\mathbb{F}_{p^m}[A])[\mathbb{Z}_{p^s}] \cong \prod_{i=1}^{t} \mathbb{F}_{p^{m_i}}[\mathbb{Z}_{p^s}]. \tag{29}$$

Under the ring isomorphism that fixes the elements in $\mathbb{F}_{p^{m_i}}$ and $Y^1 \mapsto u + 1$, it is not difficult to see that

$$\mathbb{F}_{p^{m_i}}[\mathbb{Z}_{p^s}] \cong \mathbb{F}_{p^{m_i}} + u\mathbb{F}_{p^{m_i}} + \dots + u^{p^s-1}\mathbb{F}_{p^{m_i}} \tag{30}$$

as rings. Therefore,

$$\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}] \cong \prod_{i=1}^{t} \left(\mathbb{F}_{p^{m_i}} + u\mathbb{F}_{p^{m_i}} + \dots + u^{p^s-1}\mathbb{F}_{p^{m_i}}\right) \tag{31}$$

as desired. □

For each finite abelian group B of order n, every $A \times \mathbb{Z}_{p^s}$-quasi-abelian code in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ can be viewed as a linear code of length n over $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}]$ by Lemma 4.1. The next corollary follows directly from Theorem 4.3.

Corollary 4.4. *Let s and m be positive integers. Let A and B be finite abelian groups such that $|B| = n$ and $p \nmid |A|$. Then every $A \times \mathbb{Z}_{p^s}$-quasi-abelian code $\mathcal{C}$ in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ can be viewed as*

$$\mathcal{C} \cong \prod_{i=1}^{t} \mathcal{C}_i,$$

where $\mathcal{C}_i$ is a linear code of length n over $\mathbb{F}_{p^{m_i}} + u\mathbb{F}_{p^{m_i}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m_i}}$ for all $i = 1, 2, \ldots, t$.

The enumeration of $A \times \mathbb{Z}_{p^s}$-quasi-abelian code in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is given as follows.

Theorem 4.5. *Let s and m be positive integers. Let A and B be finite abelian groups such that $|B| = n$ and the exponent of A is M and $p \nmid M$. Then the number of $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is*

$$\prod_{d|M} \left(N_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n) \right)^{\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}},$$

where $\mathcal{N}_A(d)$ is the number of elements of order d in A determined in [1] and $N_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n)$ is the number of linear codes of length n over $\mathbb{F}_{p^{m \cdot \mathrm{ord}_d(p^m)}} + u\mathbb{F}_{p^{m \cdot \mathrm{ord}_d(p^m)}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m \cdot \mathrm{ord}_d(p^m)}}$ determined in [4, Lemma 2.2].

Proof. From Theorem 4.3, it suffices to determine the number of linear codes of length n over the ring $\mathbb{F}_{p^{m_i}} + u\mathbb{F}_{p^{m_i}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m_i}}$ for all $i = 1, 2, \ldots, t$.

For each divisor d of M, each p^m-cyclotomic class containing an element of order d has $\mathrm{ord}_d(p^m)$ elements and the number of such p^m-cyclotomic classes is $\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}$. By Theorem 4.3, it follows that the number of linear codes of length n over $\mathbb{F}_{p^{m \cdot \mathrm{ord}_d(p^m)}} + u\mathbb{F}_{p^{m \cdot \mathrm{ord}_d(p^m)}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m \cdot \mathrm{ord}_d(p^m)}}$ corresponding to d is

$$\left(N_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n) \right)^{\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}}.$$

By taking the summation over all the divisors d of M, the desired result follows. $\square$

Example 4.6. *Let $A \leq H \leq G$ be finite abelian groups such that $A \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, $H \cong A \times \mathbb{Z}_3$, and $G \cong H \times \mathbb{Z}_4$. Then the 3-cyclotomic classes of A are $S_3((0,0)) = \{(0,0)\}$, $S_3((0,2)) = \{(0,2)\}$, $S_3((1,0)) = \{(1,0)\}$, $S_3((1,2)) = \{(1,2)\}$, $S_3((0,1)) = \{(0,1),(0,3)\}$, and $S_3((1,1)) = \{(1,1),(1,3)\}$. It follows that $\mathrm{ord}_{\mathrm{ord}((0,0))}(3) = \mathrm{ord}_{\mathrm{ord}((0,2))}(3) = \mathrm{ord}_{\mathrm{ord}((1,0))}(3) = \mathrm{ord}_{\mathrm{ord}((1,2))}(3) = 1$ and $\mathrm{ord}_{\mathrm{ord}((0,1))}(3) = \mathrm{ord}_{\mathrm{ord}((1,1))}(3) = 2$. By Proposition 4.2, $\mathbb{F}_3[A]$ has 4 primitive idempotents e_i such that $\mathbb{F}_3[A]e_i \cong \mathbb{F}_3$ and 2 primitive idempotents e_j such that $\mathbb{F}_3[A]e_j \cong \mathbb{F}_9$. Such primitive idempotents are induced by the above 6 cyclotomic classes while their explicit forms can be determined using [5, Proposition II.4]. Consequently,*

$$\mathbb{F}_3[A] \cong \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_9 \times \mathbb{F}_9$$

and

$$\mathbb{F}_3[H] = \mathbb{F}_3[A \times \mathbb{Z}_3] \cong \mathbb{F}_3[\mathbb{Z}_3] \times \mathbb{F}_3[\mathbb{Z}_3] \times \mathbb{F}_3[\mathbb{Z}_3] \times \mathbb{F}_3[\mathbb{Z}_3] \times \mathbb{F}_9[\mathbb{Z}_3] \times \mathbb{F}_9[\mathbb{Z}_3].$$

By Proposition 4.3, we have $\mathbb{F}_3[\mathbb{Z}_3] \cong \mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3$ and $\mathbb{F}_9[\mathbb{Z}_3] \cong \mathbb{F}_9 + u\mathbb{F}_9 + u^2\mathbb{F}_9$, where $u^3 = 0$. Hence,

$$\mathbb{F}_3[H] \cong \prod_{i=1}^{4} (\mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3) \times \prod_{j=1}^{2} (\mathbb{F}_9 + u\mathbb{F}_9 + u^2\mathbb{F}_9). \tag{32}$$

Using Corollary 4.4, every H-quasi-abelian code in $\mathbb{F}_3[G]$ is isomorphic to a code of the form

$$\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \times \mathcal{C}_5 \times \mathcal{C}_6,$$

where $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ are linear codes of length $|\mathbb{Z}_4| = 4$ over $\mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3$, and $\mathcal{C}_5$ and $\mathcal{C}_6$ are linear codes of length 4 over $\mathbb{F}_9 + u\mathbb{F}_9 + u^2\mathbb{F}_9$.

In the next subsections, we focus on self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ with respect to both the Euclidean and Hermitian inner products.

4.3. Euclidean self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$

Euclidean self-dual $A\times\mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A\times\mathbb{Z}_{p^s}\times B]$ is studied in terms of the following types of p^m-cyclotomic classes. A p^m-cyclotomic class $S_{p^m}(a)$ is said to be of *type* I if $a = -a$ (in this case, $S_{p^m}(a) = S_{p^m}(-a)$), *type* II if $S_{p^m}(a) = S_{p^m}(-a)$ and $a \neq -a$, or *type* III if $S_{p^m}(-a) \neq S_{p^m}(a)$. The primitive idempotent e induced by $S_{p^m}(a)$ is said to be of type $\lambda \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ if $S_{p^m}(a)$ is a p^m-cyclotomic class of type λ.

Rearrange the terms in the decomposition in Theorem 4.3 based on the p^m-cyclotomic classes of types I, II and III, we have the next theorem.

Theorem 4.7. *Let m and s be positive integers and let A be a finite abelian group such that $p \nmid |A|$. Then*

$$\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}] \cong \left(\prod_{i=1}^{r_{\mathrm{I}}} \mathcal{R}_i\right) \times \left(\prod_{j=1}^{r_{\mathrm{II}}} \mathcal{S}_j\right) \times \left(\prod_{l=1}^{(r_{\mathrm{III}})/2} (\mathcal{T}_l \times \mathcal{T}_l)\right),$$

where $r_{\mathrm{I}}, r_{\mathrm{II}}$ and r_{III} are the numbers of elements in a complete set of representatives of p^m-cyclotomic classes of A of types I, II, and III, respectively, $\mathcal{R}_i = F_{p^m} + uF_{p^m} + \cdots + u^{p^s-1}\mathbb{F}_{p^m}$ for all $i = 1, 2, \ldots, r_{\mathrm{I}}$, $\mathcal{S}_j = \mathbb{F}_{p^{m_{r_{\mathrm{I}}+j}}} + u\mathbb{F}_{p^{m_{r_{\mathrm{I}}+j}}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m_{r_{\mathrm{I}}+j}}}$ for all $j = 1, 2, \ldots, r_{\mathrm{II}}$, and $\mathcal{T}_l = \mathbb{F}_{p^{m_{r_{\mathrm{I}}+r_{\mathrm{II}}+l}}} + u\mathbb{F}_{p^{m_{r_{\mathrm{I}}+r_{\mathrm{II}}+l}}} + \ldots u^{p^s-1}\mathbb{F}_{p^{m_{r_{\mathrm{I}}+r_{\mathrm{II}}+l}}}$ for all $l = 1, 2, \ldots, (r_{\mathrm{III}})/2$.

Using Theorem 4.7 and the analysis similar to those in [11, Section II.D], a $A \times \mathbb{Z}_{p^s}$-quasi-abelian code $\mathcal{C}$ in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ and its Euclidean dual are given.

Proposition 4.8. *Let s and m be positive integers. Let A and B be finite abelian groups such that $|B| = n$ and $p \nmid |A|$. Then an $A \times \mathbb{Z}_{p^s}$-quasi-abelian code in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ can be viewed as*

$$\mathcal{C} \cong \left(\prod_{i=1}^{r_{\mathrm{I}}} \mathcal{B}_i\right) \times \left(\prod_{j=1}^{r_{\mathrm{II}}} \mathcal{C}_j\right) \times \left(\prod_{l=1}^{(r_{\mathrm{III}})/2} (\mathcal{D}_l \times \mathcal{D}_l')\right), \tag{33}$$

where $\mathcal{B}_i$, $\mathcal{C}_j$, $\mathcal{D}_l$ and $\mathcal{D}_l'$ are linear codes of length n over $\mathcal{R}_i$, $\mathcal{S}_j$, $\mathcal{T}_l$ and $\mathcal{T}_l$, respectively, for all $i = 1, 2, \ldots, r_{\mathrm{I}}$, $j = 1, 2, \ldots, r_{\mathrm{II}}$ and $l = 1, 2, \ldots, (r_{\mathrm{III}})/2$.

Furthermore, the Euclidean dual of $\mathcal{C}$ in (33) is of the form

$$\mathcal{C}^{\perp_{\mathrm{E}}} \cong \left(\prod_{i=1}^{r_{\mathrm{I}}} \mathcal{B}_i^{\perp_{\mathrm{E}}}\right) \times \left(\prod_{j=1}^{r_{\mathrm{II}}} \mathcal{C}_j^{\perp_{\mathrm{H}}}\right) \times \left(\prod_{l=1}^{(r_{\mathrm{III}})/2} \left((\mathcal{D}_l')^{\perp_{\mathrm{E}}} \times \mathcal{D}_l^{\perp_{\mathrm{E}}}\right)\right).$$

The characterization of Euclidean self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is established in terms of a product of linear codes, Euclidean self-dual linear codes, and Hermitian self-dual linear codes over Galois extensions of the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{p^s-1}\mathbb{F}_{p^m}$.

Corollary 4.9. *Let s and m be positive integers. Let A and B be finite abelian groups such that $|B| = n$ and $p \nmid |A|$. Then a $A \times \mathbb{Z}_{p^s}$-quasi-abelian code $\mathcal{C}$ in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is Euclidean self-dual if and only if in the decomposition (33),*

i) $\mathcal{B}_i$ is a Euclidean self-dual linear code of length n over $\mathcal{R}_i$ for all $i = 1, 2, \ldots, r_{\mathrm{I}}$,

ii) $\mathcal{C}_j$ is a Hermitian self-dual linear code of length n over $\mathcal{S}_j$ for all $j = 1, 2, \ldots, r_{\mathrm{II}}$, and

iii) $\mathcal{D}'_l = \mathcal{D}_l^{\perp_{\mathrm{E}}}$ is a linear code of length n over $\mathcal{T}_l$ for all $l = 1, 2, \ldots, (r_{\mathrm{III}})/2$.

From Corollary 4.9, the Euclidean self-duality of $A \times \mathbb{Z}_{p^s}$-quasi-abelian code $\mathcal{C}$ in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ depends only on the structure of $A \times \mathbb{Z}_{p^s}$ and the index $n = |B|$ but not the structure of B itself.

Given positive integers m and j, the pair (j, p^m) is said to be *good* if j divides $p^{mt} + 1$ for some positive integer t, and *bad* otherwise. This notion have been introduced in [8] and [11] for the enumeration of self-dual cyclic codes and self-dual abelian codes over finite fields and it is completely determined in [9]. Let χ be a function defined on pairs (j, p^m) as follows.

$$\chi(j, p^m) = \begin{cases} 0 & \text{if } (j, p^m) \text{ is good,} \\ 1 & \text{otherwise.} \end{cases} \tag{34}$$

The number of Euclidean self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian code $\mathcal{C}$ in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ can be determined as follows.

Theorem 4.10. *Let s and m be positive integers. Let A and B be finite abelian groups such that $|B| = n$ is even and the exponent of A is M and $p \nmid M$. Then the number of Euclidean self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is*

$$\left(NE_{p^s}(p^m, n)\right)^{\sum\limits_{d|M, \mathrm{ord}_d(p^m)=1} (1-\chi(d,p^m))\mathcal{N}_A(d)} \times \prod_{\substack{d|M \\ \mathrm{ord}_d(p^m) \neq 1}} \left(NH_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n)\right)^{(1-\chi(d,p^m))\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}}$$
$$\times \prod_{d|M} \left(N_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n)\right)^{\chi(d,p^m)\frac{\mathcal{N}_A(d)}{2\mathrm{ord}_d(p^m)}},$$

where $\mathcal{N}_A(d)$ denotes the number of elements in A of order d determined in [1].

Proof. From Corollary 4.9, it suffices to determine the numbers of linear codes $\mathcal{B}_i$'s, $\mathcal{C}_j$'s, and $\mathcal{D}_l$'s such that $\mathcal{B}_i$ and $\mathcal{C}_j$ are Euclidean and Hermitian self-dual, respectively.

From [12, Remark 2.5], the elements in A of the same order are partitioned into p^m-cyclotomic classes of the same type. For each divisor d of M, a p^m-cyclotomic class containing an element of order d has cardinality $\mathrm{ord}_d(p^m)$ and the number of such p^m-cyclotomic classes is $\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}$. We consider the following 3 cases.

Case 1: $\chi(d, p^m) = 0$ and $\mathrm{ord}_d(3^k) = 1$. By [11, Remark 2.6], every 3^k-cyclotomic class of A containing an element of order d is of type I. Since there are $\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}$ such p^m-cyclotomic classes, the number of Euclidean self-dual linear codes B_i's of length n corresponding to d is

$$\left(NE_{p^s}(p^m, n)\right)^{\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}} = \left(NE_{p^s}(p^m, n)\right)^{(1-\chi(d,p^m))\mathcal{N}_A(d)}.$$

Case 2: $\chi(d, p^m) = 0$ and $\mathrm{ord}_d(p^m) \neq 1$. By [11, Remark 2.6], every p^m-cyclotomic class of A containing an element of order d is of type II and of even cardinality $\mathrm{ord}_d(p^m)$. Hence, the number of Hermitian self-dual linear codes $\mathcal{C}_j$'s of length n corresponding to d is

$$\left(NH_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n)\right)^{\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}} = \left(NH_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, n)\right)^{(1-\chi(d,p^m))\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}}.$$

Case 3: $\chi(d, p^m) = 1$. By [11, Lemma 4.5], every p^m-cyclotomic class of A containing an element of order d is of type III. Then the number of linear codes $\mathcal{D}_l$'s of length n corresponding to d is

$$\left(N_{p^s}(p^{m\cdot \mathrm{ord}_d(p^m)}, n)\right)^{\frac{\mathcal{N}_A(d)}{2\mathrm{ord}_d(p^m)}} = \left(N_{p^s}(p^{m\cdot \mathrm{ord}_d(p^m)}, n)\right)^{\chi(d,p^m)\frac{\mathcal{N}_A(d)}{2\mathrm{ord}_d(p^m)}}.$$

The formula for the number of Euclidean self-dual $A\times\mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A\times\mathbb{Z}_{p^s}\times B]$ follows since d runs over all divisors of M. $\square$

Remark 4.11. *In general, the numbers $NE_{p^s}(p^m, n)$ and $NH_{p^s}(p^m, n)$ in Theorem 4.10 have not been well studied. In the case where the field characteristic is 3, we have the following conclusions.*

1. *The numbers $N_3(3^m, n)$, $NE_3(3^m, n)$ and $NH_3(3^m, n)$ have been determined in Proposition 2.1, [3, Theorem 1] and Theorem 3.4. By Theorem 4.10, the enumeration for Euclidean self-dual $A\times\mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A\times\mathbb{Z}_3\times B]$ is completed. .*

2. *The construction/characterization of linear, Euclidean self-dual and Hermitian self-dual codes of length n over $\mathbb{F}_{3^m}+u\mathbb{F}_{3^m}+u^2\mathbb{F}_{3^m}$ have been given in [3], [4] and in the proof of Proposition 3.3. Hence, the construction/characterization of Euclidean self-dual $A\times\mathbb{Z}_3$-quasi-abelian code in $\mathbb{F}_{3^m}[A\times\mathbb{Z}_3\times B]$ can be obtained from Corollary 4.9.*

3. *Note that, if n is odd, there are no Hermitian self-dual linear codes of length n over $\mathbb{F}_{3^m}+u\mathbb{F}_{3^m}+u^2\mathbb{F}_{3^m}$ by Theorem 3.4. Hence, there are no Euclidean self-dual $A\times\mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A\times\mathbb{Z}_3\times B]$ for all abelian groups B of odd order.*

Example 4.12. *Let $A\le H\le G$ be finite abelian groups such that $A\cong\mathbb{Z}_2\times\mathbb{Z}_4$, $H\cong A\times\mathbb{Z}_3$, and $G\cong H\times\mathbb{Z}_4$. Form Example 4.6, it is easily seen that the 3-cyclotomic classes $S_3((0,0))=\{(0,0)\}$, $S_3((0,2))=\{(0,2)\}$, $S_3((1,0))=\{(1,0)\}$, $S_3((1,2))=\{(1,2)\}$ of $A\cong\mathbb{Z}_2\times\mathbb{Z}_4$ are of type I, the 3-cyclotomic classes $S_3((0,1))=\{(0,1),(0,3)\}$ and $S_3((1,1))=\{(1,1),(1,3)\}$ are of type II, and there are no 3-cyclotomic classes of type III. Then $r_{\mathrm{I}}=4$, $r_{\mathrm{II}}=2$, and $r_{\mathrm{III}}=0$. In view of Theorem 4.7, (32) is recalled as*

$$\mathbb{F}_3[H]\cong\prod_{i=1}^{4}(\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3)\times\prod_{j=1}^{2}(\mathbb{F}_9+u\mathbb{F}_9+u^2\mathbb{F}_9).$$

Hence, by Corollary 4.9, each Euclidean self-dual H-quasi-abelian code in $\mathbb{F}_3[G]$ is isomorphic to a code of the form

$$\mathcal{C}_1\times\mathcal{C}_2\times\mathcal{C}_3\times\mathcal{C}_4\times\mathcal{C}_5\times\mathcal{C}_6,$$

where $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ are Euclidean self-dual linear codes of length 4 over $\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$, and $\mathcal{C}_5$ and $\mathcal{C}_6$ are Hermitian self-dual linear codes of length 4 over $\mathbb{F}_9+u\mathbb{F}_9+u^2\mathbb{F}_9$.

4.4. Hermitian self-dual $A\times\mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A\times\mathbb{Z}_{p^s}\times B]$

In this subsection, we focus on the case where m is even and study Hermitian self-dual $A\times\mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A\times\mathbb{Z}_{p^s}\times B]$.

The characterization and enumeration of Hermitian self-dual $A\times\mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A\times\mathbb{Z}_{p^s}\times B]$ are given based on the decomposition of a group algebra $\mathbb{F}_{p^m}[A\times\mathbb{Z}_{p^s}]$ in terms of the following types of p^m-cyclotomic classes of A. A p^m-cyclotomic class $S_{p^m}(a)$ is said to be of *type* I' if $S_{p^m}(a)=S_{p^m}(-p^{\frac{m}{2}}a)$ or *type* II' if $S_{p^m}(a)\neq S_{p^m}(-p^{\frac{m}{2}}a)$. The primitive idempotent e induced by $S_{p^m}(a)$ is said to be of type $\lambda\in\{\mathrm{I}',\mathrm{II}'\}$ if $S_{p^m}(a)$ is a p^m-cyclotomic class of type λ.

Rearrange the terms in the decomposition in Theorem 4.3 based on the p^m-cyclotomic classes of types I' and II', the next theorem follows.

Theorem 4.13. *Let m be an even positive integer and let A be a finite abelian group such that $p \nmid |A|$.*

$$\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s}] \cong \left(\prod_{j=1}^{r_{\mathrm{I}'}} \mathcal{S} \right) \times \left(\prod_{l=1}^{(r_{\mathrm{II}'})/2} (\mathcal{T}_l \times \mathcal{T}_l) \right),$$

where r_{I}' and $r_{\mathrm{II}'}$ are the numbers of elements in a complete set of representatives of p^m-cyclotomic classes of A of types I' and II', respectively, $\mathcal{S}_j = \mathbb{F}_{p^{m_j}} + u\mathbb{F}_{p^{m_j}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m_j}}$ for all $j = 1, 2, \ldots, r_{\mathrm{I}'}$ and $\mathcal{T}_l = \mathbb{F}_{p^{m_{r_{\mathrm{I}'}+l}}} + u\mathbb{F}_{p^{k_{r_{\mathrm{I}'}+l}}} + \cdots + u^{p^s-1}\mathbb{F}_{p^{m_{r_{\mathrm{I}'}+l}}}$ for all $l = 1, 2, \ldots, (r_{\mathrm{II}'})/2$.

Using Theorem 4.13 and the analysis similar to those in [12, Section II.D], the $A \times \mathbb{Z}_{p^s}$-quasi-abelian code $\mathcal{C}$ in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ and its Hermitian dual are given.

Proposition 4.14. *Let s and m be positive integers such that m is even. Let A and B be finite abelian groups such that $|B| = n$ and $p \nmid |A|$. Then an $A \times \mathbb{Z}_{p^s}$-quasi-abelian code in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ can be viewed as*

$$\mathcal{C} \cong \left(\prod_{j=1}^{r_{\mathrm{I}'}} \mathcal{C}_j \right) \times \left(\prod_{l=1}^{(r_{\mathrm{II}'})/2} (\mathcal{D}_l \times \mathcal{D}_l') \right), \tag{35}$$

where $\mathcal{C}_j$, $\mathcal{D}_l$ and $\mathcal{D}_l'$ are linear codes of length n over $\mathcal{S}_j$, $\mathcal{T}_l$ and $\mathcal{T}_l$, respectively, for all $j = 1, 2, \ldots, r_{\mathrm{I}'}$ and $l = 1, 2, \ldots, (r_{\mathrm{II}'})/2$.

Furthermore, the Hermitian dual of $\mathcal{C}$ in (35) is of the form

$$\mathcal{C}^{\perp_{\mathrm{H}}} \cong \left(\prod_{j=1}^{r_{\mathrm{I}'}} \mathcal{C}_j^{\perp_{\mathrm{H}}} \right) \times \left(\prod_{l=1}^{(r_{\mathrm{II}'})/2} \left((\mathcal{D}_l')^{\perp_{\mathrm{E}}} \times \mathcal{D}_l^{\perp_{\mathrm{E}}} \right) \right).$$

The characterization of Hermitian self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ in term of a product of linear codes, and Hermitian self-dual linear codes over Galois extensions of the ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{p^s-1}\mathbb{F}_{p^m}$ is established.

Corollary 4.15. *Let s and m be positive integers such that m is even. Let A and B be finite abelian groups such that $|B| = n$ and $p \nmid |A|$. Then an $A \times \mathbb{Z}_{p^s}$-quasi-abelian code in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is Hermitian self-dual if and only if in the decomposition (35),*

i) $\mathcal{C}_j$ is a Hermitian self-dual linear code of length n over $\mathcal{S}_j$ for all $j = 1, 2, \ldots, r_{\mathrm{I}'}$, and

ii) $\mathcal{D}_l' = \mathcal{D}_l^{\perp_{\mathrm{E}}}$ is a linear code of length n over $\mathcal{T}_l$ for all $l = 1, 2, \ldots, (r_{\mathrm{II}'})/2$.

From Corollary 4.15, it follows that the Hermitian self-duality of $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ depends only on the structure of $A \times \mathbb{Z}_{p^s}$ and the index $n = |B|$ but not the structure of B itself.

Given a positive integer m and a positive integer j, the pair (j, p^m) is said to be *oddly good* if j divides $p^{mt} + 1$ for some odd positive integer t. This notion has been introduced in [12] for characterizing the Hermitian self-dual abelian codes in principal ideal group algebra and completely determined in [9].

Let λ be a function defined on the pair (j, p^m) as

$$\lambda(j, p^m) = \begin{cases} 0 & \text{if } (j, p^m) \text{ is oddly good,} \\ 1 & \text{otherwise.} \end{cases} \tag{36}$$

The number of Hermitian self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ can be determined as follows.

Theorem 4.16. *Let s and m be positive integers such that m is even. Let A and B be finite abelian groups such that $|B| = n$ is even and the exponent of A is M and $p \nmid M$. Then the number of Euclidean self-dual $A \times \mathbb{Z}_{p^s}$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times \mathbb{Z}_{p^s} \times B]$ is*

$$\prod_{d|M} \left(NH_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, p^s) \right)^{(1-\lambda(d,p^{\frac{m}{2}}))\frac{\mathcal{N}_A(d)}{\mathrm{ord}_d(p^m)}} \times \prod_{d|M} \left(N_{p^s}(p^{m \cdot \mathrm{ord}_d(p^m)}, p^s) \right)^{\lambda(d,p^{\frac{m}{2}})\frac{\mathcal{N}_A(d)}{2\mathrm{ord}_d(p^m)}},$$

where $\mathcal{N}_A(d)$ denotes the number of elements of order d in A determined in [1].

Proof. By Corollary 4.15, it is enough to determine the numbers linear codes $\mathcal{C}_j$'s and $\mathcal{D}_l$'s of length n in (35) such that $\mathcal{C}_j$ is Hermitian self-dual. The result can be deduced using arguments similar to those in the proof of Theorem 4.10, where [12, Lemma 3.5] is applied instead of [11, Lemma 4.5]. □

Remark 4.17. *In general, the number $NH_{p^s}(p^m, n)$ of Hermitian self-dual linear codes of length n over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{p^s-1}\mathbb{F}_{p^m}$ in Theorem 4.16 has not been well studied. In the case where the field characteristic is 3, we have the following results.*

1. *The numbers $N_3(p^m, n)$ and $NH_3(3^m, n)$ have been determined in Proposition 2.1 and Theorem 3.4. Hence, the complete enumeration of Hermitian self-dual $A \times \mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ follows.*

2. *The construction/characterization of linear and Hermitian self-dual dual linear codes of length n over $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m} + u^2\mathbb{F}_{3^m}$ have been given in [4] and in the proof of Proposition 3.3. Hence, the construction/characterization of Hermitian self-dual $A \times \mathbb{Z}_3$-quasi-abelian code in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ can be obtained from Corollary 4.15.*

3. *Note that, if n is odd, there are no Hermitian self-dual linear codes of length n over $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m} + u^2\mathbb{F}_{3^m}$ by Theorem 3.4. Hence, there are no Hermitian self-dual $A \times \mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ for all abelian groups B of odd order.*

5. Conclusion and remarks

By extending the technique used in the study of Euclidean self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ in [3], complete characterization and enumeration of Hermitian self-dual linear codes over $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$ have been established for all square prime powers q. An algorithm for constructions of such self-dual codes has veen provided as well as an illustrative example. Subsequently, algebraic characterization of H-quasi-abelian codes in $\mathbb{F}_{p^m}[G]$ has been studied, where $H \leq G$ are finite abelian groups and the Sylow p-subgroup of H is cyclic, or equivalently, $\mathbb{F}_{p^m}[H]$ is a principal ideal group algebra. In the special case where $H \cong A \times \mathbb{Z}_3$ with $3 \nmid |A|$, characterization and enumeration of H-quasi-abelian codes and self-dual H-quasi-abelian codes in $\mathbb{F}_{3^m}[H \times B]$ have been completely determined for all finite abelian group B. As applications, characterization and enumeration of self-dual $A \times \mathbb{Z}_3$-quasi-abelian codes in $\mathbb{F}_{3^m}[A \times \mathbb{Z}_3 \times B]$ can be presented in terms of linear codes and self-dual linear codes over some extensions of $\mathbb{F}_{3^m} + u\mathbb{F}_{3^m} + u^2\mathbb{F}_{3^m}$ determined in [3], [4] and Section 3.

In general, it would be interesting to studied $A \times P$-quasi-abelian codes and self-dual $A \times P$-quasi-abelian codes in $\mathbb{F}_{p^m}[A \times P \times B]$ for all primes p and finite abelian p-groups P. For $e \geq 4$, characterization and enumeration of self-dual linear codes over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{e-1}\mathbb{F}_{p^m}$ are other interesting problems.

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สมการไดโอเฟนไทน์ $8^x + 61^y = z^2$ และ $8^x + 67^y = z^2$

On The Diophantine Equations $8^x + 61^y = z^2$ and $8^x + 67^y = z^2$

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บทคัดย่อ

ในงานวิจัยนี้ได้ศึกษาผลเฉลย (x, y, z) เมื่อ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบของสมการไดโอเฟนไทน์ $8^x + 61^y = z^2$ และ $8^x + 67^y = z^2$ โดยพบว่าสมการทั้งสองมีผลเฉลยเพียงผลเฉลยเดียว คือ $(x, y, z) = (1,0,3)$

คำสำคัญ: สมการไดโอเฟนไทน์ ข้อคาดการณ์ของคาตาลาน

ABSTRACT

In this paper, we study solutions (x, y, z) where x, y and z are non - negative integers of Diophantine equations $8^x + 61^y = z^2$ and $8^x + 67^y = z^2$. We find that both of them have a unique solution, that is $(x, y, z) = (1,0,3)$.

Keywords: Diophantine equation, Catalan's conjecture

1. บทนำ

สมการไดโอเฟนไทน์ เป็นสมการที่ได้ศึกษากันอย่างมากมาย และศึกษาในหลายรูปแบบ รูปแบบหนึ่งที่ได้ศึกษากันอย่างกว้างขวางคือ สมการที่อยู่ในรูป $a^x + b^y = z^2$ เช่น

ในปี ค.ศ. 2011 Suvarnamani [9] ได้ศึกษาผลเฉลยของสมการไดโอเฟนไทน์ $2^x + p^y = z^2$ เมื่อ x, y, z เป็นจำนวนเต็มที่ไม่เป็นลบและ p เป็นจำนวนเฉพาะ

ในปี ค.ศ. 2012 Chotchaisthit [1] ได้ศึกษาผลเฉลยทั้งหมดที่เป็นจำนวนเต็มที่ไม่เป็นลบของสมการไดโอเฟนไทน์ $4^x + p^y = z^2$ โดยที่ p เป็นจำนวนเฉพาะ และในปีเดียวกัน Sroysang [4] ได้พิสูจน์ว่า $(x, y, z) = (1,0,3)$ เป็นผลเฉลยเดียวที่ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบของสมการไดโอเฟนไทน์ $8^x + 19^y = z^2$

ในปี ค.ศ. 2013 Sroysang [5], [6] ได้แสดงว่าสมการไดโอเฟนไทน์ $2^x + 3^y = z^2$ มีเพียงสามผลเฉลยคือ $(0,1,2), (3,0,3)$ และ $(4,2,5)$ และได้ศึกษาสมการไดโอเฟนไทน์ $7^x + 8^y = z^2$ โดยที่ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ และพบว่าสมการดังกล่าวมีผลเฉลยเพียงผลเฉลยเดียว นั่นคือ $(x, y, z) = (0,1,3)$

ในปี ค.ศ. 2014 Sroysang [8], [7] ได้แสดงให้เห็นว่า $(1,0,3)$ เป็นผลเฉลยเดียวของสมการไดโอเฟนไทน์ $8^x + 13^y = z^2$ และ $8^x + 59^y = z^2$ เมื่อ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ

ในงานวิจัยนี้ได้ศึกษาสมการไดโอเฟนไทน์ที่อยู่ในรูป $8^x + p^y = z^2$ เมื่อ p เป็นจำนวนเฉพาะอีกสองจำนวนที่อยู่ถัดจาก 59 นั่นคือ สมการ $8^x + 61^y = z^2$ และ $8^x + 67^y = z^2$

2. ผลลัพธ์หลัก

ทฤษฎีบท 2.1 [3] (ข้อคาดการณ์ของคาตาลาน)
สำหรับ a, b, x และ y ที่เป็นจำนวนเต็ม ซึ่ง $\min\{a, b, x, y\} > 1$
สมการไดโอเฟนไทน์ $a^x - b^y = 1$ มีผลเฉลยเพียงผลเฉลยเดียวคือ $(a, b, x, y) = (3,2,2,3)$

ทฤษฎีบท 2.2 [4] สำหรับ x และ z ที่เป็นจำนวนเต็มที่ไม่เป็นลบ
สมการไดโอเฟนไทน์ $8^x + 1 = z^2$ มีผลเฉลยเพียงผลเฉลยเดียว คือ $(x, z) = (1,3)$

ทฤษฎีบท 2.3 [2] สำหรับ x และ z ที่เป็นจำนวนเต็มที่ไม่เป็นลบ
สมการไดโอเฟนไทน์ $1 + p^x = z^2$ โดยที่ p เป็นจำนวนเฉพาะที่เป็นจำนวนคี่ จะมีผลเฉลยเพียงผลเฉลยเดียวคือ $(p, x, z) = (3,1,2)$

บทแทรก 2.4 สมการไดโอเฟนไทน์ $1 + 61^x = z^2$ ไม่มีผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบ

บทพิสูจน์ โดยทฤษฎีบทที่ 2.3 เนื่องจาก 61 เป็นจำนวนเฉพาะคี่ และ $61 \neq 3$ $\square$

บทแทรก 2.5 สมการไดโอเฟนไทน์ $1 + 67^x = z^2$ ไม่มีผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบ

บทพิสูจน์ โดยทฤษฎีบทที่ 2.3 เนื่องจาก 67 เป็นจำนวนเฉพาะคี่ และ $67 \neq 3$ $\square$

ทฤษฎีบท 2.6 สำหรับ x, y และ z ที่เป็นจำนวนเต็มที่ไม่เป็นลบ
สมการไดโอเฟนไทน์ $8^x + 61^y = z^2$ มีผลเฉลยเพียงผลเฉลยเดียวคือ $(x, y, z) = (1,0,3)$

บทพิสูจน์ ให้ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ ที่ทำให้ $8^x + 61^y = z^2$ (1)

สมมติให้ $x = 0$ จะได้ $1 + 61^y = z^2$ (2)

โดยบทแทรกที่ 2.4 จะได้ว่า (2) ไม่มีผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบ

สมมติให้ $x \geq 1$ เราจะพิจารณา y เป็น 2 กรณีดังต่อไปนี้

<u>กรณี 1</u> ถ้า y เป็นจำนวนคี่ แล้วจะมีจำนวนเต็มที่ไม่เป็นลบ r ที่ทำให้ $y = 2r + 1$

จาก (1) จะได้ $8^x + 61^{2r+1} = z^2$ นั่นคือ $8^x + 61(3721)^r = z^2$

เนื่องจาก $8^x + 61^y$ เป็นจำนวนคี่ จะได้ว่า z^2 เป็นจำนวนคี่ นั่นคือ z เป็นจำนวนคี่

ให้ $z = 2q + 1$ เมื่อ q เป็นจำนวนเต็มที่ไม่เป็นลบ

ทำให้ได้ $z^2 = 4q(q + 1) + 1$ (3)

จะได้ว่า $z^2 \equiv 1 \pmod 8$

เนื่องจาก $3721 \equiv 1 \pmod 8$ จะได้ $3721^r \equiv 1 \pmod 8$

เนื่องจาก $61 \equiv 5 \pmod 8$ ทำให้ได้ $61(3721^r) \equiv 5 \pmod 8$

ดังนั้น $8^x + 61(3721^r) \equiv 5 \pmod 8$ นั่นคือ $z^2 \equiv 5 \pmod 8$ เกิดข้อขัดแย้ง

<u>กรณี 2</u> ถ้า y เป็นจำนวนคู่

<u>กรณีย่อย 2.1</u> $y = 0$

จาก (1) จะได้ $8^x + 1 = z^2$ (4)

โดยทฤษฎีบทที่ 2.2 จะได้ว่า (4) มีผลเฉลยเพียงผลเฉลยเดียวคือ $(x, z) = (1,3)$

นั่นคือ (1) มีผลเฉลยคือ $(x, y, z) = (1,0,3)$

<u>กรณีย่อย 2.2</u> $y > 1$

ให้ $y = 2s$ เมื่อ s เป็นจำนวนนับ

จะได้ว่า (1) สามารถเขียนเป็น $8^x + 61^{2s} = z^2$

นั่นคือ $2^{3x} = (z - 61^s)(z + 61^s)$ (5)

ให้ u เป็นจำนวนเต็มที่ไม่เป็นลบ ที่ทำให้ $2^u = z - 61^s$ และ $2^{3x-u} = z + 61^s$

โดยที่ $3x > 2u$

เราพิจารณา $2^{3x-u} - 2^u = (z + 61^s) - (z - 61^s) = 2(61^s)$

เราจะได้ $2^u(2^{3x-2u} - 1) = 2(61^s)$ (6)

จาก (6) จะเป็นไปได้เพียง 1 กรณี นั่นคือ $2^u = 2$ และ $2^{3x-2u} - 1 = 61^s$ (7)

นั่นคือจะได้ $u = 1$ ทำให้ได้ว่า $2^{3x-2} - 1 = 61^s$ (8)

ถ้า $s = 1$ จะได้ $2^{3x-2} = 62$ ซึ่งกรณีนี้เป็นไปไม่ได้

ดังนั้น $s > 1$ จะได้ว่า $61^s > 61$

นั่นคือ $61^s + 1 > 62 > 32$

จาก (8) เราจะได้ $2^{3x-2} = 61^s + 1 > 2^5$ ดังนั้น $3x - 2 > 5$

ซึ่งจะได้ว่า $\min\{2, 61, 3x - 2, s\} > 1$ และโดยทฤษฎีบทที่ 2.1 จะได้ว่า

สมการ $2^{3x-2} - 61^s = 1$ ไม่มีผลเฉลย

นั่นคือ $(x, y, z) = (1,0,3)$ เป็นเพียงผลเฉลยเดียวของสมการไดโอเฟนไทน์ $8^x+61^y = z^2$
โดยที่ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ ☐

ทฤษฎีบท 2.7 สำหรับ x, y และ z ที่เป็นจำนวนเต็มที่ไม่เป็นลบ
สมการไดโอเฟนไทน์ $8^x+67^y = z^2$ มีผลเฉลยเพียงผลเฉลยเดียวคือ $(x, y, z) = (1,0,3)$

บทพิสูจน์ ให้ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ ที่ทำให้ $8^x+67^y = z^2$ (9)

สมมติให้ $x = 0$ จะได้ $1+67^y = z^2$ (10)

โดยบทแทรกที่ 2.5 จะได้ว่า (10) ไม่มีผลเฉลยที่เป็นจำนวนเต็มที่ไม่เป็นลบ

สมมติให้ $x \geq 1$

เนื่องจาก 8^x+67^y เป็นจำนวนคี่ จะได้ว่า z^2 เป็นจำนวนคี่ นั่นคือ z เป็นจำนวนคี่

ให้ $z = 2t + 1$ เมื่อ t เป็นจำนวนเต็มที่ไม่เป็นลบ

จะได้ว่า

$$8^x+67^y = (2t + 1)^2$$
$$8^x+67^y = 4(t^2 + t) + 1$$
$$67^y - 1 = 4(t^2 + t) - 8^x$$
$$67^y - 1 = 4[(t^2 + t) - 2 \cdot 8^{x-1}]$$

ดังนั้น $67^y \equiv 1 \pmod 4$

เราจะแสดงว่า y เป็นจำนวนคู่

สมมติให้ y เป็นจำนวนคี่ แล้วจะมีจำนวนเต็ม k ที่ไม่เป็นลบที่ทำให้ $y = 2k + 1$

เนื่องจาก $67 \equiv 3 \pmod 4$ จะได้ว่า $67^2 \equiv 9 \pmod 4$

ทำให้ $67^2 \equiv 1 \pmod 4$ ดังนั้น $67^{2k} \equiv 1 \pmod 4$

จะได้ว่า $67^{2k+1} \equiv 3 \pmod 4$ นั่นคือ $67^y \equiv 3 \pmod 4$ จึงเกิดข้อขัดแย้ง

ทำให้ y เป็นจำนวนคู่ เราจะแบ่งพิจารณาออกเป็น 2 กรณี ต่อไปนี้

<u>กรณี 1</u> $y = 0$

จะได้ว่า $8^x + 1 = z^2$ (11)

โดยทฤษฎีบทที่ 2.2 จะได้ว่า (11) มีผลเฉลยเพียงผลเฉลยเดียวคือ $(x, z) = (1,3)$

นั่นคือ สมการ $8^x+67^y = z^2$ มีผลเฉลยคือ $(x, y, z) = (1,0,3)$

<u>กรณี 2</u> $y > 1$

ให้ $y = 2m$ เมื่อ m เป็นจำนวนนับ

เราจะได้ (9) เขียนเป็น $8^x+67^{2m} = z^2$

นั่นคือ $2^{3x} = (z - 67^m)(z + 67^m)$ (12)

ให้ u เป็นจำนวนเต็มที่ไม่เป็นลบที่ทำให้ $2^u = z - 67^m$ และ $2^{3x-u} = z + 67^m$ โดยที่ $3x > 2u$

เราพิจารณา $2^{3x-u}-2^u = (z + 67^m) - (z - 67^m) = 2(67^m)$

เราจะได้ $2^u(2^{3x-2u} - 1) = 2(67^m)$ (13)

จาก (13) จะเป็นไปได้เพียง 1 กรณี นั่นคือ $2^u = 2$ และ $2^{3x-2u} - 1 = 67^m$

ทำให้ $u = 1$ ดังนั้น $2^{3x-2} = 67^m + 1$

ถ้า $m = 1$ จะได้ $2^{3x-2} = 68$ ซึ่งเป็นไปไม่ได้

ทำให้ $m > 1$ และ $2^{3x-2} = 67^m + 1 > 67 + 1 > 64 = 2^6$

จะได้ว่า $3x - 2 > 6$

ทำให้ $\min\{2,67,3x - 2, m\} > 1$

โดยทฤษฎีบทที่ 2.1 จะได้สมการ $2^{3x-2} - 67^m = 1$ ไม่มีผลเฉลย

นั่นคือ $(x, y, z) = (1,0,3)$ เป็นเพียงผลเฉลยเดียวของสมการไดโอเฟนไทน์ $8^x+67^y = z^2$

โดยที่ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบ □

3. สรุป

สำหรับในงานวิจัยนี้เราได้ศึกษาผลเฉลย (x, y, z) เมื่อ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นลบของสมการไดโอเฟนไทน์ $8^x+61^y = z^2$ และ $8^x+67^y = z^2$ ซึ่งเราได้พบว่า $(x, y, z) = (1,0,3)$ เป็นผลเฉลยเพียงผลเฉลยเดียวของสมการไดโอเฟนไทน์ทั้งสอง

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